2
Preliminaries and literature survey

In this chapter, we survey the existing literature regarding the problems considered in this thesis followed by a brief review of standard concepts in network science that are used in the thesis. The chapter consists of three major parts: first part contains basic definitions of terminology and notations, second part provides a review of existing network models and network reconstruction techniques, and the third part is related to the existing diffusion protocols on networks.

2.1 BASIC DEFINITIONS

In this section, we recall some basic notations in network theory following [Newman, 2010]. We also introduce the notations that are used throughout the thesis.

2.1.1 Graph theoretic concepts

A graph or network $G = (V, E)$ is defined as a collection of nodes and edges, where V is the set of nodes and $E \subseteq V \times V$ is the set of edges. The number of nodes and edges in the network G are denoted by $|V|$ and $|E|$ respectively. $|V|$ is called the size of the network. A network can be stored in the form of an edge list or a matrix known as adjacency matrix A, defined as follows. The ij^{th} entry a_{ij} of A represents the existence of an edge between the nodes i and j. Thus $a_{ij} = 1$ if nodes i and j are linked otherwise $a_{ij} = 0$. Obviously A is a $|V| \times |V|$ matrix.

Directed and Undirected network- If the edge set contains a set of ordered pair of nodes then it is called directed otherwise the network is called undirected. For instance, if $(i, j) \in E$ in a directed network then there is an edge from i in to the direction j. We consider only undirected network in the thesis unless mentioned otherwise. In case of undirected network, the adjacency matrix associated with a network is a symmetric matrix *i.e.* $A^T = A$ or $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq |V|$.

Complete network- A network is said to be a complete network if any two distinct nodes in the network are linked by an edge. Thus the ij^{th} entry of the adjacency matrix A, that is $a_{ij} = 1$ if and only if $i \neq j$. We denote a complete network on n nodes as K_n. The number of edges in K_n is $n(n-1)/2$.

Degree- Degree of a node i in a network, denoted by k_i, is the number of nodes adjacent to it. Degree vector of a network is denoted by $\mathbf{d} = A\mathbf{1}$, where A is the adjacency matrix associated with the network and $\mathbf{1}$ denotes the all one vector of compatible dimension.

Degree distribution- The degree distribution of G is $p(k)$ which equals the fraction of nodes in G of degree k. In other words, for any node i in the network picked uniformly at random, $P(k_i = k) = p(k)$, *i.e.* the probability that the degree of node i is k equals $p(k)$.

Path- A path (self-avoiding path) in a network is a sequence of distinct nodes such that every consecutive pair of nodes in the sequence is linked by an edge in the network.

Connected network- A network said to be connected if there exist a path between any pair

of distinct nodes in the network.

Clustering coefficient (CC)- The clustering coefficient of a node i in a network is defined by

$$CC(i) = \frac{(A^3)_{ii}}{k_i(k_i-1)},$$

where $(A^3)_{ii}$ defines the i^{th} diagonal entry of A^3. $CC(i)$ represents the average probability that a pair of i's friends are friends of one another. Clustering coefficient of a network is defined as

$$\overline{CC} = \sum_i \frac{CC(i)}{|V|}.$$

Spectrum of a network- The spectrum of a network is the multi-set of eigenvalues of the adjacency matrix associated with the network. Since A is a symmetric matrix, all the eigenvalues of A are real and can be ordered as $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$. The maximum eigenvalue $\lambda_1(A)$ of the network is called the spectral radius (SR) of the network.

Scree plot- It is the plot of the eigenvalues (or singular values) of the adjacency matrix of a network, versus their rank [Leskovec, 2008].

Network values- Entries of the leading eigenvector which is defined as the eigenvector corresponding to maximum eigenvalue of the adjacency matrix associated with a network are known as the network values.

Algebraic connectivity- Let A be the adjacency matrix of a network and D is a diagonal matrix such that $D_{ii} = k_i$. The graph Laplacian of the network is defined by

$$L = D - A.$$

It is easy to verify that L is a symmetric positive semi-definite matrix. Thus eigenvalues of L are non-negative and can be ordered as $\lambda_1(L) = 0 \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$. $\lambda_2(L)$ is the minimum non-zero eigenvalue of the graph Laplacian of a connected network which is known as algebraic connectivity (AC) of the network.

Geodesic path (Shortest path)- Geodesic path or shortest path between a pair of nodes i and j is a path which connect them with minimum number of edges and denoted by $dist(ij)$.

Diameter- The diameter of a connected network is the length of the longest geodesic path (shortest path) between any pair of vertices in the network.

Expected diameter- Expected diameter of a network, denoted by D_G is the maximum of the average distance between any pair of nodes in the network defined by

$$D_G = max_{ij}\{l_{ij}\},$$

where l_{ij} is defined as

$$l_{ij} = \sum_{k=1}^{\infty} k\, z(k),$$

where $z(k)$ is the distribution of path length between a pair of nodes i and j.

Average path length-(APL) Average path length of a network is defined as the average number of steps to reach a node from any other node via shortest route. Thus

$$APL = \sum_{ij} \frac{dist(ij)}{|V|\,(|V|-1)}.$$

Small-world network- In general, a network is called a small world network if the diameter of the network is small. In particular, if $l_{ij} \propto \log|V|$ for any two nodes $i \neq j$ in G then G is called small world network, while clustering coefficient of G is not small.

Community structure- Communities in G are groups of nodes with high concentration of edges in each of these groups and low concentartion of edges between any two such groups. Communities are also called clusters or modules. The strength of community structure in a network is measured by modularity index Q defined by

$$Q = \frac{1}{2m}\sum_{ij}\left[A_{ij} - \frac{k_i k_j}{2m}\right]\delta_{c_i c_j}$$

where m is the number of edges in the network, k_i and k_j are the degrees of the nodes i and j which belong to communities c_i and c_j respectively, and δ is the Kronecker delta function. Q varies from -1 to 1.

Power-law- A function defined by

$$f(x) = C\,x^{\gamma}$$

is called a power-law function. Here C and γ are constants and γ is called the power-law exponent of f. Note that a power-law function is scale invariant *i.e.*

$$f(ax) = C(ax)^{\gamma} = Ca^{\gamma}x^{\gamma} = a^{\gamma}f(x) \propto f(x).$$

Thus a real network is called a scale-free network if its degree distribution follows power-law in its tail.

Triads- Triad is closed structure among three entities. Let v_1, v_2, and v_3 be three nodes in G. Then a triad (v_1, v_2, v_3) exist, if v_i and v_j are linked for any $i \neq j$. Thus a triad is a complete network on three nodes.

Assortativity- The Assortative Index (AI) of a network G is defined by

$$AI(G) = \frac{\sum_{ij}(a_{ij} - \frac{k_i k_j}{2m})k_i k_k}{\sum_{ij}(k_i\delta_{ij} - \frac{k_i k_j}{2m})k_i k_j}$$

where a_{ij} is the ij^{th} entry of the adjacency matrix associated with G, δ_{ij} is the Kronecker delta function. Obviously $-1 \leq AI(G) \leq 1$. A positive value of $AI(G)$ signifies nodes with similar degrees are linked whereas a negative value of $AI(G)$ implies that similar degree nodes are not linked.

2.2 DYNAMICS OF THE NETWORKS

In this section, a brief review of some existing network models and network reconstruction techniques.

2.2.1 Network models

There are two types of network modelling approaches in general. In the first approach, the number of nodes is fixed before the formation of links in the network model whereas in the second approach the model starts with a small graph (possibly with a node or a link) then new nodes appear in the initial network and the process of link formation starts.

In the following we first discuss two fundamental random network models in which the number of nodes is fixed in the beginning of the formation of the network. These models are known as ER (Erdős-Rényi) model [Erdös and Rényi, 1959], WS (Watts and Strogatz) model [Watts and Strogatz, 1998a]. We follow ER random network from [Chung and Lu, 2006]

Classical ER random model

The classical ER model for random networks has simple assumption that each pair of nodes is equally probable to generate a link independently with probability p for some fixed value. The mathematical notation of the model is $G(n, p)$, where n is the number of nodes and p is the probability to generate links. Classical ER model is effective to calculate expected properties of model networks but not able to capture the properties of real networks. The generation of ER model is as follows:

Consider a biased coin which has probability $p, 0 \leq p \leq 1$, of coming up heads. In a network on n nodes under ER model, each link is determined by flipping the coin. If head comes, link is formed between the considered pair of nodes. The process of flipping the coin is repeated for each pair of nodes. The resulting network is a ER random network. For details see Chapter 5 in [Chung and Lu, 2006].

WS model (small world model)

In 1998 Watts and Strogatz gave a hypothesis that networks are neither totally random as ER model nor perfectly structured similar to lattice structure. They proposed that real-world networks are the combination of both, the regular networks and the random networks. Regular network maintains high clustering, while random network maintains the low average path length and diameter. Thus WS model exhibits small-world phenomena. They provided a time line structure of the networks to be produced in which one end is the perfectly structured regular network and the other end is the totally random ER model. Real-world networks lie between these two ends.

In the generation of the networks under WS model, a regular network of desired regularity is considered first, then to introduce randomness in the network, each link is rewired with probability p and the resulting network would be a WS model network which shows high clustering and low average path length. In another approach, a WS model can be implemented by considering a regular network G_{re} and combining with a random network G_{ra}. The resulting network $G = G_{re} \cup G_{ra}$ exhibits the properties of WS model. The variation in the properties depends on the mixing parameter or rewiring probability p. WS model also shows binomial degree distribution in contrast to many real-world networks which have power-law degree distribution and modular structure. Still, WS model is widely implemented and known for significant contribution in network science.

Scale-free network models

As defined earlier, a network is called a scale-free network if its degree distribution follows power-law, at least asymptotically. In 1999 Barabasi and Albert introduced a preferential attachment model for growing networks which exhibits power-law degree distribution with power-law exponent 3. Consequently, several models have been proposed in literature for scale-free networks. In the following we review some popular scale-free network generative models.

We mention that if we characterize real networks on the basis of degree distribution, a big class of real-world networks follow scale-free degree distribution [Newman, 2010; Barabási *et al.*, 2009; Eguiluz *et al.*, 2005]. For example, power-law degree distributions have been observed

in phones call graphs [Abello *et al.*, 1998], the Internet [Faloutsos *et al.*, 1999], the Web [Barabási and Albert, 1999b; Huberman and Adamic, 1999; Kumar *et al.*, 1999b], click-stream data [Bi *et al.*, 2001], and for a who-trusts-whom social network [Chakrabarti *et al.*, 2004]. We mention that real scale-free networks also exhibit small-world phenomenon in many contexts (see [Barabási and Albert, 1999b; Bollobás and Riordan, 2004; Broder *et al.*, 2000; Chung and Lu, 2002; Kleinberg, 2002; Watts and Strogatz, 1998a]).

It is speculated that '*rich get richer*' phenomena is responsible for scale-free behaviour of the networks which have power-law degree distribution [Simon, 1955]. Most of the existing network models use the preferential scheme for generation of scale-free networks [Chakrabarti and Faloutsos, 2006]. To explain the evolution phenomena of growing networks, development of the models based on preferential attachment mainly started in 1999 when Albert and Barabasi applied degree based preferential attachment to explain the evolution of the networks like WWW and citation networks [Barabási and Albert, 1999b]. Indeed we mention that Herbert Simon already confirmed the existence of power law tail in [Simon, 1955] due to '*rich get richer*' phenomena before Barabasi *et al.*. Later Derek Price applied this idea in the context of networks [Price, 1976]. Here, the basic idea behind the preferential attachment is that the probability of connection from a new appeared node i to pre-existing node j is given by

$$p_{ij} = \frac{k_j + x_j}{\sum_j k_j + x_j}. \tag{2.1}$$

where x_j is a constant. Derek Price used it to generate a directed network in which each time a node appears with constant out-degree and join the existing network using probability p_{ij} in which k_j is the in-degree of node j and $x_j = 0$ for all j. But initially, each node has zero in-degree that prevents the growth of the network. To solve this problem Price replaced x_j with constant k_0 for all j. Barabasi developed this model, known as BA model, to produce undirected networks in which $x_j = 0$ for all j because the network is undirected that's why it does not confront the problem what Price faced initially. BA model has constant value 3 of the power-law exponent γ and clustering of network produced under this model has $O(1/n)$ which closes to zero as the network grows.

Several parametrized models for generation of complex networks have been proposed in literature and a list of interesting network generative models can be found in [Chakrabarti and Faloutsos, 2006; Bollobás and Riordan, 2003]. For example, Albert and Barabasi introduced a parametrized model using both random and preferential attachment for evolving networks [Albert and Barabási, 2000]. In Edge Coping Models (ECM) the nodes from an existing network are copied and edges are added preferentially [Kleinberg *et al.*, 1999; Kumar *et al.*, 1999a, 2000]. Node Fitness Model (NFM) is defined by introducing the fitness value η_i associated with each node i [Bianconi and Barabási, 2001]. The linking probability in NFM is defined as $p_{ij} = \eta_j k_j / \sum_j (\eta_j k_j)$ for any two nodes $i \neq j$, where k_i is the degree of node i in the network. In PageRank based preferential attachment models, the crux is to match the PageRank distribution of a real-world network in addition to the degree distribution [Pandurangan *et al.*, 2002]. In random walker growing network models, a new node is linked to a randomly selected node in an existing network. Then the new node attempts to make links with one of the first neighbors of the randomly selected existing node with some probability and the process stops if the attempt to make such a link fails [Vázquez, 2003; Vazquez, 2001]. Dorogovtsev et al. proposed a variety of one parameter models, see [Dorogovtsev and Mendes, 2001b,a; Dorogovtsev *et al.*, 2000; Dorogovtsev and Mendes, 2013]. In [Dorogovtsev *et al.*, 2000], the model given by Dorogovtsev et al. (known as DMS model) considers initial attractiveness A_0 assigned to each node and the probability of link formation is defined by

$$p_{ij} = \frac{A_0 + k_j}{\sum_j (A_0 + k_j)}, \tag{2.2}$$

which is similar to the Price model [Price, 1976]. They have shown that this model can produce

networks whose degree distribution follow power-law in the tail and the power-law exponent varies from 2 to ∞ depending on the initial attractiveness. Later, Broder et al. [Broder *et al.*, 2000] and Faloutsos et al. [Faloutsos *et al.*, 1999] reported the increase of the average degree over time on the Web and the Internet that motivated Dorogovtsev and Mendes to modify their model in order to show a similar behavior in their proposed models [Dorogovtsev and Mendes, 2001a]. They introduced degree dependent and time dependent attractiveness to forcefully incorporate the densification in network generated by their model. Here, they have shown that the number of edges grows polynomially with the number of nodes which is called 'accelerated growth' [Dorogovtsev and Mendes, 2001a]. for Some interesting multi-parameter network models can be found in [Xuan *et al.*, 2007, 2006; Ravasz *et al.*, 2002]. The above discussion describes the great efforts made in literature for generation of network models which inherit properties of real networks.

Each of the above mentioned models captures some collective structural properties of real networks. However, given a specific real network, the potential of these models to generate a network which can capture the structural properties of the given network is largely unexplored.

A big chunk of network related empirical studies is devoted to finding the structural pattern in a static network which is the snapshot of a dynamic network at some time, see a detailed discussion in [Ntoulas *et al.*, 2004]. Nevertheless, there are some exceptional studies that lead to the analysis of dynamic networks in which the work of Katz [Katz, 2005] and Redner [Redner, 2004] are primary. Katz has discovered the densification power law in citation networks, and Render studied the evolution of the citation graph of Physical Review papers over the past century. Katz's work builds on his earlier research on power-law relationships between the size and the recognition of professional communities [Katz, 1999]; his work on densification is focused specifically on citations. In [Chung and Lu, 2006] (chapter 3), Fan Chung and Linyuan Lu presented a generative model (called as Chung-Lu model) for scale-free networks which is also based on preferential attachment scheme but the model includes internal growth of the network as well. During the internal growth of the network, links can be generated without addition of new nodes. Chung-Lu model is facing the same problem of community structure as most of the others.

In community guided attachment (CGA) model, community structure generated recursively, community within community, and the probability of link between a pair of nodes defined by an inverse function, also known as difficulty function, of standard tree distance of the considered nodes. Branching factor decides the maximum height of the hierarchical structure of the network and recursive structure of communities. Forest-Fire-Model (FFM) is inspired by natural phenomena of spreading of fire in a forest. In FFM, a new node gets attached with an old node randomly, then selects neighbours of previous neighbours recursively until the processes die out. A *forward burning probability p*, and a *backward burning ratio r* are selected for each node to generate out-links and in-links respectively. For more details, see [Leskovec *et al.*, 2007].

Community guided attachment model (CGA) [Leskovec *et al.*, 2007], coping model [Kleinberg *et al.*, 1999; Kumar *et al.*, 1999a, 2000], Forest Fire Model (FFM) [Leskovec *et al.*, 2007] and Kronecker Graphs based model [Leskovec *et al.*, 2005; Mahdian and Xu, 2007] are some examples which contributed significantly in the field of study of dynamic networks. In [Seshadhri *et al.*, 2012], Block Two-Level ER model (BTER model) is presented which is based on the idea that networks are the collection of ER communities. In this model first nodes are divided into small groups which are converted into dense ER sub-networks and finally dense ER sub-networks (communities) are linked. BTER model is well fitted to represent the structure of real networks. Indeed the BTER model accommodate the growth process which are observed in real networks.

2.2.2 Structural Network reconstruction

A few reconstruction methods are introduced recently for structural reconstruction of networks. For example, a reconstruction method is proposed using betweenness centrality in Comellas and Paz-Sánchez [2008], spectral reconstruction of complex network is introduced in Comellas and Diaz-Lopez [2008], and evolutionary reconstruction of networks is considered in Ipsen and Mikhailov [2002]. Indeed a common feature in these proposed approaches is the consideration of an initial (random) network and then the given network is reconstructed by rewiring the edges of the initial network. Here we mention that network reconstruction (NR) methods are proposed in the context of accessing the dynamics on the nodes of a network. This approach approximately determines the interaction pattern between the nodes during the dynamical process Angulo *et al.* [2015] Shen *et al.* [2014]. However, the authors in Angulo *et al.* [2015] discussed the fundamental limitations of NR.

In [Comellas and Diaz-Lopez, 2008; Comellas and Paz-Sánchez, 2008; Ipsen and Mikhailov, 2002], the authors have attempted to reconstruct a network by incorporating a given structural or spectral property of the network. Francesc Comellas et al. have proposed to reconstruct a network by using its Laplacian eigenvalues. The structural reconstruction is done by considering an initial random network on which edge rewiring process is applied to match the Laplacian eigenvalues of the given network [Comellas and Diaz-Lopez, 2008]. The process of rewiring terminates when a stopping criteria is met. They introduce a cost function and use the *tabu* search combinatorial optimization method to investigate the performance of the proposed method. A similar process called simulated annealing is used in [Comellas and Paz-Sánchez, 2008] for structural reconstruction of a network from the values of the betweenness centrality of the network. The method starts with an initial random network and at each time step rewiring of an edge is accepted if betweenness of the network after rewiring gets closer to that of the original network. In [Ipsen and Mikhailov, 2002], a Metropolis algorithm is used to reconstruct networks from their Laplacian spectra. The reconstruction of a given network is done by first selecting an arbitrary initial network and then mutation and selection process is applied. Indeed a node is selected randomly from the initial network with zero degree (incident edges of the selected node are deleted) and then new edges incident to this node are formed to create a mutated network. Comparing the spectral distance between these networks a criteria for accepting a mutation is defined and if the criteria is not met, the mutation process is applied to an another node until the desired result is obtained.

Besides, the configuration model is extended to incorporate the degree-dependent clustering and degree-degree correlations in [Serrano and Boguná, 2005] and [Weber and Porto, 2007] respectively. Here random networks are generated with additional structural properties. Pusch et al. have provided a random network generation method that incorporates both the degree-degree correlations and degree-degree clustering [Pusch *et al.*, 2008]. Further, these approaches are exploited in [Karalus and Krug, 2016] for reconstruction of evolved dynamic networks.

However, the above mentioned methods require a lot of verification in each step of the algorithm and thus may need a little bit of luck in order to reconstruct a given network within a given time frame. These methods are also adhoc and lack intuition about the evolution process during its formation as a growing network to achieve a particular structure in the desired network. These limitations in solving the problem of structural reconstruction of networks trigger-off the following question. Can structural reconstruction of network be done with the help of parametric modeling approach? This calls for development of parametric network generative models such that a network with pre-determined structural properties can be constructed by choosing suitable values of the parameters involved in the model.

2.3 DIFFUSION PROCESSES ON NETWORKS

In this section, we briefly review three existing models of diffusion protocols on networks. Diffusion on networks is a process in which a diffusive material spread from regions of high intensity or density to the regions of low and affected by many factors such as structural properties of networks and applied diffusion protocols. It could be the spreading of an idea, information or disease.

2.3.1 Agreement protocol

Consider a connected network $G = (V, E)$ of size $n = |V|$ in which observers or sensors are attached to each node that are also called as agents. Here $V = \{1, 2, 3, \ldots, n\}$ and $E \subset V \times V$ is the set of connected pairs of nodes in the network G. Let $x_i(t)$ be a variable associated with an observer i at time t. Here on, we use node in place of observer to avoid ambiguity. Let $x_i(t)$ is the status of a diffusive material in the node i at time t. The diffusion dynamics at node i depends on the status of its neighbours which is given by

$$\dot{x}_i(t) = \sum_j a_{ij} \left(x_j(t) - x_i(t) \right).$$

where a_{ij} is ij^{th} entry of the adjacency matrix A corresponding to the network. Thus the collective dynamics of the networked system is given by

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t). \tag{2.3}$$

where $\mathbf{x}(t) = [x_1(t),\ x_2(t),\ \ldots x_n(t)]^T$, and $L = D - A$ is the Laplacian matrix of the network G. Solution of Eq. (2.3) is given by

$$\mathbf{x}(t) = e^{-Lt}\mathbf{x}_0,$$

where $\mathbf{x}_0 = [x_1(0),\ , x_2(0),\ , \ldots x_n(0)]^T$ is initial state vector of the network. In an undirected network, if $\mathbf{v}_i$ is an eigenvector of the matrix L corresponding to the eigenvalue $\lambda_i(L)$ then the collective dynamics of the networked system defined by Eq. (2.3) at time t is

$$\mathbf{x}(t) = \sum_{i \in V} e^{-\lambda_i(L)t} (\mathbf{v}_i^T \mathbf{x}_0) \mathbf{v}_i.$$

It is well known that $\lambda_1(L) = 0$ is the smallest eigenvalue of the Laplacian matrix L and $\mathbf{v}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ is corresponding unit eigenvector irrespective of network structure, where $\mathbf{1}$ is all one vector of length n. In a connected network, as $t \to \infty$, $\mathbf{x}(t) \to \bar{x}_0\mathbf{1}$ where $\bar{x}_0 = \frac{1}{n}\sum x_i(0)$ is the average of the initial states of the nodes of the network.

The converging state of any connected networked system is invariant irrespective of the topology of the underlying network under the agreement protocol given by Eq. (2.3). Convergence rate of the agreement protocol is key to many real-time applications. Time of reaching consensus (agreement) depends on the second smallest eigenvalue of the graph Laplacian L that is known as algebraic connectivity of the network. Apparently, consensus problem reduces to the spectral analysis of Laplacian of network G. Other variants of consensus problem are also examined in [Olfati-Saber and Murray, 2004; Ren and Beard, 2004; Moreau, 2003; Hatano and Mesbahi, 2005; Eren *et al.*, 2002].

2.3.2 Random Walk

Random walk dynamics is often used to study different diffusion dynamics. Dynamics of a random walker has been investigated in many contexts [Nash-Williams, 1959; Tetali, 1991], for example, navigation and centrality of networks [Perra *et al.*, 2012; Starnini *et al.*, 2012], routing of

packets in Internet, and diffusion in communication networks. Traffic in transportation networks can also be represented as phenomena of random walkers [Wang *et al.*, 2006]. Random walk dynamics has been applied to explore the dynamics of wealth's distribution in economic networks, gene expression pathways in biological networks and search (navigation) strategies in Internet [Adamic *et al.*, 2001; Tadić and Rodgers, 2002; Tadić and Thurner, 2004; Kim *et al.*, 2002; Rosvall *et al.*, 2005; Germano and de Moura, 2006]. In random walks, a walker positioned at a node i can move to any node j which is linked to the node i in the network with equal probability. Thus the selection of a node for a move is uniform among the neighbors of the current node occupied by a random walker [Akyildiz *et al.*, 2000; Noh and Rieger, 2004].

Let $p_i(t)$ is the occupation probability of a node i at time t then,

$$p_i(t+1) = \sum_j a_{ij} \frac{p_j(t)}{k_j},$$

and the collective dynamics of the random walking in a network is given, [Redner, 2001], by

$$\mathbf{p}(t+1) = AD^{-1}\mathbf{p}(t), \tag{2.4}$$

where a_{ij} is the ij^{th} entry of adjacency matrix A of the network. $a_{ij} = 1$, if nodes i and j are connected otherwise 0, D is the degree matrix. Solution of the Eq. (2.4) is given by

$$\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}_\infty = \frac{\mathbf{d}}{\mathbf{1}^T\mathbf{d}},$$

where $\mathbf{d} = D\mathbf{1}$ and $\mathbf{1}$ is a all-one column vector of appropriate length.

2.3.3 Susceptible-Infected-Susceptible (SIS) model

SIS is a simple model to study epidemics like diffusion. Here, we consider SIS diffusion model at node level, assuming $x_i(t)$ is the probability that node i is infected at time t, the diffusion model is given by

$$\dot{x}_i(t) = \gamma \sum_j a_{ij} x_j(t) - \delta x_i(t),$$

where a_{ij} is ij^{th} element of the connection (adjacency) matrix A of the considered network. γ is the diffusion probability of an edge and δ is recovery probability of an infected node that directly goes to susceptible class. In this model, a node can be either in the susceptible state (S) or infected state (I). $x_i(t)$ is the probability of the node i being infected or informed at time t. The collective dynamics of SIS model is given by,

$$\dot{\mathbf{x}}(t) = \gamma A\mathbf{x}(t) - \delta I\mathbf{x}(t), \tag{2.5}$$

$\mathbf{x}(t) = [x_i(t), \ , ..., \ , x_n(t)]^T$ is the state vector. For a given time series, $x(t)$ and $\dot{x}(t)$ are known. Eq. (2.5) is linear in coefficient of matrix A.

2.3.4 Network reconstruction using diffusion dynamics

Previously, we discussed about the existing methods for network reconstruction based on a given structural property. In this section, we summarize the methods of reconstruction of network topology using the diffusion dynamics on the network.

Many dynamical systems are studied in the context of network reconstruction [Han *et al.*, 2015; Zhang *et al.*, 2016]. Methods based on phase synchronization [Shandilya and Timme, 2011], Lyapunov exponent [Aniszewska and Rybaczuk, 2008; Zhou and Lu, 2007; Comellas and Diaz-Lopez, 2008], feedback control [Yu, 2010; Yu and Parlitz, 2011] and compressed sensing

theory (CST) [Ma *et al.*, 2015; Tang *et al.*, 2015] have been developed previously to reconstruct the underlying network [Han *et al.*, 2016; Guo *et al.*, 2016]. Reconstruction of network topology using phase oscillator is based on deterministically known coupling strength and nodal dynamics in advance. Later on, authors have considered some uncertainty as an improvement and modification of the network reconstruction procedures [Timme, 2007; Shandilya and Timme, 2011]. CST is used to reconstruct the interaction pattern of a network of coupled oscillators and games from time series [Wang *et al.*, 2011c,d,b; Wu *et al.*, 2016]. Further, CST is also used to uncover the interaction pattern of nodes in a network given that the time series has only binary data [Shen *et al.*, 2014]. In [Li *et al.*, 2014], network reconstruction is successfully done by phase synchronization with linearly and non-linearly coupled systems based on *Kuramoto* phase oscillators.

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