4

Parametric model for signed networks: A 2-layered network formation

The past of the research in the area of network science witnesses the domination of the study of the unsigned or more homogeneous structure of real-world networks which comprises the study of unsigned undirected or directed networks [Boccaletti *et al.*, 2006; Bornholdt and Schuster, 2002; Newman, 2010]. The study of the dynamics of social structure might be biased due to the consideration of only positive relations between social objects (people). The evolution of the structure of social network and the dynamics of diffusion processes on unsigned social networks overrated by positive links while the studies in this area confirm the existence of multiple types of relationships among the people which can be broadly categorized into three types according to their nature of actions. For example, if the relationship between two persons affect the action of each other positively then it is treated as a positive relation and represented by $+1$; negative relations have opposite nature of actions between linked nodes and denoted by -1 and the third one is the ignorance between two nodes that is represented by 0. Exemption of the negative links on social networks does not provide the real picture of diffusion phenomena including influence maximization, innovation adaptation, and information spreading. Negative ties also play important role in influence propagation and innovation adaptation. In a simple example of voting dynamics, two negatively added persons have a tendency to oppose each other. So to get a more realistic picture of diffusion dynamics on social networks, consideration of negative links is important. Thus a network is called a signed network if it has both positive and negative links.

For example, it was reported in [Zachary, 1977] that in the network of karate club the division of the club members into two groups indicates the existence of negative relations that is conflict between members of two groups which is a type of negative relationship that is ignored while studying the structure of the network of karate club [Girvan and Newman, 2002]. The process of group formation in karate club network redirects our thought of community formation in social networks. Grouping can be explained as an effect of conflict between nodes which reform the network into groups of people who share the similar thought of actions. In this dimension, a community is a maximal group of people which have minimal conflicts. There are multiple examples of social networks which have negative and positive relations between the nodes in similar context [Leskovec and Krevl, 2014]. Youtube users may like or dislike the videos; facebook users can do supportive or opposing comments on some topic or post. Ratings given by product users have similar network structure between users and products. Lower ratings have a negative impact on the selling of that product and higher ratings do the reverse. The network constructed by these users and products have the different type of links which can be represented by $+1$, 0 and -1.

In the context of signed networks, [Anchuri and Magdon-Ismail, 2012; Chiang *et al.*, 2011; Doreian, 2004; Doreian and Mrvar, 2009; Leskovec *et al.*, 2010] *Structural balance theory* introduced by Heider in 1940s [Heider, 1946a], first in the context of social psychology which is converted in the graph-theoretic language by Cartwright and Harary [Cartwright and Harary, 1956a], has played a pivotal role in to analysis of real-world signed networks. The theory follows that there could be four type of triads which are defined as $T_0(---)$, $T_1(+--)$, $T_2(++-)$ and $T_3(+++)$

where subscript represents the number of positive signed edges in the triad [Leskovec *et al.*, 2010]. Later on Davis explores the theory of structural balance and formulated a variant of it, known as weak structural balance [Wasserman and Faust, 1994]. In structural balance theory, T_1 and T_3 are considered as balanced triad which are permissible in the social networks and T_0 and T_2 are unbalanced triads. Davis relaxed this assumption with the inclusion of T_0 in balanced triad and permissible in the social networks [Davis, 1967].

In [Leskovec *et al.*, 2010], three real world signed networks, Epinions, Slashdot, and Wikielection are considered for the analysis of these networks in the shade of balance theory and status theory which has been introduced for directed signed networks [Guha *et al.*, 2004]. Balance theory deals with the structure of the triads formed by the nodes in a network and explains the conflict among a social group of three connected people. Under the balance theory, two nodes are connected positively, if they have no social conflict and vice versa. Similar to balance theory, status theory defines the sign of links in the context of the social status of a node with respect to other connected nodes. For example, if a person (a node) says that he/she is superior to a set of people (called S_1) and inferior to another set of people (called S_2), then people from S_1 are positively connected and members of set S_2 are negatively connected to that node. It is observed that the distribution of different type of triads is not random, T_3 has the high concentration as compared to the random network of the same number of signed edges of each type, and the counts of T_0 is minimum [Leskovec *et al.*, 2010]. In all the considered data sets, T_3 has more density than expected in the random network which has the random distribution of edges keeping the fraction of positive and negative edges constant. These data sets have almost $\sim 20\%$ negative edges [Leskovec *et al.*, 2010]. In [Hassan *et al.*, 2012], authors did the sentiment analysis of text shared by the participants of the online discussion forums and generated a signed network between participants in the forums on the basis of the sentiment of their posts and comments. The signed network is generated between the participants of the discussion and two are connected via a positive or negative link based on the positive or negative sentiments of the replies given by one to other. Based on the analysis of signed and unsigned network, it is advocated that the existence of signed social networks is more realistic as compared to the unsigned version of the same network. There are some other approaches to model and study the dynamics of signed networks. In [Malekzadeh *et al.*, 2011], a game theoretic approach is adopted to model a signed network. In [Doreian, 2008], sign network is considered as the collection of blocks of positive edges and these blocks are connected with negative edges for modeling signed networks. In another approach, trust and distrust have been considered as positive and negative edges which are known as low-rank modeling of signed network [Hsieh *et al.*, 2012]. In [Ludwig and Abell, 2007], a social balance theory based evolutionary model (EM) is discussed in which a random network of size n nodes is considered. Link between a pair of nodes is formed by the probability $p = (1+\alpha)/2$ where $-1 \leq \alpha \leq 1$ is the friendliness index. Then in the next step, a number of balanced triangles attached with a node are redistributed according to balance index $\beta_i = \frac{\Delta^+ - \Delta^-}{\Delta^+ + \Delta^-}$, where Δ^+ and Δ^- are the number of balanced and imbalanced triangles attached to the node i. In EM modeling approach, we need the information of triangles' distribution of real-world network to generate the similar model network. Still, EM is neither a growing modeling approach (size of the network grows with time) that is the fundamental property of the social networks nor a social dynamics dependent model, for example, preferential attachment, internal growth or edge densification etc. These are some handful attempts made by network scientist to model a signed network which follow the structural balance theory of Heider [Heider, 1946a].

The problem of modeling of signed-social-network is considered in this chapter. The question of 'why model social networks?' is answered in [Robins *et al.*, 2007]. It can be said that modeling of a complex system or network provides a comparatively easy understanding of that system. We present a dynamic growing model for signed networks based on preferential attachment and a random attachment having local growth with structural balance. In this model, we also consider that older nodes have the capability to initiate link formation with the other older nodes based on preferences. The problem of signed network modeling is viewed as a multiplex network modeling having two layers in which the first layer is assumed as the collection of positive edges and the second layer consists of negative edges.

Multiplex network is defined as a network which has multiple layers with the same number of nodes in each layer. Each node participates in making connections with other nodes in each layer. We assume that in a multiplex network, there are two layers l_1 and l_2. As discussed in [Kim and Goh, 2013], we consider the degree growth equation of a node i in the layer l_1 at time t as

$$\frac{\partial k_i^{l_1}}{\partial t} = m_1 \left(\alpha_{11} \frac{k_i^{l_1}}{t} + \alpha_{12} \frac{k_i^{l_2}}{t} \right). \tag{4.1}$$

where α_{11}, α_{12} and m_1 are constants. Similarly for layer l_2

$$\frac{\partial k_i^{l_2}}{\partial t} = m_2 \left(\alpha_{21} \frac{k_i^{l_2}}{t} + \alpha_{22} \frac{k_i^{l_1}}{t} \right). \tag{4.2}$$

where α_{21}, α_{22} and m_2 are constants.

In a network, α_{11} and α_{22} correspond to the effect of intra-layer connections in the evolution of a node, while α_{12} and α_{21} control the inter-layer effect in the growth of a node. In a simple example, consider a politician which has supporters as well as haters (both type of connections, positive and negative). A new member can be supporter or hater. α_{11} and α_{22} signifies that if a politician has more supporters (haters) then new member will support (hate) him. In this case, α_{12} and α_{21} will be less effective as we can observe that a new supporter does not prefer to support a politician on the basis of the count of his haters and vice versa. In another example, consider a researcher who is connected with other researchers and investors to finance the research. In this case also α_{11} and α_{22} has the similar effect but α_{12} and α_{21} has opposite effect as compared to the previous example. If a researcher is connected with a big team of good researchers then he can get a number of investors and vice versa.

Now Eqs. (4.1) and (4.2) can be combined in the form of a single dynamical system which is represented by,

$$\dot{\mathbf{k}}_i = \frac{1}{t} C \mathbf{k}_i. \tag{4.3}$$

where $m_1 = m_2 = 1$, $\dot{\mathbf{k}}_i = \left[\frac{\partial k_i^{l_1}}{\partial t}, \frac{\partial k_i^{l_2}}{\partial t} \right]^T$, $\mathbf{k}_i = \left[k_i^{l_1}, k_i^{l_2} \right]^T$ and $C = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$. Consider that C is a symmetric matrix with the condition $\alpha_{ii} > |\alpha_{ij}|$ for $i \neq j$. Due to diagonal dominance, C is a positive definite (PD) matrix. In the discussed model of multiplex network, if we assume that l_1 represents a layer which contains all positive connections and l_2 has all negative connections then it would be a model corresponding to a signed network.

The rest of the chapter is organized in the following way. Next section provides a complete picture of network modeling process adopted in this chapter and underlying assumptions. Section 4.2 is dedicated to showing the novelty of the model on the basis of compared properties of real-world networks and corresponding model network defined in this chapter. The chapter is completed by the discussion and conclusion of the work presented in this article.

4.1 THE MODEL

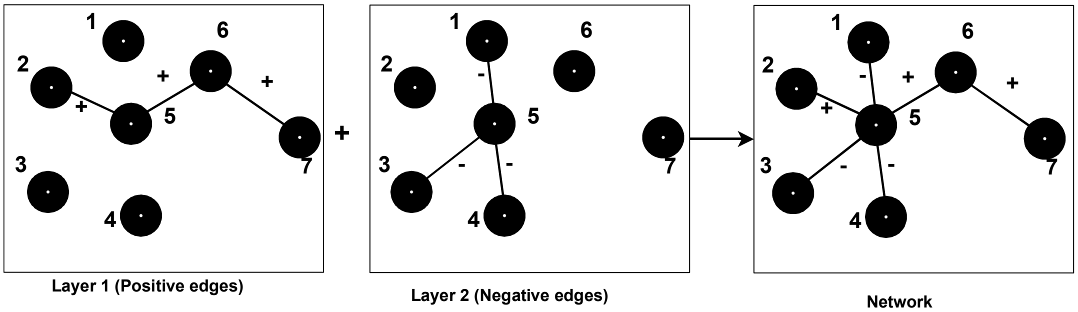


Figure 4.1 : Example of signed network and its layers.

Networks with preferential attachment growth process have few nodes (called *hubs*) of larger connectivity. In the proposed model, we combine three dimensions of growth of a network; preferential attachment, random attachment with local growth, and internal growth of a network. In the proposed model, two layers are considered; one is the collection of positive edges and second is comprised of negative edges. An example is shown in Fig. 4.1. The growth process of the network is as follows:

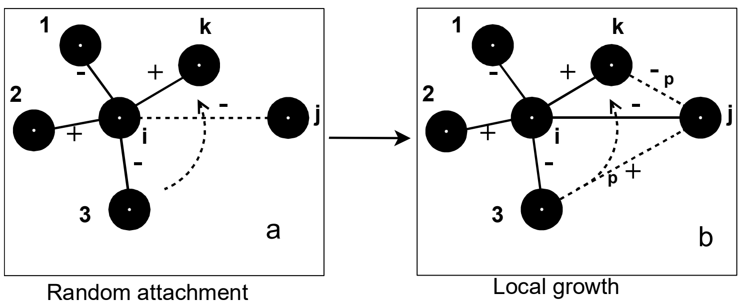


Figure 4.2 : Step 2- Random attachment with local growth.

Step 1- Preferential attachment (**PA**). Under this step, a newly appeared node j gets attached with a pre-existing node i either in l_1 layer of positive edges or in l_2 layer of negative edges using the probabilities defined by

$$p_i^{l_1} = \left(\alpha_{11} \frac{k_i^{l_1}}{t} + \alpha_{12} \frac{k_i^{l_2}}{t} \right), \tag{4.4}$$

$$p_i^{l_2} = \left(\alpha_{21} \frac{k_i^{l_2}}{t} + \alpha_{22} \frac{k_i^{l_1}}{t} \right) \tag{4.5}$$

where α_{11}, α_{12}, α_{21} and α_{22} are controlling parameters. α_{11} and α_{22} control the effect of intra-layer connections while α_{12} and α_{21} control the effect of inter-layer connections in the evolution of a node.

Step 2– **Random attachment with local growth** (**RA-LG**). A new node j gets linked with a pre-existing node i via a positive or negative link randomly in the respective layer, and then by probability p, it generates balance triangles with the immediate neighbors of node i. The pictorial representation of the random attachment with local growth process is shown in Fig. 4.2.

In Fig. 4.2(a), the random attachment is shown in which a new node j get attached via negative edge with the node i. After that the node j attempts to connect with the neighbors of node i with probability p. The connections made in such a way that the triangle formed by nodes i, j and a neighboring node of i is balanced, see Fig. 4.2(b).

Step 3− **Internal growth** (**IG**). During the growing process of the network, it also grows internally by adding edges preferentially without addition of any new node to the network.

These three steps are followed by each node in both the layers. In the proposed evolution process of a signed network, a node appears with probability $(1-\varepsilon)$ at each time step and goes for either step 1 with probability β or goes for step 2 with probability $(1-\beta)$. The network grows internally with probability ε without the addition of a new node. The combined process of network growth is shown in Fig. 4.3 and defined by

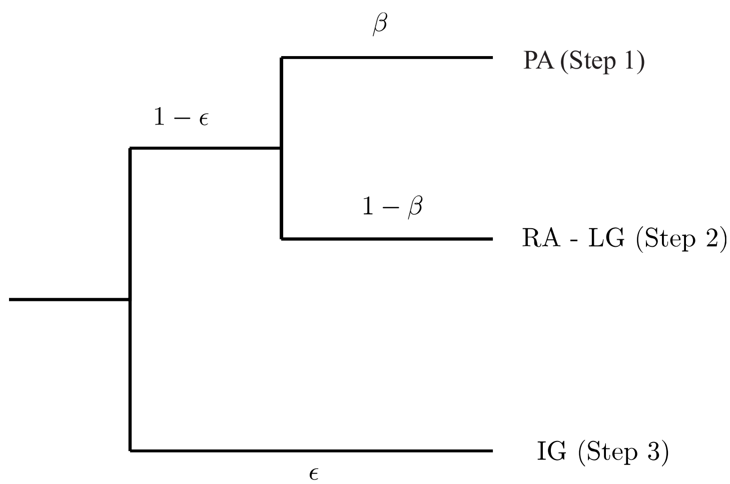


Figure 4.3 : Graphical representation of addition of a new node.

$$\dot{\mathbf{k}}_i = (1-\varepsilon)\frac{\beta}{2t}\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\mathbf{k}_i + (1-\varepsilon)(1-\beta)\begin{bmatrix}\frac{.5+pk_i^{l_1}}{t}\\ \frac{.5+pk_i^{l_2}}{t}\end{bmatrix} + \varepsilon\begin{bmatrix}\frac{k_i^{l_1}}{t}\\ \frac{k_i^{l_2}}{t}\end{bmatrix},$$

Dynamics of higher degree nodes (high positive as well as negative degree) is given by,

$$\dot{\mathbf{k}}_i \approx (1-\varepsilon)\frac{\beta}{2t}\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\mathbf{k}_i + (1-\varepsilon)\frac{(1-\beta)}{t}\begin{bmatrix}p & 0\\ 0 & p\end{bmatrix}\mathbf{k}_i + \frac{\varepsilon}{t}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{k}_i.$$

$$\dot{\mathbf{k}}_i = \frac{1}{t}\mathscr{C}\mathbf{k}_i. \tag{4.6}$$

where $\mathscr{C} = \begin{bmatrix}(1-\varepsilon)\beta\alpha_{11}/2 + (1-\varepsilon)(1-\beta)p + \varepsilon & (1-\varepsilon)\beta\alpha_{12}/2\\ (1-\varepsilon)\beta\alpha_{21}/2 & (1-\varepsilon)\beta\alpha_{22}/2 + (1-\varepsilon)(1-\beta)p + \varepsilon\end{bmatrix}$.

$\mathscr{C}$ is a diagonal dominant matrix and Eq. (4.6) represents a dynamical system similar to Eq. (4.3) that has scale-free behaviour and power-law exponent is $\gamma = 1 + \frac{1}{\lambda_1(\mathscr{C})}$.

Assume that at each time step either a node appears with probability $(1-\varepsilon)$ or network grows internally under step-3. Newly appeared node makes connections with the pre-existing nodes in the network. Let t_i be the time when node i participate in network formation process.

The solution of the Eq. (4.6) is given by

$$\mathbf{k}_i(t) = \left(\frac{t}{t_i}\right)^{\lambda_1(\mathscr{C})} \left(\mathbf{u}_1^T \mathbf{k}_i^0\right) \mathbf{u}_1 + \left(\frac{t}{t_i}\right)^{\lambda_2(\mathscr{C})} \left(\mathbf{u}_2^T \mathbf{k}_i^0\right) \mathbf{u}_2. \tag{4.7}$$

where $\mathbf{u}_i$ is an eigenvector of the matrix C and $\lambda_i(\mathscr{C})$ is corresponding eigenvalue. $\mathbf{k}_i^0$ is an initial condition.

Theorem 4.1.1. *A system given by*

$$\dot{x} = \frac{1}{t}\mathscr{C}\boldsymbol{x}, \tag{4.8}$$

has unique solution given that $\boldsymbol{x}_0$ and t_0 are initial conditions, where $\mathscr{C}$ is either a symmetric matrix or full rank matrix.

Proof. Let $\mathbf{x} = \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)}$ be the solution of Eq. (4.8), where $\mathbf{u}_i$ is an eigenvector of the matrix $\mathscr{C}$ and $\lambda_i(\mathscr{C})$ is corresponding eigenvalue. Consider

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)} \frac{d\left(\lambda_i(\mathscr{C})\log_e(t/t_0)\right)}{dt}, \\
\dot{\mathbf{x}} &= \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)} \lambda_i(\mathscr{C}) \frac{t_0}{t} \frac{d\left(t/t_0\right)}{dt}, \\
\dot{\mathbf{x}} &= \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)} \lambda_i(\mathscr{C}) \frac{t_0}{t} \frac{1}{t_0}, \\
\dot{\mathbf{x}} &= \frac{1}{t}\lambda_i(\mathscr{C}) \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)}, \\
\dot{\mathbf{x}} &= \frac{1}{t}\mathscr{C} \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)}, \\
\dot{\mathbf{x}} &= \frac{1}{t}\mathscr{C}\mathbf{x}.
\end{aligned}$$

Now the general solution of the Eq. (4.8) will be

$$\mathbf{x} = \sum_i a_i \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)},$$

where a_i is a constant. By satisfying the initial conditions ($\mathbf{x}_0$ and t_0), we get $a_i = \mathbf{u}_i^T \mathbf{x}_0$, $\forall i$.

$$\mathbf{x} = \sum_i \left(\mathbf{u}_i^T \mathbf{x}_0\right) \mathbf{u}_i e^{\lambda_i(\mathscr{C})\log_e(t/t_0)}, \tag{4.9}$$

It proves that a solution exist and it is a unique solution (proof is given in Chapter 1, page-13, Theorem 1) [Brockett, 2015]. □

Put $\mathbf{x} = \mathbf{k}_i$, $\mathbf{x}_0 = \mathbf{k}_i^0$ and $t_0 = t_i$ in Eq. (4.9), and we get the Eq. (4.7). If node i has total degree $k_i = \mathbf{1}^T \mathbf{k}_i$ at time t then

$$k_i = a_1 \left(\frac{t}{t_i}\right)^{\lambda_1(\mathscr{C})} + a_2 \left(\frac{t}{t_i}\right)^{\lambda_2(\mathscr{C})}.$$

where $a_1 = (\mathbf{u}_1^T \mathbf{k}_i^0)(\mathbf{1}^T \mathbf{u}_1)$ and $a_2 = (\mathbf{u}_2^T \mathbf{k}_i^0)(\mathbf{1}^T \mathbf{u}_2)$. If $\lambda_1(\mathscr{C}) > \lambda_2(\mathscr{C})$ then for older nodes, $\left(\frac{t}{t_i}\right)$ will be large and

$$k_i \approx a_1 \left(\frac{t}{t_i}\right)^{\lambda_1(\mathscr{C})}$$

this leads to

$$P(k) \approx \frac{1}{a_1} \left(\frac{k}{a_1}\right)^{-\gamma}.$$

where $\gamma = 1 + \frac{1}{\lambda_1(\mathscr{C})}$, $\lambda_1(\mathscr{C})$ is the largest eigenvalue of the matrix $\mathscr{C}$.

Note: $\mathscr{C}$ should have positive eigenvalues.

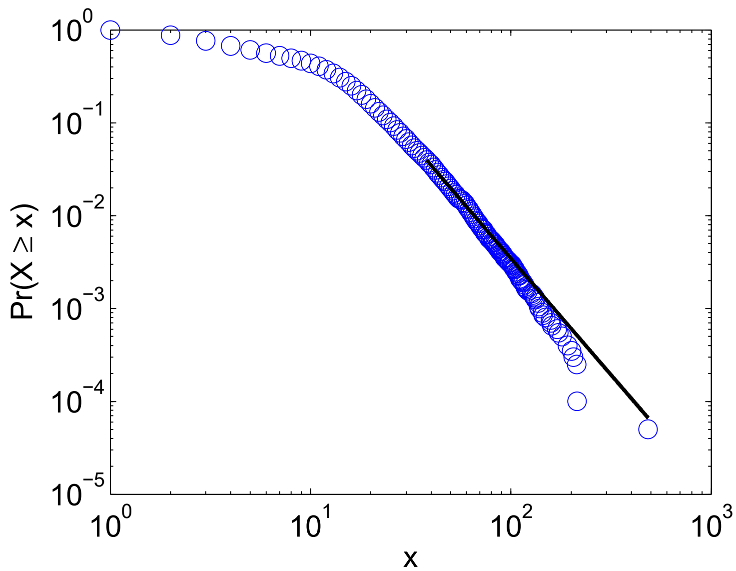


Figure 4.4 : Degree distribution of signed network generated by the model defined by Eq.(4.6).

Example 4.1.2. *An example of model network named as Signed-Social-Network (SSN) is considered by setting the values of parameters:* $\beta = 0.4$, $\varepsilon = 0.005$, $p = 0.1$, $\alpha_{11} = 0.83$, $\alpha_{22} = 0.42$, $\alpha_{12} = 0.6 \times \alpha_{11}$ *and* $\alpha_{21} = 0.6 \times \alpha_{22}$. *The produced matrix* $\mathscr{C}$ *has eigenvalues* $\lambda_1(\mathscr{C}) = 0.27$ *and* $\lambda_1(\mathscr{C}) = 0.10$ *for the given set of parameters. Cumulative degree distribution of the generated model network is plotted in the Fig.4.4 that is showing a strait line in the upper tail of the distribution. It corresponds to power-law behaviour of degree distribution of the network generated by the model proposed in this chapter.*

As we know that the behavior of the SSN depends on the settings of the parameter values. Further, we considered some specific cases to analyze the behavior and dependency of the power-law exponent in the parameters. Some special case is discussed with their special structure of the matrix $\mathscr{C}$. These cases are summarized in Table 4.1.

Case-1 It is considered that network does not grow internally. Preferential attachment scheme which is explained in step-1, is the only process applied in the network formation. It is assumed that the connections in one layer do equal contribution in the growth of other layers (linking probability). The specified dynamics is obtained by setting $\varepsilon = 0,\ \beta = 1,\ \alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha \leq 0.5$ in Eq. (4.6), corresponding to which the matrix $\mathscr{C} = \alpha/2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Matrix $\mathscr{C}$ has eigenvalues α, 0 and corresponding eigenvectors are $u_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$ respectively. Networks generated by the given settings of parameters have power-law exponent $1 + 1/\alpha$.

Table 4.1 : Some special cases

Parameter settings	γ
$\varepsilon = 0,\ \beta = 1,\ \alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha \leq 0.5$	$1 + \frac{1}{\alpha}$
$\varepsilon = 0,\ \beta = 0$	$1 + \frac{1}{p}$
$\varepsilon = 0,\ \beta = 1,\ = \alpha_{21} = \alpha_{12} = 0,\ \alpha_{11} = \alpha_{22} = \alpha \leq 1$	$1 + \frac{2}{\alpha}$
$\varepsilon = 0,\ \beta = 1,\ = \alpha_{21} = \alpha_{12} = r\alpha,\ \alpha_{11} = \alpha_{22} = \alpha \leq 0.5$	$1 + \frac{2}{(1+r)\alpha},\ r \leq 1$
$\varepsilon = 0,\ \beta = 1,\ = \alpha_{21} = \alpha_{12} = -\alpha/2,\ \alpha_{11} = \alpha_{22} = \alpha \leq 1$	$1 + \frac{4}{\alpha}$
$\beta = 0$	$1 + \frac{1}{\varepsilon + (1-\varepsilon)p}$

Case-2 It corresponds to the second row of the table in that a network does not grow internally and it follows only the local growth scheme of balanced triangles. The preferential attachment has zero contribution to the evolution of the network. The whole dynamics is obtained by setting $\varepsilon = 0,\ \beta = 0$, corresponding to which the matrix $\mathscr{C} = p\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Matrix $\mathscr{C}$ has both the eigenvalues p, and corresponding eigenvectors are $u_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$ respectively. The networks generated by the given settings of parameters have power-law exponent $1 + 1/p$.

Case-3 This case is given in the third row of the table. Here also, network does not grow internally. It follows only the preferential attachment scheme in individual layer. Connections in a layer does not affect the dynamics of other layers. The complete explained dynamics is obtained by setting $\varepsilon = 0,\ \beta = 1,\ \alpha_{21} = \alpha_{12} = 0,\ \alpha_{11} = \alpha_{22} = \alpha \leq 1$, corresponding to which the matrix $\mathscr{C} = \alpha/2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Matrix $\mathscr{C}$ has both the eigenvalues same, $\alpha/2$, and corresponding eigenvectors are $u_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$. The networks generated by the given settings of parameters have power-law exponent $1 + 2/\alpha$.

Case-4 In this case also, network does not grow internally, it follows only the preferential attachment scheme in both the layers. The connections from one layer has fixed contribution in the growth of other layers. The inter-layers dependency of growth process is defined by $1 \geq r > 0$. The specified model is obtained by setting $\varepsilon = 0,\ \beta = 1,\ \alpha_{11} = \alpha_{22} = \alpha,\ \alpha_{21} = \alpha_{12} = r\alpha,\ \alpha \leq 0.5$, corresponding to which the matrix $\mathscr{C} = \alpha/2\begin{bmatrix}1 & r\\ r & 1\end{bmatrix}$. Matrix $\mathscr{C}$ has eigenvalues $(1+r)\alpha/2$, $(1-r)\alpha/2$ and corresponding eigenvectors are $u_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$ respectively. The networks generated by the given settings of parameters have power-law exponent $1 + 2\alpha^{-1}/(1+r)$.

Case-5 Here, network follows only preferential attachment scheme in both the layers without internal and local growth. The connections from one layer have negative effect in the growth of other layers. The inter-layers dependency of growth process is defined by $0 > r \geq -1$. The specified model is obtained by setting $\varepsilon = 0$, $\beta = 1$, $\alpha_{11} = \alpha_{22} = \alpha$, $\alpha_{21} = \alpha_{12} = r\alpha$, $\alpha \leq 0.5$, corresponding to which the matrix $\mathscr{C} = \alpha/2 \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$. $\mathscr{C}$ has eigenvalues $(1+r)\alpha/2$, $(1-r)\alpha/2$ and corresponding eigenvectors are $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. The networks generated by the given settings of parameters have power-law exponent $1 + 2\alpha^{-1}/(1-r)$. If $r = -1/2$, then $\gamma = 1 + 4/\alpha$ (Fifth row in Table 4.1).

Case-6 In this case, network evolution process considers only local growth with balanced triangles in both the layers having internal growth. Preferential attachment scheme does not contribute anything in the network formation. The specified model is obtained by setting $\beta = 0$, corresponding to which the matrix $\mathscr{C} = (\varepsilon + (1-\varepsilon)p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\mathscr{C}$ has both ethe eigenvalues $(\varepsilon + (1-\varepsilon)p)$, and corresponding eigenvectors are $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. The networks generated by the given settings of parameters have power-law exponent $1 + 1/(\varepsilon + (1-\varepsilon)p)$.

4.2 RESULTS

In the previous section, a mathematical representation of considered social activities is modeled with the discussion of 6 special cases. In this section, we are providing a comparison between the structural properties of the considered real-world signed social network, corresponding network generated using the model defined in the previous section and networks generated under the evolutionary model (EM) without considering the balance index given in [Ludwig and Abell, 2007]. We do not consider other discussed model to compare with SSN. SSN is a dynamic model while others are not. EM is an evolutionary model but does not capture the dynamics of new node addition in a network. The model defined in this chapter is growing random network model for signed networks which is based on observed characteristics of real-world social networks, for example, real-world networks follow balance theory, internal growth, and preferential attachment. Other considered models only focus on balance theory for signed networks. We considered some basic properties, for example, distribution of different type of balanced and unbalanced triangles, and some other measures to analyze the structure of a signed networks from the perspective of characteristics of the nodes. We define two quantities that are stability s, and diversity Φ of the nodes in the network for the same purpose. Before that, we discuss the notion of the degree in the signed networks. Two notions are adopted to measure the degree of a node in a signed network, one is *unsigned-degree* in which all the connections of a node is counted irrespective of sign of the edge attached to the node, and second is *signed-degree* that is the summation of the signed weights of the edges attached to a node.

Definition 4.2.1. *If A is the adjacency matrix of a signed network G and $\mathscr{A} = abs(A)$, then the stability of a node i in the network G is defined by*

$$s_i = \frac{(A^3)_{i,i}}{(\mathscr{A}^3)_{i,i}}.$$

It represents the contribution of node i in the balance triangles.

Definition 4.2.2. *If A is the adjacency matrix of a signed network G and $\mathscr{A} = abs(A)$, then the diversity of a node i in the network G is defined by*

$$\Phi_i = \frac{(\mathscr{A}\mathbf{1})_i - |(A\mathbf{1})_i|}{max(\mathscr{A}\mathbf{1})}.$$

Stability measure of a node i (which is also known as balance index in [Ludwig and Abell, 2007]) is the relative excess of balance triangles attached to that node. A node of high stability has neighbors of less social conflict. The diversity of a node i is the relative difference of unsigned-degree to the absolute value of signed-degree of the node. A node is highly polar that have an equal number of friends and enemies, and less polar if it has either friends or enemies only.

Nature of the edges and their pattern in the network has the important role in the stability of the structure of the networks. Structural balance of the triangles and their distribution play a key role in the study the dynamics of the signed-networks. Nodes are the actors in the networks and the growth of networks depend on the actions taken by these nodes. They participate in stabilizing the structure of networks by sharing balanced and unbalanced triangles. In these networks, nodes have positive as well as negative connections which drive the dynamics of nodes that is different from unsigned networks.

In this section, we are presenting a comparative study of a considered real world network, corresponding model network generated by Eq.(4.6) and another considered model, EM. A real world signed network is considered and by tuning the values of the parameters of the models, signed networks are generated. Networks are compared on the basis of triangle distribution, the degree of balance of the network and other properties of the networks already defined in this section.

Table 4.2 : Fraction of balanced and imbalanced triangles of different types in the voting network of Wikielection, network generated using (4.6) and EM network is given in [Ludwig and Abell, 2007]. In case of model networks, all the provided values are averaged over 10 networks of same size and same parameter values.

Triad T_i	$p(T_i)$	$p(T_i)$ Model (4.6)	$p(T_i)$ Model (EM)
T_3 $(+++)$	0.67	0.7647	0.4752
T_1 $(+--)$	0.09	0.1159	0.1100
T_2 $(++-)$	0.22	0.1178	0.4043
T_0 $(---)$	0.01	0.0017	0.0105

We considered the data of voting in Wikielection to study the distribution of stability and diversity of social actors in social networks. In the network, users are the nodes and edges are the positive or negative votes given by users to promote or demote another user for the selection of admin. The network is considered as undirected and signed that has strong positive correlation ≈ 0.6 between positive and negative degrees of the nodes. In Fig. 4.6(a), stability has broad range of distribution in which some nodes are perfect stable ($s_i = 1$) and some are perfect unstable ($s_i = -1$). The majority of the social actors have the positive score of the stability which indicates the stability of the social network. Diversity distribution also supports the balancedness of the social networks. Fewer people are highly polarized and a large number of people are less polar. Diversity and stability, both have the same conclusion on balancedness of the signed networks. Now the question can be raised that the same results from the measures, diversity, and stability, can be due to the similarity in measures. There is a very less positive correlation (0.0529) between the measures, stability, and diversity, which indicates that there is no interdependence between s and Φ. These are independent quantities but tell the similar story of balancedness in social networks.

For the comparative study, networks of size 5000 nodes is generated using the model defined in this chapter by setting the parameter values, $\beta = 0.4$, $p = 0.2$, $\alpha_{11} = 0.6$, $\alpha_{22} = 0.35$,

$\alpha_{12} = .6\alpha_{11}$, $\alpha_{21} = .6\alpha_{22}$ and $\varepsilon = 0.04$. In $\alpha_{12} = .6\alpha_{11}$ and $\alpha_{21} = .6\alpha_{22}$, the multiplier 0.6 is considered from the correlation between positive and negative degrees of the nodes in the voting network of Wikielection and another EM model networks of size 7000 considering 0.56 as friendliness index. The comparison of stability distribution (in Figs. 4.6(a), 4.6(b) and 4.6(c)), diversity distribution (in Figs. 4.5(a), 4.5(b) and 4.5(c)), distribution of different type of balanced and imbalanced triangles in (Figs. 4.7(a), 4.7(b) and 4.7(c)) shared by the nodes and the correlation (in Table 4.3) between the distribution of different type of triangles are compared. Behaviour of the plots of different considered structural measures in the model network (SSN) are similar to the considered real world network, Wikielection. EM network has highly polar nodes as compared to SSN and Wikielection, see Fig. 4.5. SSN has similarity to Wikielection which have a small fraction of highly polar nodes, see Figs. 4.6(a) and 4.6(b). Distribution of s or balance index, in Fig. 4.6 shows that EM has highest number of perfectly unstable nodes, $s = -1$. SSN and Wikielection have small fraction of unstable nodes which have negative value of measure s, see Figs. 4.6(a) and 4.6(b).

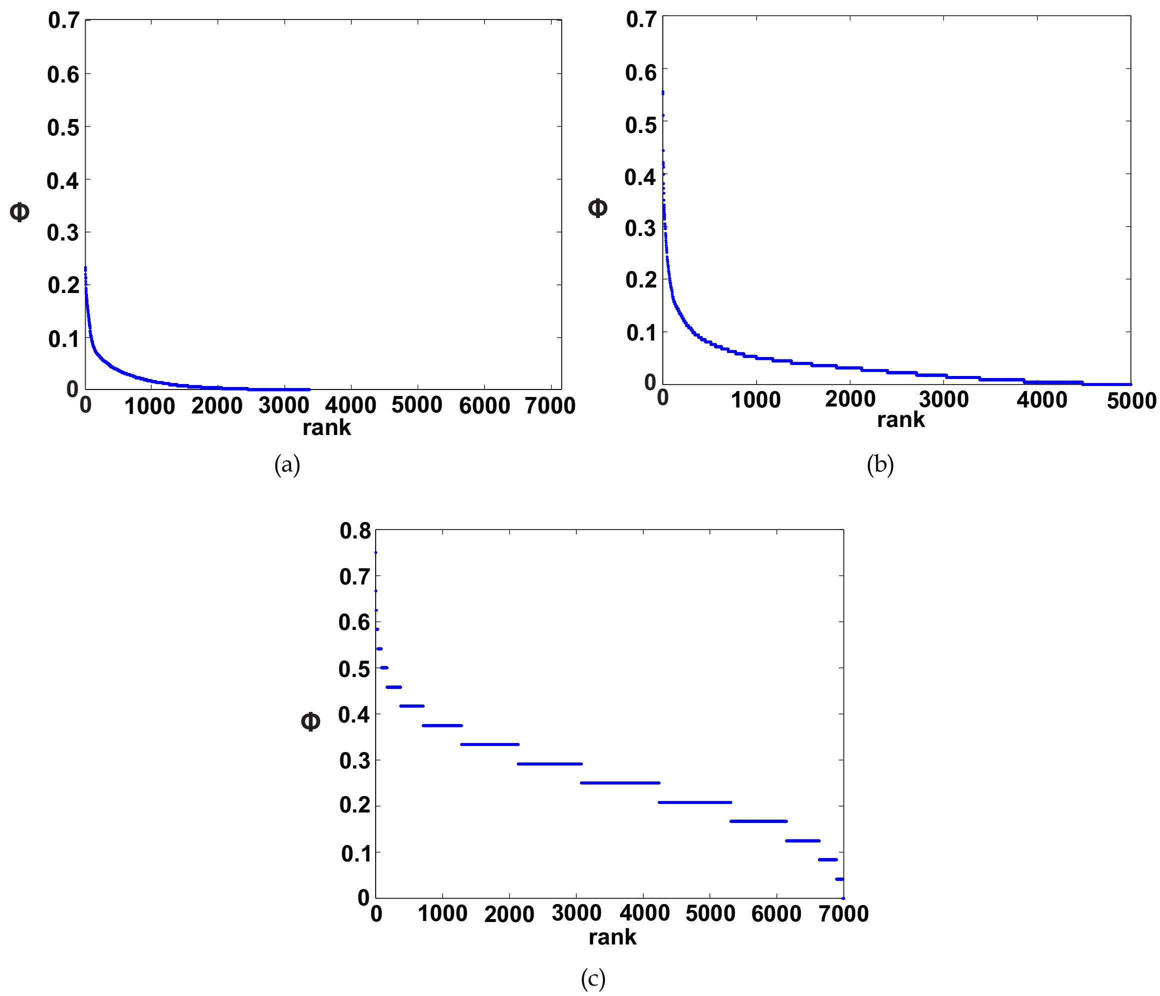
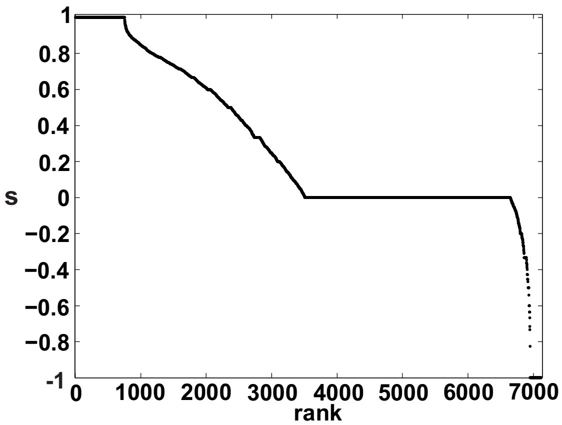
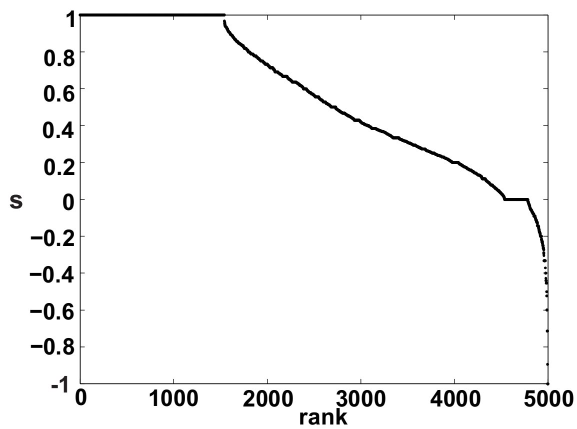


Figure 4.5 : Distribution of diversity of nodes, Φ, in network of (a) Wikielection (b) model network defined in the chapter and (c) EM model network given in [Ludwig and Abell, 2007].

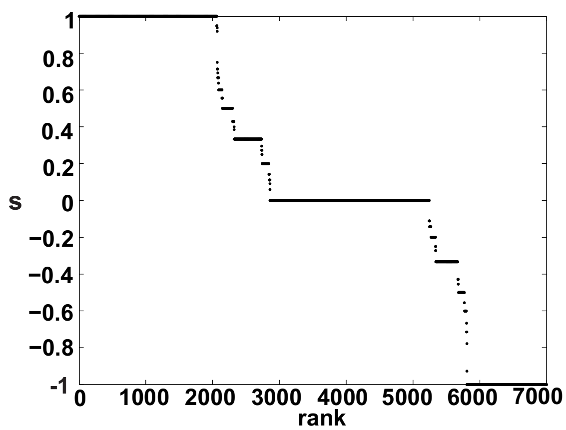
Both the networks, Wikielection and SSN, have a larger fraction of balanced triangles, 0.76 and ~ 0.88 respectively and a small fraction of imbalanced triangles (second and third column in Table 4.2, Figs. 4.5(a) and 4.5(b)), while EM has almost equal amount of balanced (~ 0.58) and imbalanced (~ 0.41) triangles. Details are given in Table 4.3. The model is able to capture the structural properties of the real world signed network. The distribution of balanced



(a)

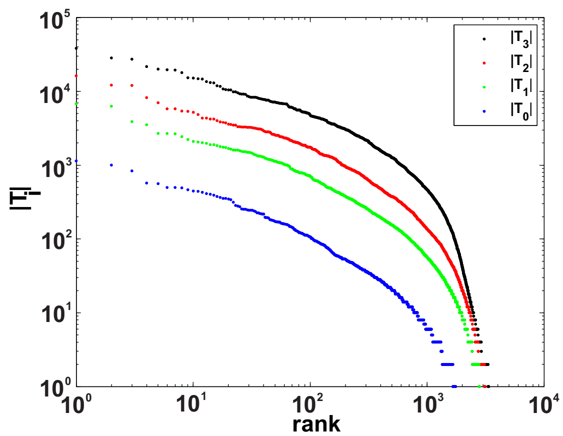


(b)

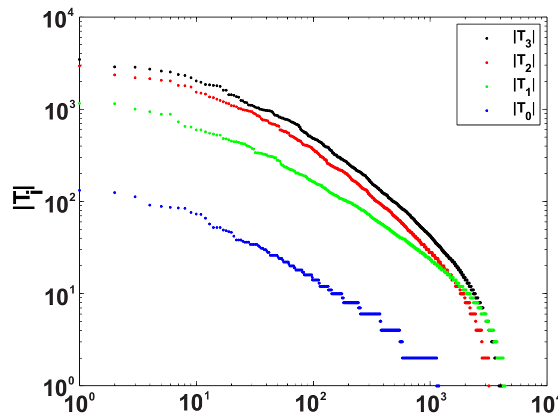


(c)

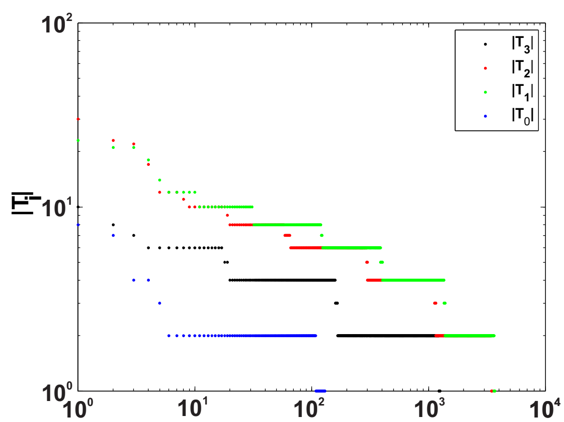
Figure 4.6 : Distribution of stability of nodes, s, in network of (a) Wikielection, (b) model network defined in the chapter and (c) EM network given in [Ludwig and Abell, 2007].



(a)



(b)



(c)

Figure 4.7 : Distribution of triangles shared by nodes in the network of (a) Wikielection (b) model network defined by Eq. (4.6) and (c) EM network given in [Ludwig and Abell, 2007].

Table 4.3 : Correlation between the distribution of balanced and imbalanced triangles of different types in the voting network of Wikielection, corresponding model network generated by 4.6 (in small bracket) and EM network (in square bracket).

	T_3	T_1	T_2	T_0
T_3 $(+++)$	1	0.7160[0.11](0.91)	0.8592[0.09](0.97)	0.4713[−0.016](0.79)
T_1 $(+--)$	–	1	0.9389[0.05](0.96)	0.8928[0.03](0.89)
T_2 $(++-)$	–	–	1	0.7664[0.16](0.87)
T_0 $(---)$	–	–	–	1

and imbalanced triangles of different type have the same order, see in Fig. 4.7. Distribution of different type of triangles is highly correlated (see Table 4.3) in considered real world network and corresponding model network generated by Eq. (4.6). The network generated by EM model has very low values of positive correlation as compared to other networks while T_0 (all negative) and T_3 (all positive) has a negative correlation. The correlation of different type of triangles indicates that the distribution of balanced and imbalanced triangles is uniform in the nodes of the considered real world network and corresponding model network generated under Eq. (4.6). But EM produces the network of different properties as compared to Wikielection and SSN in which majority of the contribution comes from balanced triangles. In Fig. 4.7, ranking of nodes according to the number of triangles shared by them are plotted. Wikielection and SSN have same order of triangles' plot ($|T_3| \leq |T_2| \leq |T_1| \leq |T_0|$) while in EM the order is different ($|T_1| \leq |T_2| \leq |T_3| \leq |T_0|$) see in Fig 4.7(c). Triangle type which has more positive edges is more probable in the Wikielection network and similarly, in SSN, see second and third columns in Table 4.2. In EM network, balanced and imbalanced triangles are almost equally probable, see fourth column in Table 4.2.

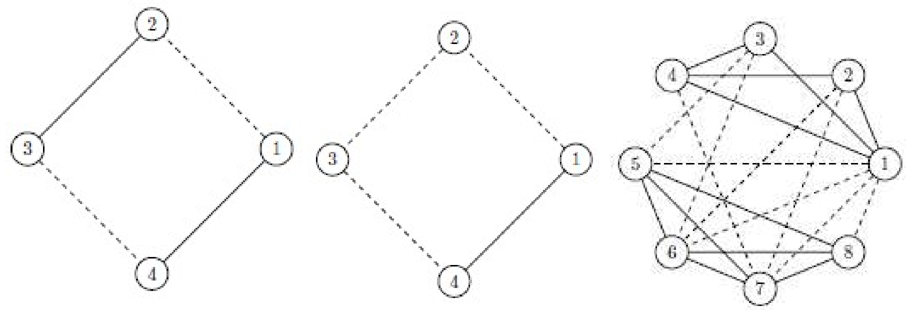
4.2.1 Balance in signed network

Balance is a very important concept in signed networks, it gives a notion of stability in signed networks. The roots of balance lie in theories of social psychology dating back to the work of Heider(1946,1958)[Heider, 1946b]. later Cartwright and Harary(1956) [Cartwright and Harary, 1956b][Davis, 1963][Harary *et al.*, 1953][Easley and Kleinberg, 2010] generalized and extended this concept in the language of graphs. The balance of signed network depends on the balance of cycles (closed walks) it contains. A cycle (closed walk) is called balanced cycle(closed walk) if it contains the even number of negative edges else it is called unbalanced cycle (closed walk). In Figs. 4.8(a) and 4.8(b), examples of balanced and unbalanced cycles on four nodes are shown respectively.

Theorem 4.2.3. *[Malekzadeh* et al.*, 2011] A signed graph is balanced iff all cycles(closed walks) are balanced.*

Theorem 4.2.4. *[Malekzadeh* et al.*, 2011] A signed graph is balanced iff the vertex set of the graph can be partitioned into two subsets such that each positive edge joins vertices in the same subset and each negative edge joins vertices in different subsets.*

An example of a balanced network is shown in Fig. 4.8(c). Most real-world networks are not balanced [Estrada and Benzi, 2014]. In order to measure how much a signed network deviate from balancedness, few measure is well established in the literature, they are as follows:



(a) Balanced cycle C_4 (b) Unbalanced cycle C_4 (c) Balanced network

Figure 4.8 : Examples of signed networks. Dark, dotted edges represent positive, negative edges, respectively.

Clustering Coefficient

Let $\triangle^+(G)$ is the number of balanced triangles and $\triangle^-(G))$ is the number of unbalanced triangles in a signed network G, then clustering coefficient $c_s(G)$ of signed network is given by [Kunegis, 2014],

$$\begin{aligned} c_s(G) &= \frac{\sum_{u\sim v\sim w\sim u}\sigma(u,v)\sigma(v,w)\sigma(w,u)}{\{u,v,w\in V|u\sim v\sim w\}} \\ &= \frac{\triangle^+(G)-\triangle^-(G)}{\{u,v,w\in V|u\sim v\sim w\}}. \end{aligned}$$

Where $u\sim v$ denotes the existence of link between u and v. The signed clustering coefficient $c_s(G)$ denotes to what extent the graph exhibits a balanced structure. Additionally, the relative signed clustering coefficient $S(G)$ is defined as follows:

$$\begin{aligned} S(G) &= \frac{\sum_{u\sim v\sim w\sim u}\sigma(u,v)\sigma(v,w)\sigma(w,u)}{\{u,v,w\in V|u\sim v\sim w\sim u\}} \\ &= \frac{\triangle^+(G)-\triangle^-(G)}{\triangle^+(G)+\triangle^-(G)}. \end{aligned}$$

Closed walk based degree of unbalance

The total number of closed walks of length k in G is given by $trA^k(G)$, where tr is trace of matrix $A(G)$ [Estrada and Benzi, 2014]. A balanced weighted closed walk (BCW) is a closed walk of length larger than zero with a positive sign. Similarly, an unbalanced weighted closed walk (UCW) is a closed walk of length larger than zero with the negative sign.

Let us take weighted sum of walks as:

$$D(G) = \sum_{k=0}^{\infty} tr[A(G)^k]/k!.$$

It converges to $D(G) = tr(e^{A(G)})$ [Estrada and Benzi, 2014]. Here $1/k!$ is weight given to every cycle of length k. Every BCW contributes positively to $D(G)$ and every UCW contributes negatively to $D(G)$. We have $tr(e^{A(G)}) = \mu^B - |\mu^U|$ as the sum of weighted balanced(unbalanced)closed walks, and $|.|$ is absolute value. Similarly we can consider the same term in the underlying graph $|G|$ which results in $tr(e^{A(|G|)}) = \mu^B + |\mu^U|$. Let us define,

$$K_e = \frac{tr(e^{A(G)})}{tr(e^{A(|G|)})}$$
$$= \frac{\sum_{j=1}^{n} e^{\alpha_j}}{\sum_{j=1}^{n} e^{\beta_j}} = \frac{\mu^B - |\mu^U|}{\mu^B + |\mu^U|}.$$

Where α_j and β_j are the eigenvalues of G and $|G|$, respectively. So this means that the ratio U_e of unbalanced to balanced CWs can be obtained as,

$$U_e = \frac{|\mu^U|}{\mu^B} = \frac{1 - K_e}{1 + K_e}.$$

Degree of unbalance using triangle is defined in similar way. Let us define:

$$K_\triangle = \frac{\triangle^+(G) - \triangle^-(G)}{\triangle^+(G) + \triangle^-(G)}.$$

Then degree of unbalance using triangle is as follows:

$$U_\triangle = \frac{1 - K_\triangle}{1 + K_\triangle}.$$

Algebraic conflict

Let $L = D - A$ be laplacian matrix of G, where A is adjacency matrix of G and D is diagonal matrix having $D_{ii} = \sum_{j=1}^{n} |A|_{ij}$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of L. λ_n is zero exactly when G is balanced. Therefore, the value λ_n can be used as an invariant of signed graphs that characterizes the conflict due to unbalanced cycles, it is called algebraic conflict of G [Kunegis, 2014].

Table 4.4 : A simple table

Balance criteria	Real network	Model network SSN	EM model network
c_s	0.0364	0.0423	2.03356×10^{-4}
S	0.3217	0.2583	0.1768
$U_\triangle$	0.5132	0.5170	0.7151
U_e	1	1	1
λ_n	0.1830	0.1871	10.74

The values of above-explained measures are calculated for the considered real world network and corresponding model network generated by Eq. (4.6) and EM. It is found that the measured criteria of degree of balance and unbalance in both the networks (Wikielection and SSN) are close enough, see the second and third column in Table 4.4 while EM has different values of balancedness given in the fourth column of the Table 4.4.

The provided analysis concludes that the model presented in this chapter is capable of capturing the properties of the given real world signed network and it is different from other considered models in the growth process. The SSN model is a dynamic network model in which nodes are continuously growing and growth process is based on observed phenomena on the social networks.

4.3 DISCUSSION AND CONCLUSION

The importance of consideration of social conflicts among the people in the society, the existence of the non-trivial pattern of signed triangles motivated us to define the rules of evolution of social network as a signed network. In the previous section, we discussed the different metrics to measure the structural properties and degree of balance of the signed network. A real world signed network is considered to evaluate the novelty of the model defined in this chapter. Distribution of different considered structural properties and degree of balance are calculated for the networks, Wikielection, corresponding model network (SSN) and EM model network. Model networks play a key role while simulating the diffusion processes over these networks to identify the pattern of diffusion phenomena under different conditions that's why modeling of social networks as a signed network is important.

For example, let a signed network G. If G which has at least one negative edge, is structurally balanced or weakly balanced, then it is positively disconnected. A balanced or weakly balanced network can be divided into clusters which have positive intra-cluster edges and negative inter-cluster edges [Davis, 1967]. It implies that clusters are not positively connected. It has a negative impact on diffusion processes. To make the diffusion widespread in the balance networks, it requires a number of seed nodes that makes the diffusion costly in highly balanced networks. A highly balanced society has more conflicts between small groups which are known as clusters [Davis, 1967]. It makes the society more polarized. In a more polarized society, clusters have a tendency either to oppose the actions of other clusters or to compete with other clusters. It may affect the diffusion in both ways. If clusters have a tendency to oppose the others then it creates chaos in society and it is unhealthy for diffusion processes as innovation adoption, marketing of a product. A sense of competition improves the output of the network of interest for example games of different types in which different teams compete with each other to win the matches and this way the performance of the teams get improve.

In a connected signed network G, if each edge is the part of a triangle type T_3 or T_2, it is positively connected. There is a signed path between each pair of nodes and each edge is sharing a triangle type T_3 or T_2. In both the type of triangle, all the three nodes are positively connected. This is true for all edges. It leads to the positive connectivity of the network. In a signed network that has all type of triangles, the distribution of unbalance triangle T_2 plays an important role in information diffusion if it is considered that negative link has a negative effect (opposite to positive edges) on information sharing.

We see that consideration of negative relations among the social objects may affect the scenario of diffusion patterns [Li *et al.*, 2013b]. Realization of more realistic social network and activities of their participants (nodes and edges)are important. It may provide more insight into the evolution of social activities and interaction patterns among the participating agents (nodes). In this chapter, we adopt a simple evolution process of social networks as a signed network on the basis of observed social activities that are preferential attachment scheme and local balance among the small group under random attachment scheme. The provided model of signed network is capable enough to capture the structural behavior of the real world signed networks that are shown in the previous section. It can be used as platform network to simulate process over the signed networks.

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