2
Literature Survey

We first provide the basic graph theoretic definitions and notations used in the thesis.

2.1 NOTATION AND BASIC DEFINITIONS

A digraph $G = \Big(V(G), E(G)\Big)$ is a collection of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$. The cardinality of the set $V(G)$ is also called the order of G. An edge (u,u) is called a loop at the vertex u. A simple graph is a special case of a digraph, where $E(G) \subseteq \{(u,v) : u \neq v\}$; and if $(u,v) \in E(G)$, then $(v,u) \in E(G)$. A weighted digraph is a digraph equipped with a weight function $f : E(G) \to \mathbb{C}$. In general weights of edges can be positive or negative, hence we call G a weighted signed digraph.

If $V(G) = \emptyset$ then, the digraph G is called a null graph. A subdigraph of G is a digraph H, such that, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subdigraph H is an induced subdigraph of G if $u, v \in V(H)$ and if $(u,v) \in E(G)$ then $(u,v) \in E(H)$. Two subdigraphs H_1 and H_2 are called vertex-disjoint subdigraphs if $V(H_1) \cap V(H_2) = \emptyset$. A path of length k between two vertices v_1 and v_k is a sequence of distinct vertices $v_1, v_2, \ldots, v_k, v_{k+1}$, such that, for all $i = 1, 2, \ldots, k$, either $(v_i, v_{i+1}) \in E(G)$ or $(v_{i+1}, v_i) \in E(G)$. Additionally, if $v_1 = v_{k+1}$ then the path is a cycle of length k. We call a digraph G connected, if there exists a path between any two distinct vertices. A component of G is a maximally connected subdigraph of G. A cut-vertex of G is a vertex whose removal results in an increase in the number of components in G.

Unless stated otherwise we consider a signed graph to be a graph G equipped with a weight function $f : E(G) \to \{-1, 0, 1\}$. Let G has n vertices $v_1, v_2, \ldots, v_n$. Then, the adjacency matrix $A(G)$=(a_{ij}) of order $n \times n$ associated with G is defined by the components

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i, v_j \text{ are connected with a positive edge} \\ -1 & \text{if the vertices } v_i, v_j \text{ are connected with a negative edge} \\ 0 & \text{if the vertices } v_i, v_j \text{ are not connected} \end{cases}$$

where, $1 \leq i, j \leq n$. Often we denote the adjacency matrix $A(G)$ simply by A when G is the only signed graph under study. Every signed graph G has a underlying graph $|G|$, in which all the negative edges are replaced by positive edges. Thus, the corresponding adjacency matrix of $|G|$ is $|A(G)|$ or simply $|A|$. Two signed graphs are called cospectral if the eigenvalues of the corresponding adjacency matrices are same including their multiplicities.

In Chapter 1, we briefly described the concepts of strong balance and weak balance in signed graphs (networks). Unless stated otherwise balance means strong balance in the thesis.

The rest of the chapter is organised as follows. In Section 2.2 we give necessary literature survey related to social signed networks, their balance and sign prediction. In Section 2.4 we state the concepts of matching and linear subdigraphs in Coates digraphs. These concepts are frequently used in the thesis to calculate the determinant, permanent of square matrices. Although

the problems of matrix determinant and permanent has huge literature, In Section 2.5 we give their brief literature survey which is used in our work.

2.2 SIGNED NETWORK, BALANCE

Despite the existence of myriad and complex relationships between social entities in the real world, the main focus in the area of social networks has been friendship relationship. A network is considered as a social network when the vertices represent social entities and a social relationship between two vertices is represented by a link in the network. In his seminal work on the analysis of balance cognitive units the Austrian psychologist, Fritz Heider distinguished between two major types of relations [Heider, 1946]. One is concerned with the relationship of love or likings, and the other one is about the relation of hate or disliking. The concept of the balanced state of a relationship between three social entities was introduced by considering certain combinations of these relations. Later, in 1956, the American mathematician Frank Harary modeled the cognitive structure of balance which is consistent with Heider's concept of balance by introducing the concept of signed graphs [Cartwright and Harary, 1956; Harary et al., 1953]. In a signed graph, the vertices represent individuals and a positive link (link with a positive sign) between two vertices reflects the existence of liking relationship, whereas, a negative link (link with a negative sign) represents disliking.

In the context of understanding relationships of a person with others, the concept of balanced state was introduced by Heider [Heider, 1946]. Besides, the attitude of a person towards other persons was assessed by two types of triples which are involved in signed relations: those involving three individuals and those of two individuals and a social object such as a belief. Later, Cartwright and Harary [Cartwright and Harary, 1956] generalized and extended this concept in the language of signed graphs in 1956, where such a graph structure was proposed in order to obtain a mathematical model of the balanced cognitive unit. In their study of structural balance, the concept of balance for a triad was proposed. A triad, which is a completely connected graph on three vertices, is called balanced when the product of signs of its edges is positive. Otherwise, a triad is called unbalanced. When this notion of balanced cycles is applied to signed networks it leads to obtaining the following theorems for structural balance. As an outcome of the study of Heider, it is believed that a signed social network evolves towards a balanced state otherwise a state of unbalance will produce tension.

Theorem 2.1. *[Harary et al., 1953] A signed network is balanced if and only if for each pair of distinct vertices u and v all paths connecting u and v have same sign.*

Theorem 2.2. *[Acharya, 1980] Graphs G and |G| are isospectral if and only if G is balanced.*

It may be noted that the theorems mentioned above can be used as criteria to determine whether a given signed graph is balanced. Nonetheless, for substantially large networks, it could be a computationally challenging task to verify these criteria, especially in dynamic networks. Further, if a very small fraction of cycles in a large signed network is unbalanced it would be unfair to call the entire network unbalanced. It is also shown in [Estrada and Benzi, 2014] that the undirected versions of some real world signed social networks are not structurally balanced. Indeed, Epinions: a trust-distrust network among users of the product review site Epinions[Guha et al., 2004], Slashdot: a friend-foe network in the technological news site Slashdot [Lampe et al., 2007], and WikiElection: a network representing the votes of the election of administrators in Wikipedia [Burke and Kraut, 2008] are all unbalanced real networks. These observations trigger off the following question: what is the level of balance exists in these networks. This calls for new criteria to measure the balance of a signed networks, especially for large signed networks.

We recall that several metrics are proposed in the literature for measuring structural

balance. For example, closed cycle-based methods are proposed in [Estrada and Benzi, 2014; Cartwright and Harary, 1956].

One of such measures which is used in model conflict dynamics [Pelino and Maimone, 2012] is given by the ratio of the number of signed to unsigned triangles in a signed network as follows

$$K_{\triangle} = \frac{\text{trace}(A^3)}{\text{trace}(|A|^3)} \tag{2.1}$$

where A denotes the adjacency matrix of the signed network and $|A|$ is the adjacency matrix of the unsigned network constructed from the signed network by replacing all negative edges in to positive edges. Obviously, $-1 \leq K_{\triangle} \leq 1$. Indeed, $K_{\triangle} = -1$ when the all triangles are unbalanced and $K_{\triangle} = 1$ when all triangles are balanced in the signed network. We mention that Equation (2.1) is a special case of the relative m balance which is defined as the ratio of the number of positive cycles of length at most m to the total number of cycles of length at most m introduced by Norman and Roberts in 1978. The relative balance which was proposed in [El Maftouhi et al., 2012] is given by

$$K_m = \frac{\sum_{m\geq 3} f(m) X_m^+}{\sum_{m\geq 3} f(m)(X_m^+ + X_m^-)}, \tag{2.2}$$

where X_m^+ (X_m^-) denotes the number of positive (negative) cycles of length m, and $f(m)$ is a monotonically decreasing function that weights the relative importance of cycles of length m. Recently, a closed walk-based method for measuring balance of signed networks is developed and it concludes that signed social networks are highly unbalanced [Estrada and Benzi, 2014; Estrada and Hatano, 2008; Crofts and Higham, 2009].

2.3 SIGN PREDICTION

Sign prediction problem deals with the inference of sign of an unknown link based upon observation of the entire signed network. The key idea for a prediction of the sign of an edge is the minimization of social unbalance [Chiang et al., 2014] assuming that a signed network evolves towards balance. Since signs of edges contribute to calculate the degree of lack of balance in a network, assigning of the sign of an edge and keeping the signs of other edges fixed, it either minimizes or maximizes the social balance of a given network. Indeed, let us explain the idea of using the balance for sign prediction as mentioned in [Chiang et al., 2011] as follows. Consider two vertices u and v in the network such that $A_{uv} = 0$ and we have a task to predict the sign of A_{uv}. First, add a positive edge between them and call the resulting augmented graph as $G^{+(uv)}$ and set $A_{uv} = 1$ in the original graph. Similarly, add a negative edge between u and v to construct the augmented graph $G^{-(uv)}$ and set $A_{uv} = -1$ in original graph. Let $\mu(G)$ be number of weighted unbalanced closed walks in graph G of length > 2. Then the predicted sign of a link between u and v is defined by

$$sign\left\{\mu(G^{-(uv)}) - \mu(G^{+(uv)})\right\} = sign\left(\sum_{t=3}^{k} \beta^t A_{uv}^{t-1}\right). \tag{2.3}$$

2.4 MATCHINGS AND COATES DIGRAPH

A matching in a signed graph G is a collection of edges in which no two have a vertex in common. The largest number of edges in a matching in G is the matching number $m(G)$. A matching with k edges is called a k-matching. A perfect matching of G, also called a 1-factor, is a matching that covers all vertices of G.

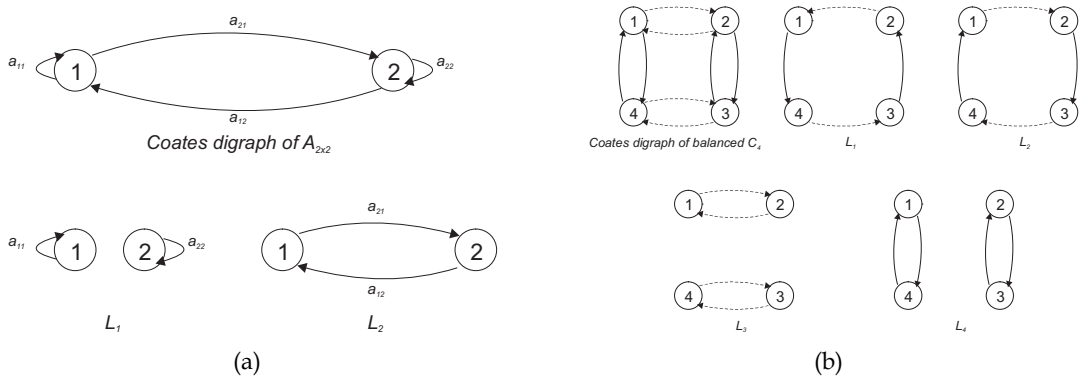

Figure 2.1 : Coates digraph and linear subdigraphs of (a) $A_{2\times 2}$, where L_1 and L_2 are linear subdigraphs (b) Balanced C_4 where L_1, L_2, L_3, and L_4 are linear subdigraphs.

The Coates digraph $D(A)$, generated from a matrix A of order n, has n vertices labeled as $1, 2, \ldots, n$ and for each pair of such vertices i, j a directed edge exists from j to i of weight $A_{i,j}$ [Brualdi and Cvetkovic, 2008][1]. The elements of the main diagonal of A corresponds to loops at vertices in $D(A)$. If diagonal elements of A are zero, then no loops are considered on corresponding vertices of $D(A)$. A linear subdigraph of $D(A)$ is a spanning subdigraph of $D(A)$ in which each vertex has indegree 1 and outdegree 1, that is, exactly one edge into each vertex and exactly one (possibly the same, in the case of the loop) out of each vertex. Thus a linear subdigraph consists of a spanning collection of pairwise vertex-disjoint cycles. The weight of a linear subdigraph is the product of the weights of edges in it. For example, the Coates digraph representation of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given in Figure 2.1(a).

By the Coates digraph of a signed graph, we mean the Coates digraph corresponding to the adjacency matrix of the signed graph. Consider a signed graph G and denote its Coates digraph by $D(G)$. For an edge between vertices i, j in G, there are two directed edges of equal weights in $D(G)$, one from i to j and other from j to i. This forms a directed cycle of length 2 which we call a directed 2-cycle. In a linear subdigraph of $D(G)$, if there are k such directed 2-cycles then these appear due to the k matchings in G. So, there is a one-one correspondence between matchings in G and directed 2-cycles in a linear subdigraph of $D(G)$ if it exists. Thus, by k-matching in linear subdigraphs, we mean the existence of k vertex-disjoint directed 2-cycles. For example, the Coates digraph of balanced C_4 in Figure 1.1(a) is depicted in Figure 2.1(b). Note that, there are two 2-matchings in balanced C_4 in Figure 1.1(a). These are $\{(1,2),(3,4)\}$, and $\{(2,3),(1,4)\}$. In Figure 2.1(b), corresponding to these matchings, there are two directed 2-cycles in linear subdigraph L_3, L_4, respectively, in Coates digraph of the balanced C_n. Now we recall the definition of the determinant of the adjacency matrix A of G in terms of its linear subdigraphs in $D(G)$ [Brualdi and Cvetkovic, 2008].

Theorem 2.3. *Let A be square matrix of order n. Then*

$$\det(A) = (-1)^n \sum_{L \in \mathscr{L}(A)} (-1)^{c(L)} w(L), \tag{2.4}$$

$$\mathrm{per}(A) = \sum_{L \in \mathscr{L}(A)} w(L), \tag{2.5}$$

[1]The direct edge for weight $A_{i,j}$ can be taken from vertex i to j, it will not change the result

where, $w(L)$ is the weight of linear subdigraph L of the Coates digraph $D(A)$, $c(L)$ is the number of directed cycles in L, and $\mathscr{L}(A)$ denotes the set of all linear subdigraphs of $D(A)$.

2.5 DETERMINANT, PERMANENT AND THEIR COMPLEXITIES

The calculation of determinant and the permanent of a matrix is a classical problem in literature [Abdollahi, 2012; Helton et al., 2009; Bibak, 2013; Pragel, 2012; Huang and Yan, 2012; Bibak and Tauraso, 2013; Bapat and Roy, 2014; Hwang and Zhang, 2003]. The digraph representation of a matrix has also been used to calculate its determinant and the permanent [Harary, 1962; Greenman, 1976; Brualdi and Cvetkovic, 2008]. In 1957, Collatz and Sinogowitz proposed a well-known problem, to characterize graphs with positive nullity [Von Collatz and Sinogowitz, 1957; Bibak, 2013]. The zero determinant of the adjacency matrix of a graph ensures its positive nullity. Nullity of graphs is applicable in various branches of science, in particular, quantum chemistry, Huckel molecular orbital theory [Lee and Li, 1994; Gutman and Borovicanin, 2011] and social network theory [Leskovec et al., 2010]. The permanent of a square matrix has significant graph theoretic interpretations. It is equivalent to finding out the number of cycle-covers in the directed graph corresponding to its adjacency matrix. Also, the permanent is equivalent to a number of the perfect matching in the bipartite graph corresponding to its biadjacency matrix. Theory of the permanent provides an effective tool in dealing with order statistics corresponding to random variables which are independent but possibly nonidentically distributed [Bapat and Beg, 1989].

The determinant of a square matrix can be solved in polynomial time using *LUP*-decomposition, whereas computing the permanent of a matrix is an "NP-hard problem" [Valiant, 1979; Wei and Severini, 2010]. The characteristic polynomial of a square matrix A, of order n, is given by the determinant of matrix $(A-\lambda I)$, where I is an identity matrix of order n. We denote the characteristic polynomial, $\det(A-\lambda I)$ by $\phi(A)$. Similarly, the permanent polynomial of A is given by the permanent of matrix $(A-\lambda I)$. We denote the permanent polynomial, $\mathrm{per}(A-\lambda I)$ by $\psi(A)$. For convenience, we can relabel the vertices in graph G. In graph theory, these relabelling are captured by permutation similarity of adjacency matrix A. The determinant of permutation matrices is equal to 1. Thus, relabelling on vertex-set keep the determinant, and characteristic polynomial unchanged.

Theorem 2.4. *[Laplace expansion] Let $A_{S,T}$ denote the submatrix of a square matrix A of order n with rows indexed by the elements in S and columns indexed by the elements in T. Let $A'_{S,T}$ denote the submatrix of A with rows indexed by the elements not in S and columns indexed by the elements not in T,*

$$\det(A)=\sum_{T}(-1)^{w(S,T)}\det(A_{S,T})\det(A'_{S,T}).$$

$$\mathrm{per}(A)=\sum_{T}\mathrm{per}(A_{S,T})\mathrm{per}(A'_{S,T}).$$

Here, S is a fixed k-subset of the rows of A; T runs over all k-subsets of the columns of A, for $k<n$. Also, $w(S,T)=|S|+|T|$, where $|S|$ and $|T|$ represent the sum of all the elements in S and T, respectively.

The theorem which relates complexity of matrix product and matrix determinant is as follows.

Theorem 2.5. *[Aho and Hopcroft, 1974] Let $M(n)$ be the time required to multiply two $n\times n$ matrices over some ring, and A is an $n\times n$ matrix. Then, we can compute* $\det(A)$ *in $O\Big(M(n)\Big)$ steps.*

In general the complexity of product of two matrices of order n is $O(n^{\varepsilon})$ where $2\leq\varepsilon\leq 3$. Complexities of the product of two n order matrices by different methods are as

follows. Schoolbook matrix multiplication: $O(n^3)$, Strassen algorithm: $O(n^{2.807})$[Aho and Hopcroft, 1974], Coppersmith-Winograd algorithm: $O(n^{2.376})$[Coppersmith and Winograd, 1990], Optimized CW-like algorithms $O(n^{2.373})$[Davie and Stothers, 2013; Williams, 2011; Le Gall, 2014]. Complexities of the determinant of n order matrix by different methods are as follows. Laplace expansion: $O(n!)$, Division-free algorithm: $O(n^4)$[Rote, 2001], LUP decomposition: $O(n^3)$, Bareiss algorithm: $O(n^3)$ [Bareiss, 1968], Fast matrix multiplication: $O(n^{2.373})$[Aho and Hopcroft, 1974]. Note that according to the Theorem 2.5 the asymptotic complexity of matrix determinant is equal to that of matrix multiplication. The complexity of matrix determinant by fast matrix multiplication is same as the complexity of Optimized CW-like algorithms for matrix multiplication, as depicted by Theorem 2.5. The fastest known method to compute permanent of matrix of order n is Ryser's method, having complexity $O(2^n n^2)$.