4

The Eigenvalues of Some Signed Graphs With Negative Cliques

In the Chapter 3, we have seen that the spectra of the signed graph play a key role in developing a measure for the degree of unbalance. Also, a signed graph is strongly balanced if and only if it is cospectral with its underlying unsigned graph. In this chapter, we calculate the spectra of some strongly as well as weakly signed graphs. A weakly balanced graph consists of $k \geq 2$ clusters of vertices, where all the edges inside the clusters are positive whereas all the edges between clusters are negative. If $k = 2$ then the graph is strongly balanced. Let G be a weakly balanced signed graph having n vertices and k clusters, $Cl_1, Cl_2, \ldots, Cl_k$. Then the adjacency matrix $A(G)$ of G is of form

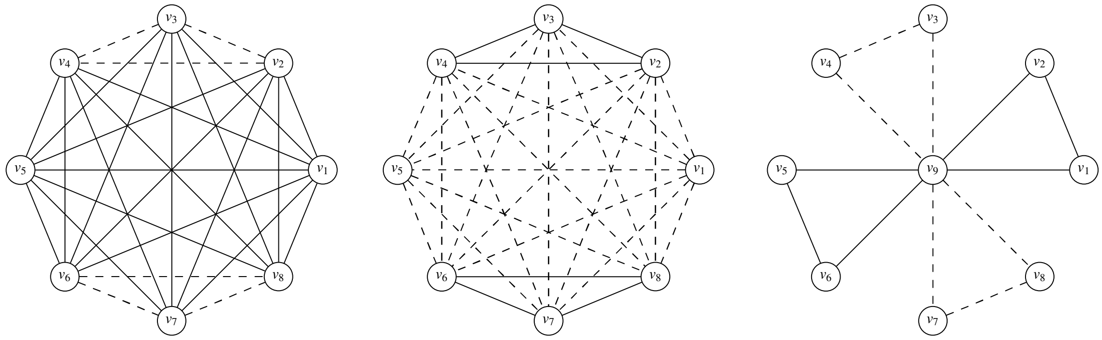
$$A(G) = \begin{bmatrix} A(Cl_1) & & \mathscr{N} & \\ & A(Cl_2) & & \\ & & \ddots & \\ & \mathscr{N}^T & & A(Cl_k) \end{bmatrix}_{n\times n}, \tag{4.1}$$

where, $A(Cl_i)$ is the adjacency matrix of the i-th cluster Cl_i . The entries of $A(Cl_i)$ are either 0 or 1, whereas the entries of $\mathscr{N}$ are either -1 or 0. Note that it is sufficient to find the spectra of $A(G)+I_n$ in order to calculate the spectra of (G). Indeed, λ is an eigenvalue of $A(G)+I_n$ corresponding to an eigenvector $X \in R^n$ if and only if $\lambda - 1$ is an eigenvalue of G corresponding to the eigenvector X. Also, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ then the eigenvalues of the matrix $-A(G)$ are $-\lambda_1, -\lambda_2, \ldots, -\lambda_n$, respectively . Observe that the matrix $-A(G)$ represents the adjacency matrix of graph having k clusters, $Cl_1, Cl_2, \ldots, Cl_k$, such that the edges inside any cluster are negative whereas the edges between any two clusters are positive. We will use these constructions frequently in this chapter.

4.0.1 Preliminaries

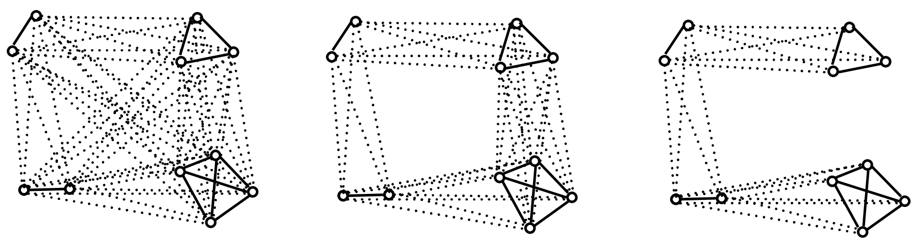
A signed cycle graph on n vertices is a signed graph having an equal number of vertices and edges with each vertex has degree equals to two. We denote a signed cycle graph by C_n or n-cycle. The adjacency matrix A of C_n is given by $A_{i,i+1} = A_{i+1,i} \in \{1,-1\}$ $i = 1,2,\ldots,n-1$ and $A_{n,1} = A_{1,n} \in \{1,-1\}$, all other entries of A are zero. Moreover, the sign of C_n is defined as the product of signs (for positive $+1$ and for negative -1) of its edges. If the sign of C_n is positive it is called balanced cycle graph, otherwise, it is called an unbalanced cycle graph. A signed tree is a connected signed graph which does not have any cycle graph as its subgraph. A signed tree is a strongly balanced graph as it has no unbalanced cycle. We denote a signed tree on n vertices by T_n. The signed path graph P_n on n vertices is a tree in which two vertices are having degree 1, and the remaining $(n-2)$ vertices are having degree 2.

A signed complete graph is a signed graph where each distinct pair of vertices is connected by an edge, positive or negative. A signed clique in signed graph G is an induced subgraph which



(a) $K_8^{2,3}$ (b) Weakly balanced graph of $K_8^{2,3}$ (c) 3-regular star block graph

Figure 4.1 : Examples: (a) A complete signed graph with negative cliques, $K_8^{2,3}$ (b) Weakly balanced signed graph corresponding to $K_8^{2,3}$, (c) 3-regular star block graph.



(a) (b) (c)

Figure 4.2 : Example of weakly balanced graphs. (a) A complete weakly balanced graph. (b) A complete-cycle weakly balanced graph. (c) A complete-path weakly balanced graph.

is a signed complete graph. In cycle graph C_n , tree T_n the only possible cliques are their edges. When each edge of a clique is negative we call it a negative clique. Similarly, if each edge of a clique is positive then, we call it a positive clique. We denote a complete graph on n vertices, having each edge positive, by K_n . By $K_n^{m,r}$, we denote a signed complete graph on n vertices, having a m number of vertex-disjoint negative cliques each of order r, and all the edges positive except those are in negatives cliques. As an example the $K_8^{2,3}$ graph is given in Figure 4.1(a), where two vertex-disjoint negative cliques, each of order 3 are there on vertex-sets $\{v_2, v_3, v_4\}$, and $\{v_6, v_7, v_8\}$ respectively. Negating the edges of $K_n^{m,r}$ gives a corresponding weakly balanced graph whose eigenvalues are equal to negative of eigenvalues of $K_n^{m,r}$. Thus, it is sufficient to calculate the eigenvalues of $K_n^{m,r}$ in order to know the eigenvalues of its corresponding weakly balanced graph. The weakly balanced graph corresponding to $K_8^{2,3}$ is given in Figure (4.1(b)) (in this case it is also strongly balanced graph). We call a graph star block graph when several cliques meet at the single cut-vertex of the graph. We consider a star block graph having a k number of blocks each having a r number of vertices. We call it r-regular star block graph. An example of a 3-regular star block graph is given in Figure 4.1(c).

The rest of the chapter is organized as follows: In the Section 4.1, we calculate the characteristic polynomials, the eigenvalues of signed cycle and path graphs using the concept of linear subdigraphs, and matching. In Section 4.2, we calculate the characteristic polynomial and the eigenvalues of complete graphs having vertex disjoint negative cliques of the same order.

In Section 4.3, we give the bounds on the eigenvalues of complete graphs having vertex disjoint negative cliques of different orders. As an example, if we negate the edges of the graph in Figure 4.2(a), we get a complete graph having negative cliques of a different order. In Section 4.4 we calculate the eigenvalues of regular star block graphs. We mentioned that the negative of the eigenvalues of graphs in these section gives the spectra of corresponding weakly balanced graphs. In later sections, we give the spectrum of some weakly balanced connected signed graphs with some special structures. In particular, we focus on completely-cycle weakly balanced and completely-path weakly balanced graphs, see Figure 4.2 (b), (c), respectively. Definitions of these graphs are given in their respective sections.

4.1 THE CHARACTERISTIC POLYNOMIAL OF SIGNED CYCLE AND PATH GRAPH

We denote the weight of signed cycle graph C_n by δ. When C_n is balanced $\delta = 1$, otherwise, $\delta = -1$. The Coates digraph corresponding to the adjacency matrix $\left(A(C_n) - \lambda I_n\right)$, is a directed graph on n vertices with

1. Loop of weight $-\lambda$ at each vertex.
2. For every adjacent vertices in cycle C_n, there are two opposite directed edges, connecting these adjacent vertices in the Coates digraph.

Next, we require number of k-matchings in C_n, which is used to find linear subdigraphs of the Coates digraph of $\left(A(C_n) - \lambda I_n\right)$. We state the following standard result [Weisstein].

Proposition 4.1. *The number of k-matching in cycle graph C_n is equal to*

$$\frac{n}{n-k}\binom{n-k}{k}. \tag{4.2}$$

For cycle graphs $m(G) = \lfloor n/2 \rfloor$, so the number of all possible matching in G is given by

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}, \tag{4.3}$$

where, $k = 0$ corresponding to no matching. Each k-matching in cycle graph corresponds to k vertex-disjoint directed 2-cycles in its Coates digraph covering $2k$ vertices. These k directed 2-cycles along with loops at remaining $n - 2k$ vertices form linear subdigraphs in Coates digraph of C_n.

Theorem 4.1. *The characteristic polynomial $\phi(C_n)$ of signed cycle graph C_n, having weight $\delta \in \{-1, 1\}$ is given by*

$$\phi(C_n) = \begin{cases} (-1)^n \left(\left(\sum_{k=1}^{(\frac{n}{2}-1)} \frac{n}{n-k}\binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} \right) + 2(-1)^{\frac{n}{2}} - 2\delta \right) & \text{if } n \text{ is even}, \\ (-1)^n \left(\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} - 2\delta \right) & \text{if } n \text{ is odd}, \end{cases}$$

Proof: In the Coates digraph of matrix $\left(A(C_n) - \lambda I_n\right)$ there will be following two type of linear subdigraphs along with their contribution to $\phi(C_n)$

1. Two directed n-cycles; one clockwise and another anticlockwise respectively, each having weight δ. Using Theorem 2.3 their contribution to $\phi(C_n)$ is

$$(-1)^n\left(2(-1)^1\delta\right) = (-1)^n(-2\delta).$$

2. Linear subdigraph having k-matching covering $2k$ vertices, and loops at remaining $n-2k$ vertices for $k = 1, 2, \dots \lfloor n/2 \rfloor$. Weight of each k-matching is 1, and weight of $n-2k$ loops is $(-\lambda)^{n-2k}$. Total number of cycles are $k+n-2k = n-k$. If,

   a) n is even: for $k = \frac{n}{2}$, there will be two linear subdigraphs having $\frac{n}{2}$ directed 2-cycles. Thus, no loop will be selected in these two linear subdigraphs. Their contribution is

   $$(-1)^n 2(-1)^{\frac{n}{2}}.$$

   b) n is odd: there will be no linear subdigraphs having $\frac{n}{2}$ directed 2-cycles.

Thus, using Proposition 4.1 and combining 1. and 2. the result follows.

Corollary 4.1. *The determinant of signed cycle C_n, having weight $\delta \in \{-1, 1\}$ is given by*

$$\det(C_n) = \begin{cases} 2-2\delta & \textit{if n is even and even multiple of 2} \\ -2-2\delta & \textit{if n is even and odd multiple of 2} \\ 2\delta & \textit{if n is odd} \end{cases}$$

Proof: To calculate the determinant we need to set $\lambda = 0$ in the characteristic polynomial. Hence, the result directly follows from Theorem 4.1.

4.1.1 The eigenvalues of signed C_n

Let us consider a matrix Q of order $n \geq 2$ such that, the $Q_{i,i+1} \in \{1, -1\}$, $i = 1, 2, ..., n-1$, the $Q_{n,1} \in \{1, -1\}$ and the remaining entries of Q are zero. The Coates digraph $D(Q - \lambda I)$ is a digraph having directed n-cycle with a loop of weight $-\lambda$ at each of its vertices. Thus, Coates diagraph $D(Q - \lambda I)$ has only two linear subdigraphs. One having the directed n-cycle without loops, and another consisting of the n loops only. Weight of the directed n-cycle is either 1 or -1. It follows that the characteristic equation of Q is given by:

$$(-1)^n\left((-1)^n(-\lambda)^n + (-1)^1\delta\right) = 0 \implies \lambda^n - \delta = 0, \tag{4.4}$$

which means the eigenvalues of Q are $1, \omega, \omega^2, ...\omega^{n-1}$, where,

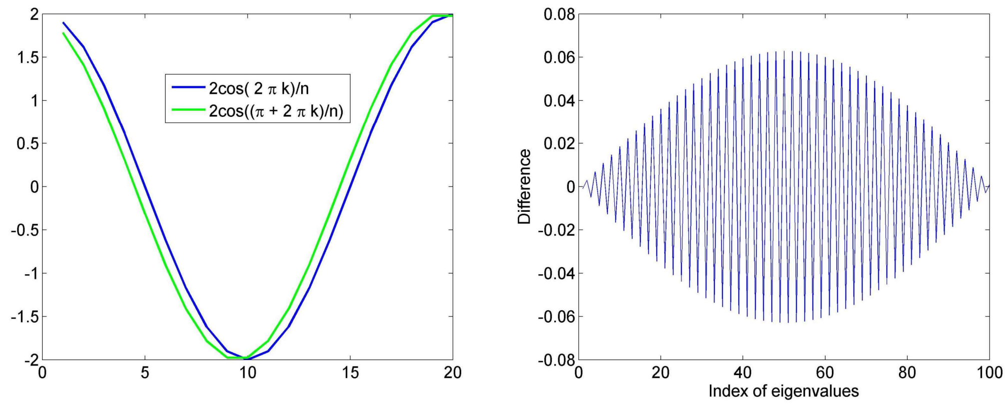
$$\omega = \begin{cases} e^{\frac{2\pi\iota}{n}} & \text{if } \delta = 1. \\ e^{\iota\frac{\pi+2\pi k}{n}} & \text{if } \delta = -1. \end{cases}$$

For a signed a cycle C_n, the adjacency matrix $A(C_n) = Q + Q' = Q + Q^{n-1}$ is a polynomial in Q [Bapat and Roy, 2014]. Thus, the eigenvalues of $A(C_n)$ are obtained by evaluating the same polynomial at each of the eigenvalues of Q, so the eigenvalues of $A(C_n)$ are $\omega^k + \omega^{n-k}, k = 1, ...n$.

Theorem 4.2. *The eigenvalues of signed C_n are*

$$\omega = \begin{cases} 2\cos\frac{2\pi k}{n} & \textit{if } C_n \textit{ is balanced} \\ 2\cos(\frac{\pi+2\pi k}{n}) & \textit{if } C_n \textit{ is unbalanced} \end{cases}$$

$k = 1, 2...n$.



(a) (b)

Figure 4.3 : (a) Eigenvalues of balanced (Blue) and unbalanced (Green) cycle graph C_{20}. (b)Difference in eigenvalues of balanced and unbalanced cycle graph C_{100}.

Proof: It is clear that the eigenvalues of $A(C_n)$ are $\omega^k + \omega^{n-k}, k = 1, ...n$. To derive adjacency matrix of balanced C_n from Q, value of δ has to be 1. Similarly, to derive adjacency matrix of unbalanced C_n from Q, value of δ has to be -1. Now, when $\delta = 1$

$$\omega^k + \omega^{n-k} = \omega^k + \omega^{-k} = e^{\frac{2\pi\imath k}{n}} + e^{-\frac{2\pi\imath k}{n}} = 2\cos\frac{2\pi k}{n}.$$

And, when $\delta = -1$

$$e^{\imath\frac{\pi+2\pi k}{n}} + e^{-\imath\frac{\pi+2\pi k}{n}} = 2\cos(\frac{\pi + 2\pi k}{n})$$

For $k = 1, 2..n$

Theorem 4.3. *Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of balanced signed cycles and $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ be the eigenvalues of unbalanced signed cycles of length $n > 2$. Then,*

$$|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|.$$

Proof:

1. If n is even: cos function lies in the range [-1 1]. The eigenvalues of the balanced and unbalanced C_n are $2\cos(\frac{2\pi k}{n})$, and $2\cos(\frac{\pi+2\pi k}{n})$, respectively for $k = 1, 2, \ldots, n$. To get $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ we need to sort the values of $2\cos(\frac{2\pi k}{n})$ and $2\cos(\frac{\pi+2\pi k}{n})$ in descending order. Also, $2\cos(\frac{2\pi k}{n}) = 2\cos(\frac{2\pi(n-k)}{n})$, and $2\cos(\frac{\pi+2\pi k}{n}) = 2\cos(\frac{-\pi-2\pi k}{n}) = 2\cos(\frac{\pi+2\pi(n-k-1)}{n})$. Sorted order of eigenvalues of balanced C_n is for sequence $k = n, 1, (n-1), 2, (n-2), \ldots, i, (n-i), \ldots, (n/2+1), n/2$. For unbalanced C_n sorted order is for sequence $k = n, (n-1), 1, (n-1-1), 2, (n-2-1), \ldots, i, (n-i-1), \ldots, (n/2-1), n/2$. Now, consider λ_i and λ_{n-i+1}. As, their corresponding k indices are at difference of $n/2$. We have, $2\cos(\frac{2\pi(k\pm n/2)}{n}) = -2\cos(\frac{2\pi k}{n})$. Hence, $\lambda_{n-i+1} = -\lambda_i$. Corresponding k indices of β_i and β_{n-i+1} are also at difference of $n/2$. Thus, $2\cos(\frac{\pi+2\pi(k\pm n/2)}{n}) = -2\cos(\frac{\pi+2\pi k}{n})$. Hence, $\beta_{n-i+1} = -\beta_i$, and $|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|$.

2. If n is odd: following similar steps as the case for even n, in this case to get $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, we need the sequence $k = n, 1, (n-1), 2, (n-2), \ldots, i, (n-i), \ldots, (n-1)/2, (n+1)/2$, and to get

$\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ we need the sequence $k = n, (n-1), 1, (n-1-1), 2, (n-2-1), \ldots, i, (n-i-1), \ldots, (n+1)/2, (n-1)/2$. The difference between k indices for β_i and k indices for α_{n-i+1} is $\pm n/2 - 1/2$. We have, $2\cos(\frac{\pi+2\pi(k\pm n/2-1/2)}{n}) = -2\cos(\frac{2\pi k}{n})$. Hence, $\lambda_i = -\beta_{n-i+1}$. Similarly, $\beta_i = -\lambda_{n-i+1}$ thus, $|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|$.

4.1.2 The characteristic polynomial of P_n

Coates digraph corresponding to adjacency matrix $A(P_n)$ of signed path graph P_n , is a directed graph having n vertices with

1. Loop of weight $-\lambda$ at each vertex.

2. For every adjacent vertices in path P_n, there are two opposite directed edges, connecting these adjacent vertices in Coates digraph.

We state the following standard result [Weisstein].

Proposition 4.2. *The number of k-matching in signed path graph P_n is equal to*

$$\binom{n-k}{k}. \tag{4.5}$$

Thus, for path graphs $m(G) = \lfloor n/2 \rfloor$ number of all possible matching in G is given by:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}. \tag{4.6}$$

Theorem 4.4. *The characteristic polynomial $\phi(P_n)$ of signed P_n is given by*

$$\phi(P_n) = \begin{cases} (-1)^n \left(\left(\sum_{k=1}^{(\frac{n}{2}-1)} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} \right) + (-1)^{\frac{n}{2}} \right) & \textit{if } n \textit{ is even} \\ (-1)^n \left(\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} \right) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof: In Coates digraph of matrix $\left(A(P_n) - \lambda I\right)$ there will be following linear subdigraph along with their contribution to $\phi(P_n)$

1. Subdigraph having k-matching covering $2k$ vertices and loops at remaining $n-2k$ vertices for $k = 1, 2, \ldots \lfloor n/2 \rfloor$. Weight of k-matching is 1, and weight of $n-2k$ loops is $(-\lambda)^{n-2k}$. Total number of cycles are $k + n - 2k = n - k$. If

   a) n is even: for $k = \frac{n}{2}$, there will be one linear subdigraphs having $\frac{n}{2}$ directed 2-cycles. Thus, no loop will be selected in this linear subdigraph. Its contribution is

   $$(-1)^n(-1)^{\frac{n}{2}}.$$

   b) n is odd: There will be no linear subdigraphs having $\frac{n}{2}$ directed 2-cycles.

Thus, using Proposition 4.2, and combining 1.a) and 1.b) the result follows.

As the characteristic polynomial of all signed path graphs P_n for a given n is same, their eigenvalues are same.

Corollary 4.2. *The determinant of signed path P_n is given by*

$$\det(P_n) = \begin{cases} 1 & \text{if } n \text{ is even and even multiple of } 2 \\ -1 & \text{if } n \text{ is even and odd multiple of } 2 \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Proof: Proof directly follows from Theorem 4.4, on setting $\lambda = 0$.

4.2 THE COMPLETE GRAPH WITH NEGATIVE CLIQUES OF SAME ORDER

In this section we derive the characteristic polynomial of $K_n^{m,r}$. Here, the determinant, and the eigenvalues are readily follows from characteristic polynomials hence, they are stated as corollaries without proofs. We first derive the result for the case when, $n = mr$, that is, when all m negative cliques each of order r cover all the n vertices of complete graph.

Theorem 4.5. *The characteristic polynomial of $A(K_{mr}^{m,r})$ is given by*

$$\phi(K_{mr}^{m,r}) = (1-\lambda)^{m(r-1)}(1-2r-\lambda)^{m-1}\Big(1+r(m-2)-\lambda\Big).$$

Proof: With suitable relabelling of vertices in $K_{mr}^{m,r}$ we have,

$$A\left(K_{mr}^{m,r}\right) = \begin{bmatrix} -A(K_r) & J & J & \cdots & J \\ J & -A(K_r) & J & \cdots & J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \cdots & -A(K_r) \end{bmatrix}_{mr\times mr},$$

where, $A(K_r)$ denotes the adjacency matrix of a positive clique K_r. Also, J is all-one matrix of order r. Then,

$$A\left(K_{mr}^{m,r}\right) - \lambda I_{mr} = \begin{bmatrix} Y & X & X & \cdots & X \\ X & Y & X & \cdots & X \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X & X & X & \cdots & Y \end{bmatrix}_{mr},$$

where,

$$Y = -A(K_r) - \lambda I_r, \;\; X = J_r,$$

and J_r is all-one matrix of order r.

In the above matrix, $A\left(K_{mr}^{m,r}\right) - \lambda I_{mr}$, subtract the last row from all the other rows. This produces

$$\begin{bmatrix} Y-X & O & O & \dots & O & X-Y \\ O & Y-X & O & \dots & O & X-Y \\ O & O & Y-X & \dots & O & X-Y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & Y-X & X-Y \\ X & X & X & \dots & X & Y \end{bmatrix},$$

Now, add first $r-1$ columns to the last column. This produce the following block lower triangular matrix,

$$\begin{bmatrix} Y-X & O & O & \dots & O & O \\ O & Y-X & O & \dots & O & O \\ O & O & Y-X & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & Y-X & O \\ X & X & X & \dots & X & \Big(Y+(m-1)X\Big) \end{bmatrix}.$$

Hence,

$$\det\left(A\left(K_{mr}^{m,r}\right)-\lambda I_{mr}\right)=\det(Y-X)^{m-1}\det\Big(Y+(m-1)X\Big). \tag{4.7}$$

Also,

$$Y-X=-2A(K_r)-(\lambda+1)I_r.$$

The eigenvalues of $A(K_r)$ are given by $-1,(r-1)$ with the multiplicity $(r-1)$, 1, respectively [Bapat, 2010]. Hence, eigenvalues of the matrix, $Y-X$, are $(1-\lambda),(-2r+1-\lambda)$ with multiplicities $(r-1)$, 1, respectively. As the determinant of a matrix is the product of its eigenvalues thus,

$$\det(Y-X)=(1-\lambda)^{r-1}(1-2r-\lambda).$$

Next,

$$Y+(m-1)X=(m-2)A(K_r)+(m-1-\lambda)I_r.$$

The eigenvalues of $Y+(m-1)X$ are $(1-\lambda)$, $\Big(1+r(m-2)-\lambda\Big)$ with multiplicity $(r-1),1$, respectively. Hence,

$$\det\Big(Y+(m-1)X\Big)=(1-\lambda)^{r-1}\Big(1+r(m-2)-\lambda\Big).$$

Thus, from Equation (4.7)

$$\begin{aligned}\phi(K_{mr}^{m,r})&=\Big((1-\lambda)^{r-1}(1-2r-\lambda)\Big)^{m-1}(1-\lambda)^{r-1}\Big(1+r(m-2)-\lambda\Big)\\&=(1-\lambda)^{m(r-1)}(1-2r-\lambda)^{m-1}\Big(1+r(m-2)-\lambda\Big).\end{aligned}$$

Corollary 4.3. *The determinant of $K_{mr}^{m,r}$ is given by*

$$(1-2r)^{(m-1)}\Big(1+r(m-2)\Big).$$

Corollary 4.4. *The eigenvalues of $K_{mr}^{m,r}$ are $1,(1-2r)$, and $\Big(1+r(m-2)\Big)$ with multiplicity $m(r-1),m-1$,and 1 respectively.*

Next we give the inverse of the matrix $A(K_{mr}^{m,r})-\lambda I_{mr}$. It is used to get characteristic polynomial of general case $A(K_n^{m,r})$.

Lemma 4.1. *The inverse of* $A(K_{mr}^{m,r}) - \lambda I_{mr}$ *is given by*

$$\frac{1}{\lambda+2r-1}\Bigg(\Big(\frac{1}{\lambda-1}\Big)I_m \otimes \Big(2A(K_r) - (\lambda+2r-3)I_r\Big) - \frac{1}{\lambda+r(2-m)-1}J\Bigg),$$

where, $\lambda \neq 1, (1-2r)$, *and* $\Big(1+r(m-2)\Big)$. *Also,* J *is all-one matrix of order* mr, *and* $\otimes$ *denotes the tensor product of matrices.*

Proof: Using the same construction as in Theorem 4.5, we can write,

$$A\Big(K_{mr}^{m,r}\Big) - \lambda I_{mr} = \Big(I_m \otimes (Y-X)\Big) + (1_{m\times m} \otimes X) = \Big(I_m \otimes (Y-X)\Big) + 1_{mr}1_{mr}^T.$$

Let $A_1 = \Big(I_m \otimes (Y-X)\Big)$. Now, recall the Sherman–Morrison formula: If A is a nonsingular square matrix and $1+v^TA^{-1}u \neq 0$ for some column vectors u, v then

$$(A+uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}.$$

In order to find A_1^{-1} we need to find $(Y-X)^{-1}$. By symmetry let, α, β be the diagonal, non-diagonal entries of $(Y-X)^{-1}$, respectively. On solving following two equations we get the values of α, β.

$$\begin{aligned} -\alpha(\lambda+1) - 2\beta(r-1) &= 1, \\ -\beta(\lambda+1) - 2\alpha - 2\beta(r-2) &= 0, \end{aligned}$$

we get,

$$\alpha = \frac{-(\lambda+2r-3)}{(\lambda-1)(\lambda+2r-1)}, \beta = \frac{2}{(\lambda-1)(\lambda+2r-1)}$$

Thus, A_1^{-1} can be written as,

$$A_1^{-1} = \frac{1}{(\lambda-1)(\lambda+2r-1)}\Bigg(I_m \otimes \Big(2A(K_r) - (\lambda+2r-3)I_r\Big)\Bigg).$$

Also,

$$A_1^{-1}1_{mr}1_{mr}^TA_1^{-1} = \frac{1}{(\lambda+2r-1)^2} \times J,$$

and

$$1 + 1_{mr}^TA_1^{-1}1_{mr} = \frac{\lambda+r(2-m)-1}{\lambda+2r-1},$$

where, J is all-one matrix of order mr.

Hence,

$$\Bigg(A\Big(K_{mr}^{m,r}\Big) - \lambda I_{mr}\Bigg)^{-1} = \frac{1}{(\lambda-1)(\lambda+2r-1)}\Bigg(I_m \otimes \Big(2A(K_r) - (\lambda+2r-3)I_r\Big)\Bigg)$$

$$-\frac{1}{(\lambda+2r-1)(\lambda+r(2-m)-1)}J$$

$$=\frac{1}{\lambda+2r-1}\left(\left(\frac{1}{\lambda-1}\right)I_m\otimes\left(2A(K_r)-(\lambda+2r-3)I_r\right)-\frac{1}{\lambda+r(2-m)-1}J\right)$$

Theorem 4.6. *The characteristic polynomial of $A(K_n^{m,r})$ is given by*

$$(1-\lambda)^{m(r-1)}(1-2r-\lambda)^{m-1}\left(\frac{-\lambda^2-r\left(2+\lambda(2-m)-m\right)+1}{\lambda+r(2-m)-1}\right)^{n-mr-1}$$
$$\times\left(n(1-2r-\lambda)+2r\left(1+m(r-1)+\lambda\right)-1+\lambda^2\right).$$

Proof: With suitable relabelling of vertices in $K_n^{m,r}$, matrix $A(K_n^{m,r})-\lambda I_n$ can be written in the form

$$A(K_n^{m,r})=\begin{bmatrix}A_1-\lambda I_{mr} & J\\ J^T & A_2-\lambda I_{n-mr}\end{bmatrix},$$

where, $A_1=A(K_{mr}^{m,r}), A_2=A(K_{n-mr})$. Also, J is all-one matrix of order $(mr)\times(n-mr)$, and J^T is the transpose of J. By Schur complement formula ([Bapat, 2010],p.4) we have,

$$\det\left(A(K_n^{m,r})-\lambda I_n\right)=\det(A_1-\lambda I_{mr})\times\det\left((A_2-\lambda I_{n-mr})-J^T(A_1-\lambda I_{mr})^{-1}J\right)$$

Using Lemma 4.1,

$$J^T(A_1-\lambda I_{mr})^{-1}J=\frac{-mr}{\lambda+r(2-m)-1}J_1,$$

and,

$$(A_2-\lambda I_{n-mr})-J^T(A_1-\lambda I_{mr})^{-1}J=\left(\frac{\lambda+2r-1}{\lambda+r(2-m)-1}\right)K_{n-mr}+\left(-\lambda+\frac{mr}{\lambda+r(2-m)-1}\right)I_{n-mr}.$$

The eigenvalues of above matrix are

$$\frac{-\lambda^2-r\left(2+\lambda(2-m)-m\right)+1}{\lambda+r(2-m)-1},\quad\frac{n(\lambda+2r-1)-2r(1+\lambda-m+mr)-\lambda^2+1}{\lambda+r(2-m)-1}$$

with multiplicity $n-mr-1,\ 1$, respectively.

From Theorem 4.5,

$$\det(A_1-\lambda I_{mr})=(1-\lambda)^{m(r-1)}(1-2r-\lambda)^{m-1}\left(1+r(m-2)-\lambda\right).$$

Hence,

$$\phi\left(A(K_n^{m,r})\right) = (1-\lambda)^{m(r-1)}(1-2r-\lambda)^{m-1}\left(\frac{-\lambda^2 - r\left(2+\lambda(2-m)-m\right)+1}{\lambda+r(2-m)-1}\right)^{n-mr-1}$$
$$\times\left(n(1-2r-\lambda)+2r\left(1+m(r-1)+\lambda\right)-1+\lambda^2\right)$$

Corollary 4.5. *The determinant of $A(K_n^{m,r})$ is given by*

$$(1-2r)^{m-1}(-1)^{n-mr-1}\left(n(1-2r)+2r\left(1+m(r-1)\right)-1\right).$$

Corollary 4.6. *The eigenvalues of $A(K_n^{m,r})$ are*

$$1,\ (1-2r),\ \frac{(n-2r)\pm\sqrt{8mr-8r-4n-8mr^2+4+(n+2r)^2}}{2}$$

and roots of the polynomial

$$\left(\frac{-\lambda^2 - r\left(2+\lambda(2-m)-m\right)+1}{\lambda+r(2-m)-1}\right)$$

with multiplicity $m(r-1),\ (m-1),\ 1,\ n-mr-1$ respectively.

4.3 THE COMPLETE GRAPH WITH NEGATIVE CLIQUES OF DIFFERENT ORDER

In this section we consider the complete graph G having vertex-disjoint negative cliques of different orders which cover the vertex-set of G. Let G have $k > 2$ number of negative cliques with order $n_1, n_2, \ldots, n_k$, respectively. Let $n_1 \leq n_2 \leq \ldots \leq n_k$. Thus, the adjacency matrix of such a graph G can be written as

$$A(G) = \begin{bmatrix} -A(K_{n_1}) & J_{12} & \ldots & J_{1k} \\ J_{12}^T & -A(K_{n_2}) & \ldots & J_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ J_{1k}^T & J_{2k}^T & \ldots & -A(K_{n_k}) \end{bmatrix}, \tag{4.8}$$

where, $A(K_{n_i})$ denotes the adjacency matrix of $K_{n_i}, i = 1, \ldots, k$ and J_{pq} denotes the all-one matrix of order $n_p \times n_q$. To calculate the eigenvalues we use approach similar to in [Esser and Harary, 1980] for complete multipartite graph. Observe that the diagonal blocks of $A(G) - I_n$ are $-J_{n_i n_i}, i = 1, \ldots, k$, and the off diagonal blocks are same as that of $A(G)$.

We first prove the following lemma which is used in the sequel.

Lemma 4.2. *Let*

$$N = \begin{bmatrix} -n_1 & n_2 & \ldots & n_k \\ n_1 & -n_2 & \ldots & n_k \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & n_2 & \ldots & -n_k \end{bmatrix} \tag{4.9}$$

be a matrix of order $k \times k$. Let $N_\lambda = N - \lambda I_k$. Then

$$\det(N_\lambda) = \left[\prod_{i=1}^{k}(-2n_i-\lambda) + \sum_{i=1}^{k} n_i \prod_{j=1, j\neq i}^{k}(-2n_j-\lambda)\right].$$

Proof: Let $\mathbf{n} = [n_1\ n_2\ \ldots\ n_k]^T \in R^k$. Then,

$$\det(N_\lambda) = \det\left(\begin{bmatrix} 1 & -\mathbf{n}^T \\ 0_k & N_\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -\mathbf{n}^T \\ \mathbf{1}_k & -2\,\mathrm{diag}(\mathbf{n}) - \lambda I_k \end{bmatrix}\right).$$

Expanding the right hand side, the desired result follows.

Now, we have the following theorem which completely characterizes the eigenvalues of $A(G) - I_n$, and hence the eigenvalues of G.

Theorem 4.7. *Let G be a complete graph on n vertices with k disjoint negative cliques of order $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \ldots + n_k = n$. Suppose $\overline{n}_i, i = 1, \ldots, t,\ t \leq k$ be the distinct numbers in the set $\{n_1, \ldots, n_k\}$. Then,*

(a) 0 is an eigenvalue of $A(G) - I_n$ with algebraic multiplicity $n - k$ corresponding to eigenvectors $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in R^{n_i}$ such that $\Sigma X_i = 0$ for all i.

(b) $-2\overline{n}_i, i = 1, \ldots, t$ are nonzero eigenvalues of $A(G) - I_n$ with multiplicity $m_i - 1$ where m_i is the number of distinct clusters in G of order $\overline{n}_i$. The other nonzero eigenvalues are the roots of the polynomial $1 + p(\lambda)$ where

$$p(\lambda) = \sum_{i=1}^{t} \frac{m_i \overline{n}_i}{-2\overline{n}_i - \lambda}.$$

Moreover, the eigenvectors corresponding to the nonzero eigenvalues of $A(G) - I_n$ are of the form $X = [\alpha_1 \mathbf{1}_{n_1}^T\ \alpha_2 \mathbf{1}_{n_2}^T\ \ldots\ \alpha_k \mathbf{1}_{n_k}^T]^T$ where $0_k \neq \alpha = [\alpha_1\ \alpha_2\ \ldots \alpha_k]^T$ satisfies $N_\lambda \alpha = 0$. Such an α determines an eigenvector corresponds to the eigenvalue λ for which $\lambda(\alpha_i - \alpha_j) = 2(n_j \alpha_j - n_i \alpha_i), i, j = 1, \ldots, k$.

Proof:

(a) Let $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in R^{n_i}$ such that $\Big(A(G) - I_n\Big)X = 0$. Then for $i, j \in \{1, \ldots, k\}$,

$$\sum_{r \neq i, r=1}^{k} \Sigma X_r - \Sigma X_i = \sum_{r \neq j, r=1}^{k} \Sigma X_r - \Sigma X_j = 0.$$

This yields $\Sigma X_i = 0$ for all $i = 1, \ldots, k$.

(b) Let $\lambda \neq 0$ and $\Big(A(G) - I_n\Big)X = \lambda X$ where $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in R^{n_i}$. For any i, consider the vector X_i, any two entries of X_i, say $x_p^{(i)}, x_q^{(i)}$, satisfy

$$\lambda x_p^{(i)} = \sum_{r \neq i, r=1}^{k} \Sigma X_r - \Sigma X_i = \lambda x_q^{(i)}. \tag{4.10}$$

Since, $\lambda \neq 0, X_i = \alpha_i \mathbf{1}_{n_i}$ for some constant α_i for all $i = 1, \ldots, k$. Setting $X = [\alpha_1 \mathbf{1}_{n_1}^T\ \alpha_2 \mathbf{1}_{n_2}^T\ \ldots\ \alpha_k \mathbf{1}_{n_k}^T]^T$, by Equation (4.10) we have

$$\lambda \alpha_i = \sum_{r \neq i, r=1}^{k} n_r \alpha_r - n_i \alpha_i. \tag{4.11}$$

For any $j \neq i$, similarly, we have

$$\lambda \alpha_j = \sum_{r \neq j, r=1}^{k} n_r \alpha_r - n_j \alpha_j. \tag{4.12}$$

Adding these above two equations, we obtain $\lambda(\alpha_i - \alpha_j) = 2(n_j\alpha_j - n_i\alpha_i)$ for any $i, j \in \{1, \ldots, k\}$.

In order to find all $\alpha_i, i = 1, \ldots, k$ which satisfy Equation (4.11) for each i, it gives the linear system $N_\lambda \alpha = 0$. Note that both λ and α are unknown in this linear system and for the existence of a nonzero solution vector α, we must have $\det(N_\lambda) = 0$. Thus, the nonzero eigenvalues of $A(G) - I_n$ are the roots of the polynomial $\det(N_\lambda)$. Now from Lemma 4.2, we have

$$\begin{aligned}
\det(T_\lambda) &= \left[\prod_{i=1}^{t}(-2\bar{n}_i - \lambda)^{m_i} + \sum_{i=1}^{t} \frac{m_i\bar{n}_i}{-2\bar{n}_i - \lambda} \prod_{j=1}^{t}(-2\bar{n}_j - \lambda)^{m_j}\right] \\
&= \prod_{i}^{k}(-2\bar{n}_i - \lambda)^{m_i - 1}\left[\prod_{i=1}^{t}(-2\bar{n}_i - \lambda) + \sum_{i=1}^{t} m_i\bar{n}_i \prod_{j=1, j\neq i}^{t}(-2\bar{n}_j - \lambda)\right].
\end{aligned}$$

Hence, the proof.

Lemma 4.3. *Let $\lambda_1^\star > \lambda_2^\star > \ldots > \lambda_{t-1}^\star > \lambda_t^\star$ be the roots of polynomial $1 + p(\lambda)$. Then*

$$\lambda_1^\star > -2\bar{n}_1 > \lambda_2^\star > -2\bar{n}_2 \ldots > \lambda_{t-1}^\star > -2\bar{n}_{t-1} > \lambda_t^\star > -2\bar{n}_t. \tag{4.13}$$

In general, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1} \geq \lambda_k$ are the nonzero eigenvalues of $A(G) - I_n$, then

$$\lambda_1 \geq -2n_1 \geq \lambda_2 \geq -2n_2 \ldots \geq \lambda_{k-1} \geq -2n_{k-1} \geq \lambda_k \geq -2n_k. \tag{4.14}$$

Proof: Polynomial $p(\lambda)$ is continuous and strictly increasing in interval $(-2\bar{n}_{i+1}, -2\bar{n}_i)$. Also, $\lim_{\lambda \to (-2\bar{n}_i)^-} p(\lambda) = +\infty$ and $\lim_{\lambda \to (-2\bar{n}_{i+1})^+} p(\lambda) = -\infty$ for $i = 1, 2 \ldots t-1$. Hence, using intermediate value theorem there exists a root λ_i^* of equation $1 + p(\lambda) = 0$ in interval $(-2\bar{n}_{i+1}, -2\bar{n}_i)$ for $i = 1, 2 \ldots t-1$, satisfying $-2\bar{n}_i > \lambda_{i+1}^* > -2\bar{n}_{i+1}$. For $i = 1$ $\lim_{\lambda \to (-2\bar{n}_1)^+} p(\lambda) = -\infty$ and $\lim_{\lambda \to +\infty} p(\lambda) = 0$. Again, using intermediate value theorem $\lambda_1^\star > -2\bar{n}_1$ which proves (4.13). Similarly, (4.14) follows from Theorem 4.7.

Corollary 4.7. *Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be eigenvalues of A, and $\alpha_1^* > \alpha_2^* > \ldots > \alpha_{t-1}^* > \alpha_t^*$ be its non-zero non-integer eigenvalues. Then,*

1.
$$\alpha_1^\star > -2\bar{n}_1 + 1 > \alpha_2^\star > -2\bar{n}_2 + 1 \ldots > \alpha_{t-1}^\star > -2\bar{n}_{t-1} + 1 > \alpha_t^\star > -2\bar{n}_t + 1. \tag{4.15}$$

2.
$$\alpha_1 \geq -2n_1 + 1 \geq \alpha_2 \geq -2n_2 + 1 \ldots \geq \alpha_{k-1} \geq -2n_{k-1} + 1 \geq \alpha_k \geq -2n_k + 1. \tag{4.16}$$

Proof: It directly follows from the fact that $\alpha_i = \lambda_i + 1 \;\forall\; i$ and Lemma 4.3.

4.4 THE CHARACTERISTIC POLYNOMIAL OF REGULAR STAR BLOCK GRAPH

In this section we calculate the eigenvalues of r-regular signed star block graph.

Theorem 4.8. *Let G be a r-regular star block graph having k blocks. If l number of blocks are negative cliques for $l \leq k$, then*

$$\phi(G) = l\Big(\phi(\tilde{K}_r)\phi(\tilde{K}_{r-1})^{l-1}\phi(K_{r-1})^{k-l}\Big) + (k-l)\Big(\phi(K_r)\phi(\tilde{K}_{r-1})^{l}\phi(K_{r-1})^{k-l-1}\Big),$$

where, $\phi(\tilde{K}_r)$ denotes the characteristic polynomial of a negative clique of order r.

Proof: In each linear subdigraphs corresponding Coates digraph of G, the cut vertex v will associate with linear subdigraphs of exactly one clique. From Theorem 2.3

$$\phi(G) = l\Big(\phi(\tilde{K}_r)\phi(\tilde{K}_{r-1})^{l-1}\phi(K_{r-1})^{k-l}\Big) + (k-l)\Big(\phi(K_r)\phi(\tilde{K}_{r-1})^{l}\phi(K_{r-1})^{k-l-1}\Big)$$

Eigenvalues of K_n are $-1,(n-1)$ while eigenvalues of $\tilde{K}_n$ are $1^{n-1}, 1-n$, with multiplicities $(n-1),1$, respectively. Hence,

$$\phi(K_n) = (-1-\lambda)^{n-1}(n-1-\lambda),\ \phi(\tilde{K}_n) = (1-\lambda)^{n-1}(1-n-\lambda)$$

$$\phi(G) = l\Big(\phi(\tilde{K}_r)\phi(\tilde{K}_{r-1})^{l-1}\phi(K_{r-1})^{k-l}\Big) + (k-l)\Big(\phi(K_r)\phi(K_{r-1})^{k-l-1}\phi(\tilde{K}_{r-1})^{l}\Big)$$

$$\phi(G) = \Big((1-\lambda)^{r-2}(2-r-\lambda)\Big)^{l-1}\Big((-1-\lambda)^{r-2}(r-2-\lambda)\Big)^{k-l-1}$$

$$\underbrace{\bigg(l\Big((1-\lambda)^{r-1}(1-r-\lambda)(-1-\lambda)^{r-2}(r-2-\lambda)\Big) + (k-l)\Big((-1-\lambda)^{r-1}(r-1-\lambda)(1-\lambda)^{r-2}(2-r-\lambda)\Big)\bigg)}_{\varphi}$$

Thus, eigenvalues of G are $1, 2-r, -1, r-2$ with multiplicities $(r-2)(l-1)$, $(l-1)$, $(r-1)(k-l-1)$, $(k-l-1)$, respectively and rest of the eigenvalues are given by roots of polynomial φ.

In the next two sections, we calculate the eigenvalues of complete-path weakly balanced and complete-cycle weakly balanced graphs. The procedure is similar to the Section 4.3, however, the non zero eigenvalues for these graphs can be calculated using numerical methods. Also, we use their weakly balanced structure as such rather than first negating their edges then using the negative cliques.

4.5 THE SPECTRUM OF COMPLETELY-PATH WEAKLY BALANCED SIGNED GRAPHS

In this section we consider weakly balanced signed graphs whose positive clusters $Cl_i = K_{n_i}, i = 1, \ldots, k$ are completely connected graphs and clusters are positioned in a path such that every vertex of a cluster Cl_j is linked to every vertex of its adjacent cluster $Cl_{j+1}, j = 1, \ldots, k-1$ by negative edges. We call it completely-path weakly balanced signed graphs. The adjacency matrix of such a graph G is a tridiagonal block matrix of the form

$$A(G) = \begin{bmatrix} A(K_{n_1}) & -J_{12} & & \\ -J_{12}^T & A(K_{n_2}) & \ddots & \\ & \ddots & \ddots & -J_{k-1k} \\ & & -J_{kk-1}^T & A(K_{n_k}) \end{bmatrix}. \tag{4.17}$$

We mention that apparently a completely-path weakly balanced signed graph seems not strongly balanced. However, by setting $\widehat{Cl}_o$ as the union of all the clusters Cl_i where i is odd and $\widehat{Cl}_e$ as the union of Cl_i where i is even, it is evident that inside $\widehat{Cl}_o$ and $\widehat{Cl}_e$ all the edges are positive and edges between vertices in $\widehat{Cl}_o$ and $\widehat{Cl}_e$ are negative. Thus a completely-path weakly balanced signed graph is strongly balanced.

First, we recall the following result from [Smith, 1985].

Theorem 4.9. *Consider the tridiagonal Toeplitz matrix*

$$T(a,b,c) = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}$$

of order k. Then the eigenvalues of $T(a,b,c)$ are given by

$$\lambda_i = a + 2b\sqrt{\frac{c}{b}}\cos\left(\frac{i\pi}{k+1}\right)$$

corresponding to the eigenvector

$$v_i = \left[\left(\tfrac{c}{b}\right)^{\frac{1}{2}} \sin\left(\tfrac{i\pi}{k+1}\right) \quad \tfrac{c}{b}\sin\left(\tfrac{2i\pi}{k+1}\right) \quad \left(\tfrac{c}{b}\right)^{\frac{3}{2}} \sin\left(\tfrac{3i\pi}{k+1}\right) \quad \ldots \quad \left(\tfrac{c}{b}\right)^{\frac{k}{2}} \sin\left(\tfrac{ki\pi}{k+1}\right) \right]^T$$

for $i = 1,\ldots,k$.

Theorem 4.10. *Let G be a completely-path weakly balanced signed graph on n vertices with $k > 2$ clusters $K_{n_i}, i = 1,\ldots,k$ such that $\sum_{i=1}^{k} n_k = n$. Then*

(a) 0 is an eigenvalue of $I_n + A(G)$ corresponding to the eigenvector $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ such that $\Sigma X_i = 0, i = 1,\ldots,k$ when $k = 3j$ or $3j+1$ for some positive integer j. If $k = 3j-1$, 0 is an eigenvalue of $I_n + A(G)$ associated with the eigenvector $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ such that $\Sigma X_i = \sin\frac{i\pi}{3}$.

(b) Let

$$N = \begin{bmatrix} n_1 & -n_2 & & 0 \\ -n_1 & n_2 & \ddots & \\ & \ddots & \ddots & -n_k \\ 0 & & -n_{k-1} & n_k \end{bmatrix}. \tag{4.18}$$

Then λ is an eigenvalue of $I_n + A(G)$ if it is a nonzero eigenvalue of N. If $\alpha = [\alpha_1\ \alpha_2\ \ldots \alpha_k]^T \in \mathbb{R}^k$ is an eigenvector corresponding to the eigenvalue λ of N then $X = [\alpha_1 \mathbf{1}_{n_1}^T\ \alpha_2 \mathbf{1}_{n_2}^T\ \ldots\ \alpha_k \mathbf{1}_{n_k}^T]^T$ is an eigenvector associated with the eigenvalue λ of $I_n + A(G)$.

Proof:

(a) Note that 0 is an eigenvalue of $I_n + A(G)$ corresponding to an eigenvector $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ if and only if $\tilde{X} = [\Sigma X_1\ \Sigma X_2\ \ldots\ \Sigma X_k]^T \in \mathbb{R}^k$ belongs to the kernel of the tridiagonal Toeplitz matrix $T(1,-1,-1)$ of order k. Then from Theorem 4.9 it follows that $\tilde{X} \neq 0$ is an eigenvector corresponding to the eigenvalue 0 of $T(1,-1,-1)$ if and only if $k = 3j-1$ for some positive integer j. Thus if $k = 3j$ or $3j+1$, $\tilde{X} = 0$. If $k = 3j-1$ then using Theorem 4.9 we obtain the desired result.

(b) Let $\lambda \neq 0$ and $(I_n + A(G))X = \lambda X$ where $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$. First consider X_1. For any two entries of X_1, say $x_p^{(1)}, x_q^{(1)}$, we have

$$\lambda x_p^{(1)} = -\Sigma X_2 + \Sigma X_1 = \lambda x_q^{(1)}. \tag{4.19}$$

Similarly, for any two entries of X_k, say $x_p^{(k)}, x_q^{(k)}$, we have

$$\lambda x_p^{(k)} = -\Sigma X_{k-1} + \Sigma X_k = \lambda x_q^{(k)}. \tag{4.20}$$

Now consider the vector X_i for $i = 2,3,\ldots,k-1$. Then for any two entries of X_i, say $x_p^{(i)}, x_q^{(i)}$, we have

$$\lambda x_p^{(i)} = -\Sigma X_{i-1} - \Sigma X_{i+1} + \Sigma X_i = \lambda x_q^{(i)}. \tag{4.21}$$

Since $\lambda \neq 0, X_i = \alpha_i \mathbf{1}_{n_i}$ for some constant α_i for all $i = 1,\ldots,k$. Setting $X = [\alpha_1 \mathbf{1}_{n_1}^T \ \alpha_2 \mathbf{1}_{n_2}^T \ \ldots \ \alpha_k \mathbf{1}_{n_k}^T]^T$, by Equations (4.19), (4.20), (4.21) we have following k equations

$$\lambda \alpha_1 = n_1 \alpha_1 - n_2 \alpha_2.$$

And,

$$\lambda \alpha_k = n_k \alpha_k - n_{k-1} \alpha_{k-1}.$$

$$\lambda \alpha_i = n_i \alpha_i - n_{i-1} \alpha_{i-1} - n_{i+1} \alpha_{i+1}.$$

for $i = 2,\ldots,k-1$.

This gives the linear system $N_\lambda \alpha = 0$, where $\alpha = [\alpha_1 \ \alpha_2 \ \ldots \alpha_k]^T$. As α is non zero vector, $\det(N_\lambda) = 0$. Solving $\det(N_\lambda) = 0$ is equivalent to finding of eigenvalues of N.

4.5.1 Eigenvalues of N

As all n_i are positive integers, N can be transformed in to a symmetric tridiagonal matrix using similarity transformation with diagonal matrix Γ. Define $\Gamma = \text{diag}\{\gamma_1,\ldots,\gamma_k\}$ as follows.

$$\gamma_i = \left(\frac{n_1}{n_i}\right)^{1/2} \quad i = 1,\ldots,k.$$

Then

$$\Gamma^{-1} N \Gamma = T = \begin{bmatrix} n_1 & -\sqrt{n_1 n_2} & & \\ -\sqrt{n_1 n_2} & n_2 & \ddots & \\ & \ddots & \ddots & -\sqrt{n_{k-1} n_k} \\ & & -\sqrt{n_{k-1} n_k} & n_k \end{bmatrix}. \tag{4.22}$$

is a tridiagonal matrix, having $T_{ii} = n_i \ \forall i$ and $T_{i,i+1} = T_{i+1,i} = -(n_i n_{i+1})^{1/2}$ for $i = 1,2,\ldots,k-1$. It is evident that N and T have the same eigenvalues including their multiplicities. Indeed we have the following theorem.

Theorem 4.11. *The following are true.*

(a) *If all the clusters have equal number of vertices i.e $\frac{n}{k}$ (whenever possible), then eigenvalue of T are $\frac{n}{k}(1 - 2\cos\frac{\pi i}{k+1})$ for $i = 1,2...k$.*

(b) *If $n_i = n_1$ for $i = 2j+1$ and $n_i = n_2$ for $i = 2j$ for $j = 1,2,\ldots,\lfloor k/2 \rfloor$. Then, eigenvalues of T are*

$$= \begin{cases} \frac{n_1+n_2}{2} \pm \sqrt{\frac{(n_1-n_2)^2}{4} + 2(n_1 n_2)^2 + 2n_1 n_2 \cos(\frac{j\pi}{r+1})} & \text{if } k = 2r+1 \\ \frac{n_1+n_2}{2} \pm \sqrt{\frac{(n_1-n_2)^2}{4} + 2(n_1 n_2)^2 + 2n_1 n_2 \cos\theta_{rj}} & \text{if } k = 2r \end{cases}$$

for $j = 1,\ldots,r$, where r is a positive integer and each θ_{rj} is a nonzero solution of the trigonometric equation $n_1 n_2(\sin(r+1)\theta + \sin(r\theta)) = 0$, with $0 < \theta < \pi$.

(c) *All the eigenvalues of T are distinct.*

(d) *T have a zero eigenvalue if and only if $k = 3j + 2$ for some positive integer j.*

Proof:

(a) As all cluster have equal size, $T = \frac{n}{k}(I - A(P_k))$, where $A(P_k)$ is adjacency matrix of path graph having k vertices. Eigenvalues of T are $\frac{n}{k}(1 - 2\cos\frac{\pi i}{k+1})$ for $i = 1, 2...k$.

(b) In this case T is tridiagonal 2-Toeplitz matrix. Proof follows from [Smith, 1985].

(c) For any eigenvalue λ of T consider the matrix $T - \lambda I$. Let us remove first row and last column of $T - \lambda I$. The resulting matrix of order $k-1$ is an upper triangular matrix having all non zero elements on diagonal. Hence it is nonsingular, and its rank is $k-1$. As this resulting matrix is a submatrix of $T - \lambda I$, hence $T - \lambda I$ will have rank at least $k-1$. But $T - \lambda I$ can have at most $k-1$ rank as λ is eigenvalue of T. As rank of the symmetric matrix equals its number of nonzero eigenvalues matrix, $T - \lambda I$ have only one zero eigenvalues. In other words, λ have algebraic multiplicity 1, hence all eigenvalues are distinct.

(d) N and its transpose matrix N^T have same rank. Consider an eigenvector $X = [x_1, x_2, \ldots, x_k]^T$ of N^T corresponding to a possible zero eigenvalue. As X is in null space of N^T, from $N^T X = 0$ we have

$$x_i = x_{i-1} + x_{i+1} \quad \text{for} \quad i = 2, 3, \ldots, k-1.$$

$$x_1 = x_2, \quad x_k = x_{k-1}$$

So, we can arrange x_is on a number line such that each x_i is sum of its adjacent entries. We need to find basis of null space of N^T. As X is non zero vector, at least one entry is non zero. Without loss of generality, let us assume $x_1 = 1$ and set its adjacent entry as $x_2 = 1$ and fill the remaining entries of X considering above equations. Let us denote $1, 1$ in pair by 1_2 and $-1, -1$ in pair by -1_2. When $k \neq 3 + 2r$ we can not fill remaining entries considering above equation. For $k = 3 + 2r$ we will get a possible X as $[1_2, 0, -1_2, 0, 1_2, \ldots, -1_2]$. Which forms the basis for null space. Hence there is a zero eigenvalue for $k = 3r + 2$.

Let λ_i are eigenvalues of T and $p_r(\lambda)$ denote the leading principal minor of order r of $(T - \lambda I)$. Assume $p_0(\lambda) = 1$, we have

$$p_1(\lambda) = -n_1 - \lambda.$$

$$p_i(\lambda) = (-n_i - \lambda)p_{i-1}(\lambda) - n_i n_{i-1} p_{i-1}(\lambda).$$

for $i = 2, 3 \ldots k$

As none of n_i is zero, then eigenvalues α_is are strictly separated (Chapter 300,[Wilkinson and Wilkinson, 1965]). Eigenvalues of N can be computed from following theorem.

Theorem 4.12. *[Wilkinson and Wilkinson, 1965] Let the quantities $p_0(\mu), p_1(\mu), \ldots, p_n(\mu)$ be evaluated for some value of μ. Then $s(\mu)$, the number of agreements in sign of consecutive members of this sequence, is the number of eigenvalues of T which are strictly greater than μ.*

The above property of sequence $p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)$ is called Sturm sequence property, which may be used to to locate any individual eigenvalue. For instance we have two values a_0 and b_0 such that

$$b_0 > a_0, \quad s(a_0) \geq k, \quad s(b_0) < k$$

This implies that $b_0 > \alpha_k > a_0$ from Theorem 4.12. We can locate α_k iteratively using method of bijection (Chapter 301,[Wilkinson and Wilkinson, 1965])

4.5.2 The spectrum of completely-cycle weakly balanced signed graph

In this section we consider weakly balanced signed graphs in which each cluster Cl_i is the complete graph K_{n_i} and each cluster are placed in cycle such that every vertex of a cluster is linked with every vertex of its adjacent cluster by negative edges. We call it completely-cycle weakly balanced signed graph. The adjacency matrix of such a graph G is given by

$$A(G) = \begin{bmatrix} A(K_{n_1}) & -J_{12} & 0 & -J_{1k} \\ -J_{12}^T & \ddots & \ddots & \\ & \ddots & A(K_{n_{k-1}}) & -J_{k-1k} \\ -J_{1k}^T & & -J_{k-1k}^T & A(K_{n_k}) \end{bmatrix} \tag{4.23}$$

where J_{ij} is the all-one matrix of order $n_i \times n_j$. Similar to previous sections we will investigate the spectra of $I_n + A(G)$ in order to investigate the spectra of G. First we recall the following result from [Yueh and Cheng, 2008].

Theorem 4.13. *Consider the perturbed tridiagonal Toeplitz matrix*

$$T(a,b,c,\gamma,\alpha,\delta,\beta) = \begin{bmatrix} a+\gamma & b & & & \alpha \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ \beta & & & c & a+\delta \end{bmatrix}$$

of order k. If

$$\gamma\delta - \alpha\beta = -c^2, \alpha+\beta = 2c, \gamma+\delta = 0$$

Then the eigenvalues of $T(a,b,c,\gamma,\alpha,\delta,\beta)$ are given by

$$\lambda_i = a + 2c\cos\theta_i$$

Where,

$$\theta_i = \frac{2i\pi}{k} \quad i = 1,2,\ldots,k.$$

Theorem 4.14. *Let G be a completely-cycle weakly balanced signed graph on n vertices with $k > 2$ clusters $K_{n_i}, i = 1,\ldots,k$ such that $\sum_{i=1}^k n_k = n$. Then*

(a) Let $X = [X_1\ X_2\ \ldots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ is an eigenvector of $I_n + A(G)$ corresponding to eigenvalue 0. Then, if $k \neq 6r$ for some positive integer r, $\Sigma X_i = 0$, $\forall\ i$.

(b) The eigenvectors corresponding to the nonzero eigenvalues of $I_n + A(G)$ are of the form $X = [\alpha_1 \mathbf{1}_{n_1}^T\ \alpha_2 \mathbf{1}_{n_2}^T\ \ldots\ \alpha_k \mathbf{1}_{n_k}^T]^T$ where $0_k \neq \alpha = [\alpha_1\ \alpha_2\ \ldots \alpha_k]^T$ satisfies $N_\lambda \alpha = 0$. Where $N_\lambda = N - \lambda I_n$ and

$$N = \begin{bmatrix} n_1 & -n_2 & 0 & -n_k \\ -n_1 & n_2 & \ddots & 0 \\ 0 & \ddots & \ddots & -n_k \\ -n_1 & 0 & -n_{k-1} & n_k \end{bmatrix} \tag{4.24}$$

Proof:

(a) Note that 0 is an eigenvalue of $I_n + A(G)$ corresponding to an eigenvector $X = [X_1\ X_2\ \dots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ if and only if $\tilde{X} = [\Sigma X_1\ \Sigma X_2\ \dots\ \Sigma X_k]^T \in \mathbb{R}^k$ belongs to the kernel of the tridiagonal Toeplitz matrix $T(1,-1,-1,0,-1,0,-1)$ of order k. Then from Theorem 4.13 it follows that $\tilde{X} \neq 0$ is an eigenvector corresponding to the eigenvalue 0 of $T(1,-1,-1,0,-1,0,-1)$ if and only if $k = 6r$ for some positive integer r. Setting $i = r$ in Theorem 4.13 we obtain the desired result. Let $X = [X_1\ X_2\ \dots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$ such that $(I_n + A(G))X = 0$. Then

$$\Sigma X_i = \Sigma X_{i-1} + \Sigma X_{i+1}$$

for $i = 1,\dots,k(mod\, k)$. Obviously, $\Sigma X_i = 0$ for all i would satisfy such condition and hence could be an eigenvector. Otherwise, if $\Sigma X_i \neq 0$ for some i, at least one of ΣX_{i+1} and ΣX_{i-1} is nonzero. This implies

Let us arrange $\Sigma X_1, \Sigma X_2, \dots, \Sigma X_k$ on circle. Arrangement requires that each ΣX_i is sum of its adjacent numbers. A trivial solution is $\Sigma X_i = 0,\ \forall\, i$. For non-trivial solution without loss of generality let us consider $\Sigma X_1 = \sigma, \Sigma X_2 = \beta$ for some non zero real number σ. Arrangement will be $[\sigma,\beta,\beta-\sigma,-\sigma,-\beta,\dots,-\beta,\sigma-\beta]$ which is only possible when $k = 6r$ for any positive integer r.

(b) Let $\lambda \neq 0$ and $(I_n + A(G))X = \lambda X$ where $X = [X_1\ X_2\ \dots\ X_k]^T, X_i \in \mathbb{R}^{n_i}$. First consider X_1. For any two entries of X_1, say $x_p^{(1)}, x_q^{(1)}$, we have

$$\lambda x_p^{(1)} = -\Sigma X_k - \Sigma X_2 + \Sigma X_1 = \lambda x_q^{(1)}. \tag{4.25}$$

Similarly, for any two entries of X_k, say $x_p^{(k)}, x_q^{(k)}$, we have

$$\lambda x_p^{(k)} = -\Sigma X_1 - \Sigma X_{k-1} + \Sigma X_k = \lambda x_q^{(k)}. \tag{4.26}$$

Now consider the vector X_i for $i = 2,3,\dots,k-1$. Then for any two entries of X_i, say $x_p^{(i)}, x_q^{(i)}$, we have

$$\lambda x_p^{(i)} = -\Sigma X_{i-1} - \Sigma X_{i+1} + \Sigma X_i = \lambda x_q^{(i)}. \tag{4.27}$$

Since $\lambda \neq 0, X_i = \alpha_i \mathbf{1}_{n_i}$ for some constant α_i for all $i = 1,\dots,k$. Setting $X = [\alpha_1 \mathbf{1}_{n_1}^T\ \alpha_2 \mathbf{1}_{n_2}^T\ \dots\ \alpha_k \mathbf{1}_{n_k}^T]^T$, by Equations (4.25), (4.26), (4.27) we have following k equations

$$\lambda \alpha_1 = n_1\alpha_1 - n_2\alpha_2 - n_k\alpha_k.$$

And,

$$\lambda \alpha_k = n_k\alpha_k - n_{k-1}\alpha_{k-1} - n_1\alpha_1.$$

$$\lambda \alpha_i = n_i\alpha_i - n_{i-1}\alpha_{i-1} - n_{i+1}\alpha_{i+1}.$$

for $i = 2,\dots,k-1$.

This gives the linear system $N_\lambda \alpha = 0$, where $\alpha = [\alpha_1\ \alpha_2\ \dots \alpha_k]^T$. As α is non zero vector, $\det(N_\lambda) = 0$. Solving $\det(N_\lambda) = 0$ is equivalent to finding eigenvalues of N.

Corollary 4.8. *If all the clusters have equal number of vertices i.e $\frac{n}{k}$ (whenever possible), then eigenvalue of A are $-1^{n-k}, \frac{n}{k}(1 - 2\cos\frac{2\pi i}{k} - 1)) - 1$ for $i = 1, 2...k$.*

Proof: As all cluster have equal size, $N = \frac{n}{k}(I - A(C_k))$, where $A(C_k)$ is adjacency matrix of cyclic graph having k vertices. Eigenvalues of N are $\frac{n}{k}(1 - 2\cos\frac{2\pi i}{k}) - 1)$ for $i = 1, 2...k$. These are non zero eigenvalues of $I_n + A(G)$. Hence eigenvalues of A are $-1^{n-k}, \frac{n}{k}(1 - 2\cos\frac{2\pi i}{k}) - 1$.

In the next subsection, we will discuss a method to find the spectrum of $A(G)$ for the general case.

4.5.3 Eigenvalues of N

Proceeding as previous section, we can transform N in to a symmetric form $\hat{T}$ using similarity transformation with diagonal matrix Γ. We define Γ as follows

$$\Gamma_{ii} = (\frac{n_1}{n_i})^{1/2}$$

Then $\Gamma^{-1}N\Gamma = \hat{T}$, is a symmetric tri-diagonal matrix with perturbed corners, having $\hat{T}_{ii} = n_i$ $\forall i$, $\hat{T}_{i,i+1} = \hat{T}_{i+1,i} = -(n_i n_{i+1})^{1/2}$ where $i = 1,2,\ldots,k-1$ and $\hat{T}_{1,n_k} = \hat{T}_{n_k,1} = -(n_1 n_k)^{1/2}$.

Theorem 4.15. *N can have either full rank or $k-2$ and rank is $k-2$ if and only if $k = 6r$ for some positive integer r.*

Proof: N and its transpose matrix N^T have same rank. Consider an eigenvector $X = [x_1, x_2, \ldots, x_k]^T$ of N corresponding to a possible zero eigenvalue. As X is in null space of N^T, from $N^T X = 0$ we have

$$x_i = x_{i-1} + x_{i+1} \quad \text{for} \;\; i = 2,3,\ldots,k-1.$$

$$x_1 = x_2 + x_k$$

$$x_k = x_1 + x_{k-1}$$

So, we can arrange x_is on a cycle such that each x_i is sum of its adjacent entries. We need to find basis of null space of N^T. As X is non zero vector, at least one entry is non zero. Without loss of generality, let us assume $x_1 = 1$ and set its adjacent entries as $x_k = 0, x_2 = 1$ and fill the remaining entries of X considering above equations. Let us denote $1,1$ in pair by 1_2 and $-1,-1$ in pair by -1_2. When $k \neq 6r$ we can not fill remaining entries considering above equation, because for that be need repetition of unit $[0, 1_2, 0, -1_2]$ or $[1_2, 0, -1_2, 0]$. For $k = 6r$ we will get one possible X as $[1_2, 0, -1_2, 0, 1_2, \ldots, -1_2, 0]$. Similarly other possible value of X can be $[0, 1_2, 0, -1_2, 0, 1_2, \ldots, -1_2]$. These two vector forms basis for null space of N^T and other vectors in null space can written as linear combination of these two. We can infer from that geometric multiplicity of zero eigenvalue is 2. Now since N is similar to a symmetric matrix $\hat{T}$, so algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. So for $k = 6r$ there are two zero eigenvalues of N and hence it has rank $k-2$ else rank is k. Moreover $\sum_i^k x_k = 0$.

Using Householder symmetric transformations $\hat{T}$ can be converted to symmetric tri-diagonal T which can be solved using Sturm sequences as in the previous section.

Corollary 4.9. *The spectrum of a weakly balanced complete cycled graph is equal to that of its underlying unsigned graph if and only of the number of cliques are even.*

Proof: For an even number of cliques, all cycles are balanced, so the spectrum of the graph is equal to the spectrum of its underlying unsigned graph. For an odd number of cliques there always exists a cycle having an odd number of negative edges, hence the spectrum can not be equal to that of its underlying unsigned graph.

4.6 CONCLUSION

In this chapter, we found the characteristic polynomial of signed cycle and signed path graph. Later we calculated the spectra of different weakly balanced graphs induced from complete graphs.