5

On Characteristic and Permanent Polynomials of a Matrix

In this chapter, we will see how blocks in the signed weighted graph corresponding to the matrix, can be used to find its characteristic and permanent polynomials. Inspired by the work on the determinant of simple block graphs [Bapat and Roy, 2014], we propose a new technique for computing the characteristic and the permanent polynomials of a matrix. First of all, we derive a recursive expression for these polynomials of a matrix with respect to a pendant block in the corresponding digraph. On solving this recursive expression we find that the characteristic (permanent) polynomial of a digraph can be written in terms of the characteristic (permanent) polynomial of some specific induced subdigraphs of blocks. Interestingly, these induced subdigraphs are vertex-disjoint and they partition the digraph. Hence, this leads us to define a new partition called $\mathscr{B}$-partition of a digraph. Corresponding to every $\mathscr{B}$-partition we define the ϕ-summand and the ψ-summand. Similarly, the det-summand and the per-summand corresponding to each $\mathscr{B}$-partition is specified. Thus, we have found the characteristic and the permanent polynomials of a matrix in terms of the ϕ-summands and the ψ-summands, respectively, of the corresponding $\mathscr{B}$-partitions. Similarly, the determinant and the permanent of the matrix can be found in terms of the det-summands and the per-summands, respectively.

This new method of calculation provides a combinatorial significance of the determinant, the permanent, the characteristic and the permanent polynomials of a matrix. A singular graph has a zero eigenvalue. Classifying singular graphs is a complicated problem in combinatorics [Sciriha, 2007; Bapat, 2011; Bapat and Roy, 2014]. In this chapter, we illuminate this problem with a number of examples with the new combinatorial implication. This procedure presents a simplified proof for the determinant of simple block graphs earlier given in [Bapat and Roy, 2014]. These graph-theoretic representations would be useful in future investigations in matrix theory.

First, we define the idea of a block of a digraph, which plays a fundamental role in this chapter. It is already defined in literature for simple graphs [Bapat and Roy, 2014].

Definition 5.1. ***Block:*** *A block is a maximally connected subdigraph of G that has no cut-vertex.*

Note that, if G is a connected digraph having no cut-vertex, then G itself is a block. A block is called a pendant block if it contains only one cut-vertex of G, or it is the only block in that component. The blocks in a digraph can be found in linear time using John and Tarjan algorithm [Hopcroft and Tarjan, 1971]. We define the cut-index of a cut-vertex v as the number of blocks adjacent to v. We specifically denote a digraph having k blocks as G_k.

A square matrix $A = (a_{uv}) \in \mathbb{C}^{n\times n}$ can be depicted by a weighted digraph $G(A)$ with n vertices. If $a_{uv} \neq 0$, then $(u,v) \in E\Big(G(A)\Big)$ and $f(u,v) = a_{uv}$. The diagonal entry a_{uu} corresponds to a loop at vertex u having weight a_{uu}. If v is a cut-vertex in $G(A)$, then we call a_{vv} as the corresponding cut-entry in A. The following example will make this assertion transparent.

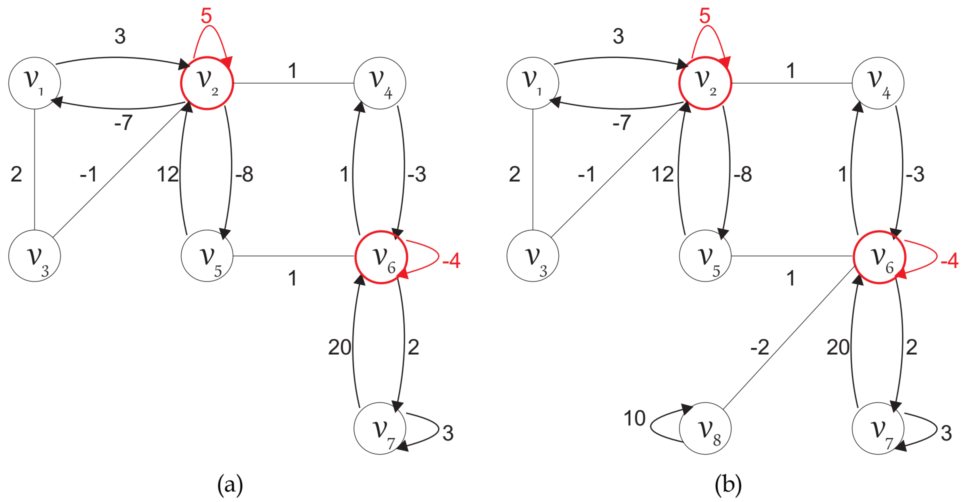


Figure 5.1 : (a) Digraph of matrix M_1 (b) Digraph of matrix M_2

Example 5.1. *The digraphs corresponding to the matrices M_1 and M_2 are presented in Figure 5.1.*

$$M_1 = \begin{bmatrix} 0 & 3 & 2 & 0 & 0 & 0 & 0 \\ -7 & 5 & -1 & 1 & -8 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 20 & 3 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ -7 & 5 & -1 & 1 & -8 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 20 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 10 \end{bmatrix}.$$

The cut-entries and the cut-vertices are shown in red in the matrices M_1 and M_2, as well as in their corresponding digraphs $G(M_1)$ and $G(M_2)$. Note that, when $a_{uv} = a_{vu} \neq 0$, we simply denote edges (u,v) and (v,u) with an undirected edge (u,v) with weight a_{uv}. As an example, in $G(M_1)$ the edge (v_1,v_3) and (v_3,v_1) are undirected edges. The digraph $G(M_1)$, depicted in the Figure 5.1(a), has blocks B_1, B_2 and B_3 which are induced subdigraphs on vertex subsets $\{v_1,v_2,v_3\}, \{v_2,v_4,v_5,v_6\}$ and $\{v_6,v_7\}$, respectively. Here, cut-vertices are v_2 and v_6 with cut-indices 2. Similarly, the digraph $G(M_2)$ in the Figure 5.1(b), has blocks B_1, B_2, B_3 and B_4 on vertex sets $\{v_1,v_2,v_3\}, \{v_2,v_4,v_5,v_6\}, \{v_6,v_7\}$ and $\{v_6,v_8\}$, respectively. Here, cut-vertices are v_2 and v_6 with cut-indices 2 and 3, respectively.

The characteristic polynomial of a digraph $G(A)$ is the characteristic polynomial of a matrix A. In other words, $\phi\Big(G(A)\Big) = \phi(A) = \det(A - \lambda I)$. Hence, the determinant of digraph $G(A)$ is the determinant of a matrix A. Similarly, the permanent polynomial of digraph $G(A)$ is the permanent polynomial of corresponding matrix A, or in other words $\psi\Big(G(A)\Big) = \psi(A) = \text{per}(A - \lambda I)$. Hence, the permanent of the digraph $G(A)$ is the permanent of the matrix A.

The chapter is organized as follows. In Section 5.1, we derive a recursive form of the characteristic and the permanent polynomials of the digraph. In Section 5.2, we define $\mathscr{B}$-partition of digraph and its corresponding ϕ-summand, ψ-summand, det-summand and the per-summand. In Section 5.3, we solve recursive expression derived in Section 5.1 to get the characteristic and the permanent polynomials of digraph in terms of the ϕ-summands and the ψ-summands, respectively. In Section 5.4, we provide results on the determinant of some simple graphs including block graphs.

5.1 RECURSIVE FORM OF THE CHARACTERISTIC AND THE PERMANENT POLYNOMIAL

In this section, we provide a recursive expression for the characteristic and the permanent polynomial of a digraph G, with respect to its pendant block. If Q is a subdigraph of G, then $G \setminus Q$ denotes the induced subdigraph of G on the vertex subset $V(G) \setminus V(Q)$. Here, $V(G) \setminus V(Q)$ is the standard set-theoretic subtraction of vertex sets. Let v be a cut-vertex of G, then the recursive expression for these polynomials can also be given with respect to subdigraph H containing v, such that, $H \setminus v$ is a union of components. For convenience, we relabel the digraph $G(A)$. In graph theory, these relabeling are captured by permutation similarity of A. Thus, relabeling on vertex set keep the determinant and the permanent unchanged. We frequently use this idea in this section.

Lemma 5.1. *Let G be a digraph having at least one cut-vertex. Let B_1 be a pendant block and v be the cut-vertex of G in B_1. Let the weight of the loop at vertex v be α. The following recurrence relation holds for the characteristic polynomial,*

$$\phi(G) = \phi(B_1) \times \phi(G \setminus B_1) + \phi(B_1 \setminus v) \times \phi\Big(G \setminus (B_1 \setminus v)\Big) + (\lambda - \alpha) \times \phi(B_1 \setminus v) \times \phi(G \setminus B_1).$$

Proof: Let $A(G)$ be the matrix corresponding to the digraph G having n vertices. Let the number of vertices in block B_1 be n_1. With suitable reordering of vertices in G, let block B_1 have vertices with the labels $\{1,2,\ldots,n_1\}$. Here, n_1-th vertex is the cut-vertex v of G in B_1. Let x and z be column vectors of order $(n_1 - 1)$ and $(n - n_1)$, respectively, such that (x, α, z) formulates n_1-th column vector of $A(G)$. Similarly, w and y are row vectors of order $(n_1 - 1)$ and $(n - n_1)$, respectively, such that (w, α, y) formulates n_1-th row vector of $A(G)$. Now,

$$A(G) - \lambda I = \begin{bmatrix} A(B_1 \setminus v) - \lambda I & x & O \\ w & \alpha - \lambda & y \\ O & z & A(G \setminus B_1) - \lambda I \end{bmatrix}. \tag{5.1}$$

Here, O is the zero matrix of appropriate size. Let x_i, w_i, z_i and y_i denote the i-th entry of the vectors x, w, z and y, respectively. Using Theorem 2.4, let us fix set $S = \{1,2,\ldots,n_1\}$, then T is n_1-subset of n columns in $A(G) - \lambda I$. Note that, T must have numbers $1,2,3\ldots,(n_1 - 1)$ otherwise it will give zero contribution to $\phi(G)$. This is because if any number $0 \le i \le (n_1 - 1)$ is missing in T, then the column corresponding to i will be in submatrix $[A(G) - \lambda I]'_{S,T}$. As this column in submatrix $[A(G) - \lambda I]'_{S,T}$ has all zero entries, hence its determinant is zero. Thus, only following sets of S, T have possible non-zero contribution in $\phi(G)$.

1. $S = \{1,2,\ldots,n_1\}$, $T = \{1,2,\ldots,n_1\}$. It contributes the following to $\phi(G)$

$$\phi(B_1) \times \phi(G \setminus B_1) \tag{5.2}$$

2. $S = \{1,2,\ldots,n_1\}$, $T = \{1,2,\ldots,n_1 - 1, r\}$, where r is a value from set $\{n_1+1, n_1+2,\ldots,n\}$. Thus, there will be $n - n_1$ such possible set of T. Let $T_i = \{1,2,\ldots,n_1 - 1, n_1 + i\}$ for $i = 1,2,\ldots,n - n_1$.

   Let c_i be the i-th column of the matrix $\Big(A(G \setminus B_1) - \lambda I\Big)$ and $\Big(A(G \setminus B_1) - \lambda I\Big) \setminus c_i$ denote the resulting submatrix after c_i is removed from $\Big(A(G \setminus B_1) - \lambda I\Big)$. Then, contribution of these sets to $\phi(G)$ is

$$
\begin{aligned}
&\sum_{i=1}^{n-n_1} (-1)^{w(S,T_i)} \det[A(G)-\lambda I]_{S,T} \times \det[A(G)-\lambda I]'_{S,T} \\
=&\sum_{i=1}^{n-n_1} (-1)^{w(S,T_i)} \det \begin{bmatrix} A(B_1\setminus v)-\lambda I & O \\ w & y_i \end{bmatrix} \times \det \begin{bmatrix} z & \left(A(G\setminus B_1)-\lambda I\right)\setminus c_i \end{bmatrix} \\
=&\det\left((B_1\setminus v)-\lambda I\right) \times \left(\sum_{i=1}^{n-n_1} (-1)^{w(S,T_i)} \times y_i \times \det \begin{bmatrix} z & \left(A(G\setminus B_1)-\lambda I\right)\setminus c_i \end{bmatrix} \right) \qquad (5.3) \\
=&\det\left((B_1\setminus v)-\lambda I\right) \times \left(\det\left(G\setminus (B_1\setminus v)-\lambda I\right) + (\lambda-\alpha)\times \det\left(A(G\setminus B_1)-\lambda I\right)\right) \\
=&\phi(B_1\setminus v)\times \left(\phi\left(G\setminus (B_1\setminus v)\right) + (\lambda-\alpha)\phi(G\setminus B_1)\right).
\end{aligned}
$$

Hence,

$$\phi(G)=\phi(B_1)\times\phi(G\setminus B_1)+\phi(B_1\setminus v)\times\phi\left(G\setminus(B_1\setminus v)\right)+(\lambda-\alpha)\times\phi(B_1\setminus v)\times\phi(G\setminus B_1) \quad (5.4)$$

Corollary 5.1. *Let G be a digraph having at least one cut-vertex. Let B_1 is a pendant block and v be the cut-vertex of G in B_1. Let the weight of loop at vertex v be α. The following recurrence relation holds for the permanent polynomial,*

$$\psi(G)=\psi(B_1)\times\psi(G\setminus B_1)+\psi(B_1\setminus v)\times\psi\left(G\setminus(B_1\setminus v)\right)+(\lambda-\alpha)\times\psi(B_1\setminus v)\times\psi(G\setminus B_1)$$

Proof: The proof is similar to Lemma 5.1.

We generalize Lemma 5.1 with respect to some subdigraphs containing cut-vertex in the next lemma.

Lemma 5.2. *Let G be a digraph with at least one cut-vertex. Let H be a non empty subdigraph of G having cut-vertex v with loop weight α, such that $H\setminus v$ is union of components. The characteristic polynomial of G,*

$$\phi(G)=\phi(H)\times\phi(G\setminus H)+\phi(H\setminus v)\times\phi\left(G\setminus(H\setminus v)\right)+(\lambda-\alpha)\times\phi(H\setminus v)\times\phi(G\setminus H).$$

Proof: Let the number of vertices in the subdigraph H be n_1. With suitable reordering of vertices in G, let n_1-th vertex be the cut-vertex v of G in H. Remaining proof is similar to Lemma 5.1.

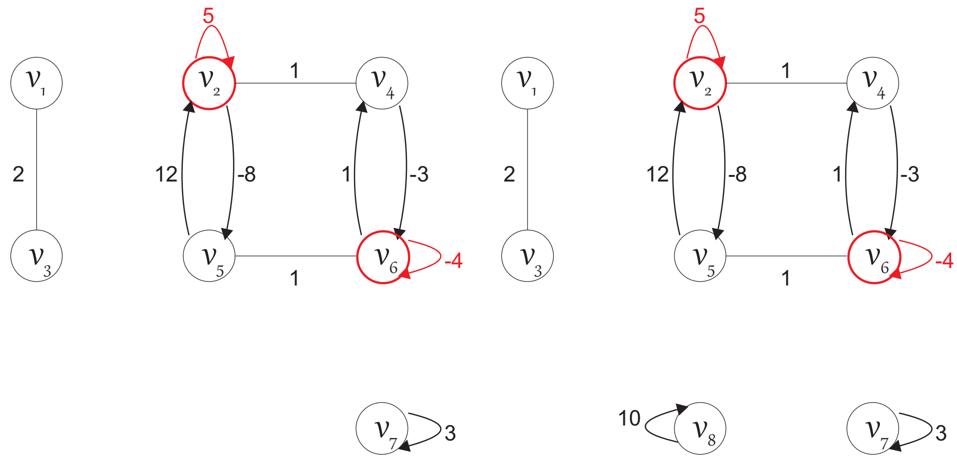
Corollary 5.2. *Let G be a digraph with at least one cut-vertex. Let H be a nonempty subdigraph of G having cut-vertex v with loop weight α, such that $H\setminus v$ is a union of components. The permanent polynomial of G,*

$$\psi(G)=\psi(H)\times\psi(G\setminus H)+\psi(H\setminus v)\times\psi\left(G\setminus(H\setminus v)\right)+(\lambda-\alpha)\times\psi(H\setminus v)\times\psi(G\setminus H).$$

Proof: The proof follows from Lemma 6.4.

5.2 $\mathscr{B}$-PARTITIONS OF A DIGRAPH

We define a new partition of digraph which helps in finding its characteristic (permanent) polynomial.



(a) (b)

Figure 5.2 : Example of a $\mathscr{B}$-partition of (a) Digraph of matrix M_1 (b) Digraph of matrix M_2

Definition 5.2. *Let G_k be a digraph having k blocks $B_1, B_2, \ldots, B_k$. Then, a $\mathscr{B}$-partition of G_k is a partition in k vertex disjoint induced subdigraphs $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$, such that, $\hat{B}_i$ is a subdigraph of B_i. The ϕ-summand and the* det*-summand of this $\mathscr{B}$-partition is*

$$\prod_i^k \phi(\hat{B}_i), \text{ and } \prod_i^k \det(\hat{B}_i),$$

respectively, where by convention $\phi(\hat{B}_i) = 1$, and $\det(\hat{B}_i) = 1$ if $\hat{B}_i$ is a null graph. Similarly, its ψ-summand and the per*-summand are given by*

$$\prod_i^k \psi(\hat{B}_i), \text{ and } \prod_i^k \mathrm{per}(\hat{B}_i),$$

respectively, where by convention $\psi(\hat{B}_i) = 1$, and $\mathrm{per}(\hat{B}_i) = 1$ if $\hat{B}_i$ is a null graph.

All possible $\mathscr{B}$-partitions of M_1, and M_2 are given in Figures 5.3 and 5.4, respectively.

Corollary 5.3. *Let G be a digraph having t cut-vertices with cut-indices $d_1, d_2, \ldots, d_t$, respectively. The number of $\mathscr{B}$-partitions of G is*

$$\prod_{i=1}^t d_i.$$

Proof: Each cut-vertex associates with an induced subdigraph of exactly one block in a $\mathscr{B}$-partition. For i-th cut-vertex there are d_i choices of blocks. Hence, the result follows.

Example 5.2. *The number of $\mathscr{B}$-partitions in digraphs corresponding to matrices M_1 and M_2 are* 4 *and* 6*, respectively.*

In the next section, we make use of $\mathscr{B}$-partitions to calculate the characteristic and the permanent polynomials. In addition to this, we also use slightly modified $\mathscr{B}$-partitions which are explained as follows. Let G_k be a digraph having k blocks $B_1, B_2, \ldots B_k$. Let the block B_i have t_i number of cut-vertices of G_k. Now consider that some or all of these cut-vertices are removed

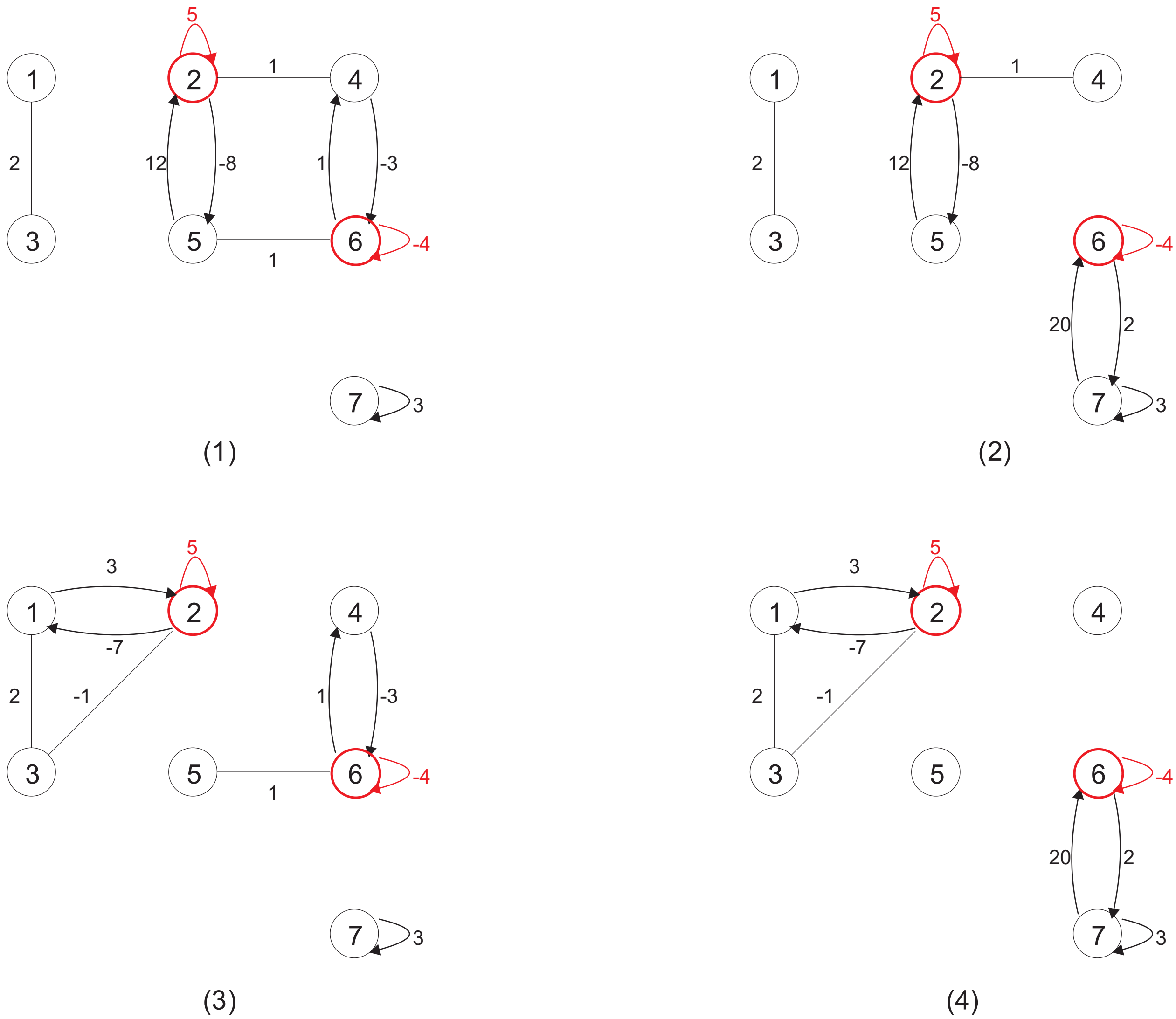


Figure 5.3 : $\mathscr{B}$-partitions of the digraph of matrix M_1

from G_k to form a desired subdigraph of it. In the process, let us call the resulting subdigraph from block B_i as cut-subdigraph of B_i. Now, in order to calculate $\mathscr{B}$-partitions of resulting subdigraph of G_k, in the Definition 5.2, we take $\hat{B}_i$ as a subdigraph of cut-subdigraph of B_i, instead of B_i. This is a modified $\mathscr{B}$-partition or $\mathscr{B}$-partition using cut-subdigraph of blocks.

Example 5.3. *Let us consider the digraph in Figure 5.1(a) corresponding to the matrix M_1. Let us remove the cut-vertex v_2 from the digraph. Then for the block B_1, the resulting cut-subdigraph is $B_1 \setminus v_2$. Similarly, for the block B_2, the resulting cut-subdigraph $B_2 \setminus v_2$. Then there are two $\mathscr{B}$-partition of the resulting digraph using cut-subdigraph; (1) $B_1 \setminus v_2,\ B_2 \setminus v_2,\ B_3 \setminus v_6$. (2) $B_1 \setminus v_2,\ B_2 \setminus \{v_2, v_6\},\ B_3$.*

5.3 THE CHARACTERISTIC AND PERMANENT POLYNOMIAL OF MATRIX

In this section, we derive an expression for the characteristic polynomial of an arbitrary square matrix using the ϕ-summands in $\mathscr{B}$-partitions of the digraph corresponding to the matrix. Similarly, we apply the ψ-summands for the permanent polynomial.

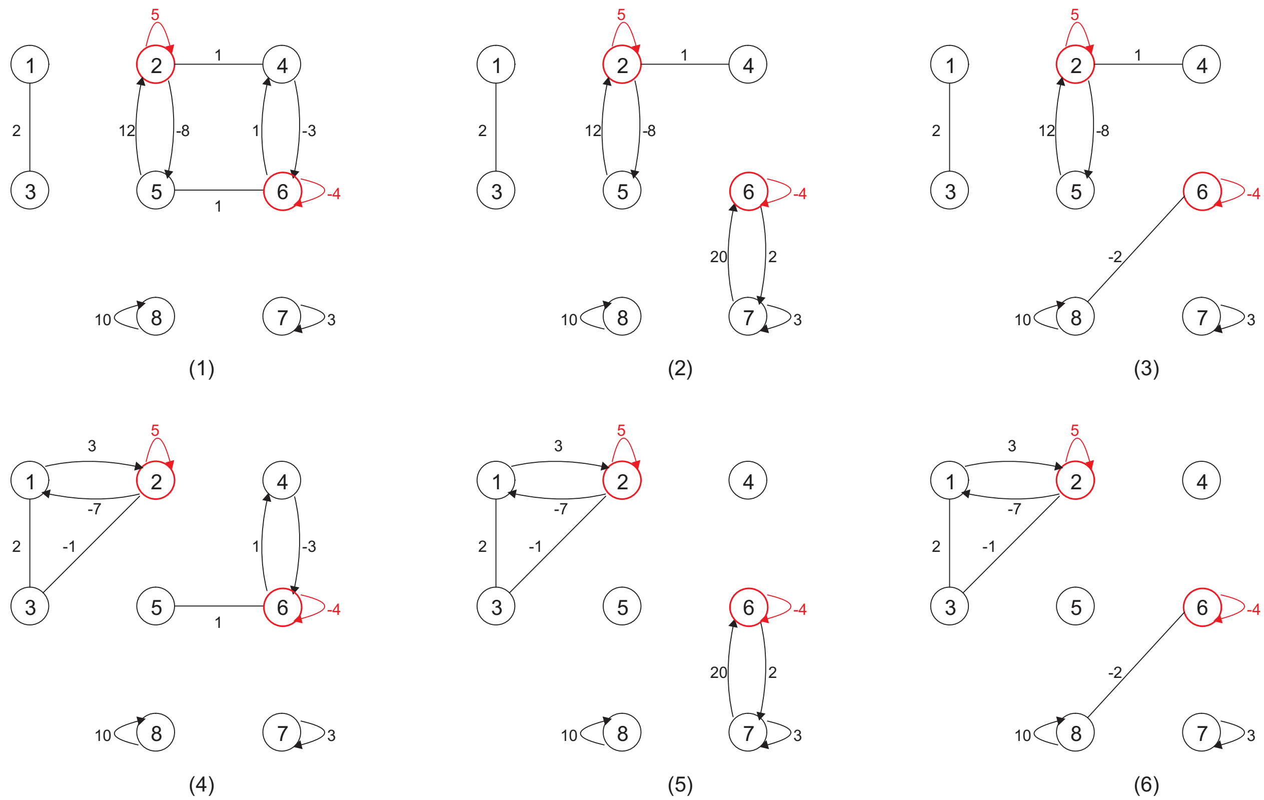


Figure 5.4 : $\mathscr{B}$-partitions of the digraph of matrix M_2

Let G_k be a digraph having k blocks $B_1, B_2, \ldots, B_k$. Let G_k have m cut-vertices with cut-indices $d_1, d_2, \ldots, d_m$. Also, assume that the weights of loops at these vertices are $\alpha_1, \alpha_2, \ldots, \alpha_m$, respectively. Then, following are the steps to calculate the characteristic (permanent) polynomials of G_k.

Procedure 5.1.

1. *Add $-\lambda$ with the loop-weight at each vertex. Whenever there is no loop at a vertex then, add a loop with weight $-\lambda$.*

2. *For $q = 0, 1, 2, \ldots, m$,*

   a) *Delete any q cut-vertices at a time from G_k to construct an induced subdigraph. In this way, construct all $\binom{m}{q}$ induced subdigraphs of G_k.*

   b) *Find all possible $\mathscr{B}$-partitions of each subdigraph constructed in (a) using cut-subdigraphs.*

   c) *For each $\mathscr{B}$-partition in (b) multiply its ϕ-summand (ψ-summand) by $\prod_i(\lambda - \alpha_i)(d_i - 1)$, where, $i = 1, 2, \ldots, q$. Also, α_i and d_i are weight and cut-index of removed i-th cut-vertex, respectively.*

3. *Sum all the terms in 2(c).*

In the following theorem, we justify the above procedure to find the characteristic polynomial of an arbitrary matrix.

Theorem 5.1. *Let G_k be a digraph with m cut-vertices and k blocks $B_1, B_2, \ldots, B_k$. Let $G_k^q =$*

$\left\{G^q_{k1}, G^q_{k2}, \ldots, G^q_{k\binom{m}{q}}\right\}$ *be set of all induced subdigraphs of* G_k *after removing any* q *cut-vertices. Also, let* $d_{i1}, d_{i2}, \ldots, d_{iq}$ *be cut-indices and* $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iq}$ *be the weights of the loops of these removed* q *cut-vertices to form* G^q_{ki}*. Let* S^q_{ki} *denotes the summation of all the* ϕ*-summands of all possible* $\mathscr{B}$*-partition of* G^q_{ki} *using cut-subdigraphs. Then, the characteristic polynomial of* G_k *is given by,*

$$\phi(G_k) = \sum_{q=0}^{m} \varphi(G^q_k), \text{ where } \varphi(G^q_k) = \sum_{i=1}^{\binom{m}{q}} \left(L^q_i S^q_{ki}\right) \text{ and } L^q_i = \prod_{t=1}^{q} (\lambda - \alpha_{it})(d_{it} - 1), \quad L^0_1 = 1.$$

Proof: We use method of mathematical induction on G_k, with $k \in \mathbb{N}$ to prove the theorem.

1. Let $k = 1$. For any digraph G_1 having only one block,

$$\phi(G_1) = \varphi(G^0_1) = L^0_1 S^0_{11} = S^0_{11}.$$

There is only one $\mathscr{B}$-partition, which is block itself, thus the ϕ-summand is $\phi(G_1)$, that is, $S^0_{11} = \phi(G_1)$. Hence, the theorem is valid in this case.

2. For $k = 2$, digraph G_2 has two blocks with one cut-vertex. Let them be B_1 and B_2. Also, let v be the cut-vertex with cut-index 2 and the loop-weight at vertex v be α_1. Thus,

$$G^0_2 = \left\{G^0_{21}\right\}, \; G^1_2 = \left\{G^1_{21}\right\},$$

where, $G^0_{21} = G_2$, $G^1_{21} = G_2 \setminus v$. Now we find $\varphi(G^0_2), \varphi(G^1_2)$.

a) Note that, to compute $\varphi(G^0_2)$ there is no cut-vertex to remove. Thus, $q = 0$, $\binom{m}{q} = 1$ and $L^0_1 = 1$. There are two $\mathscr{B}$-partition of G^0_{21}. One of them consists of induced subdigraphs B_1 and $B_2 \setminus v$. Another one contains the induced subdigraphs $(B_1 \setminus v)$ and B_2. Hence,

$$S^0_{21} = \phi(B_1)\phi(B_2 \setminus v) + \phi(B_2)\phi(B_1 \setminus v) = \varphi(G^0_2).$$

b) Also, to compute $\varphi(G^1_2)$ we need to remove the cut-vertex v. Here, $q = 1$, $\binom{m}{q} = 1$ and $d_{11} = 2$. Thus, $L^1_1 = (\lambda - \alpha_1)$. The only possible $\mathscr{B}$-partition of G^1_{21} contains the induced subdigraphs $B_1 \setminus v$ and $B_2 \setminus v$. Hence,

$$\begin{aligned} S^1_{21} &= \phi(B_1 \setminus v)\phi(B_2 \setminus v), \\ \text{therefore, } \varphi(G^1_2) &= (\lambda - \alpha_1)\phi(B_1 \setminus v)\phi(B_2 \setminus v). \end{aligned} \tag{5.5}$$

On combining (a) and (b) we observe,

$$\phi(G_2) = \phi(B_1)\phi(B_2 \setminus v) + \phi(B_2)\phi(B_1 \setminus v) + (\lambda - \alpha_1)\phi(B_1 \setminus v)\phi(B_2 \setminus v).$$

Using Lemma 5.1 on G_2 we also obtain the above expression. It proves the theorem for $k = 2$.

Hence, the theorem is true for G_2.

3. Now, we assume that the theorem is true for any G_n for $2 < n \leq k$. We need to prove the theorem for G_{n+1}. We can always select a pendant block of G_{n+1} and denote it as B_{n+1}. Let v

be the cut-vertex of G_{n+1} in B_{n+1}, having loop-weight $(\alpha-\lambda)$. Note that, the digraph G_{n+1} has an induced subdigraph $G_n = \Big(G_{n+1} \setminus (B_{i+1} \setminus v)\Big)$. The theorem is true for G_n by the assumption of induction. Then, from Lemma 5.1,

$$\phi(G_{n+1}) = \phi(G_n)\phi(B_{n+1} \setminus v) + \phi(G_n \setminus v)\phi(B_{n+1}) + (\lambda-\alpha)\Big(\phi(G_n \setminus v)\phi(B_{n+1} \setminus v)\Big). \tag{5.6}$$

In this context, there are two cases depending on whether v is also a cut-vertex of induced subdigraph G_n or not.

a) Let the vertex v is not a cut-vertex of G_n. In this case, the number of cut-vertices in G_{n+1} is one more than that of G_n. In G_{n+1} all the cut-vertices have the same cut-indices as is in G_n, except v. The cut-index of v in G_{n+1} is two.

b) Let the vertex v be a cut-vertex of G_n. In this case, the number of cut-vertices in G_{n+1} is the same as the number of cut-vertices in G_n. In G_{n+1} all the cut-vertices have the same cut-indices as is in G_n, except v. If the cut-vertex v has cut-index equal to d_v in G_n, then it has cut-index d_v+1 in G_{n+1}.

   Now, we check whether theorem for G_{n+1} is equivalent to Equation (5.6) for both the cases. For each set of q deleted cut-vertices from G_{n+1} as required by the theorem, there are two cases:

   The cut-vertex v is not in deleted cut-vertices: Note that, in this case, cut-indices and loop-weights of these removed q cut-vertices in G_{n+1} are same as those were in G_n. Thus, L_i^q value corresponding to $G_{(n+1)i}^q$ remain same as for G_{ni}^q. Also, in this case, in all the ϕ-summands of G_{n+1}, either $\phi(B_{n+1})$ or $\phi(B_{n+1} \setminus v)$ has to be there. The first term in right hand side of Equation (5.6), that is, $\phi(G_n)\phi(B_{n+1} \setminus v)$ gives all the required ϕ-summands where $\phi(B_{n+1} \setminus v)$ is there. The second term in right hand side, that is, $\phi(G_n \setminus v)\phi(B_{n+1})$ gives all the required ϕ-summands where $\phi(B_{n+1})$ is there. Thus, all the required ϕ-summands of G_{n+1} are generated when q cut-vertices are deleted from G_{n+1} and cut-vertex v is not included in these q cut-vertices.

   The cut-vertex v is in deleted cut-vertices: As v get removed, in this case, in G_{n+1} all the ϕ-summands must have term, $\phi(B_{n+1} \setminus v)$. The first term in right hand side of Equation (5.6) that is, $\phi(G_n)\phi(B_{n+1} \setminus v)$ gives the ϕ-summands where $\phi(B_{n+1} \setminus v)$ is there. Note that, in G_{n+1}, in case of (a), cut-vertex v has cut-index equal to 2, in case of (b), cut-vertex v has cut-index equal to d_v+1. Hence, in both the cases there must be a extra ϕ-summand multiplied by $(\lambda-\alpha)\phi(B_{n+1} \setminus v)$ corresponding to each ϕ-summand in G_n when these q cut-vertices being removed. The third term in right hand side gives these extra summands. Therefore, all the ϕ-partitions of G_{n+1} can be obtained when q cut-vertices are deleted from G_{n+1} and v is included in these q cut-vertices.

Hence, the statement is true for G_{n+1}. This proves the theorem.

Corollary 5.4. *Let G_k be a digraph with m cut-vertices and k blocks $B_1, B_2, \ldots, B_k$. Let $G_k^q = \Big\{G_{k1}^q, G_{k2}^q, \ldots, G_{k\binom{m}{q}}^q\Big\}$ be set of all induced subdigraphs of G_k after removing any q cut-vertices. Also, let $d_{i1}, d_{i2}, \ldots, d_{iq}$ be cut-indices and $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iq}$ be the weights of the loops of these removed q cut-vertices to form G_{ki}^q. Let S_{ki}^q denote the summation of all the ψ-summands of all possible $\mathscr{B}$-partition of G_{ki}^q using* *cut-subdigraphs. Then, the permanent polynomial of G_k is given by,*

$$\psi(G_k)=\sum_{q=0}^{m}\varphi(G_k^q),\ \textit{where}\ \varphi(G_k^q)=\sum_{i=1}^{\binom{m}{q}}\left(L_i^q S_{ki}^q\right)\ \textit{and}\ L_i^q=\prod_{t=1}^{q}(\lambda-\alpha_{it})(d_{it}-1),\quad L_1^0=1.$$

Proof: Proof is similar to Theorem 5.1.

Corollary 5.5. *Let G_k be a digraph having only one cut-vertex v and k blocks $B_1,B_2,\dots,B_k$. Let v has a loop of weight α. Then,*

$$\phi(G_k)=\sum_{i=1}^{k}\Big(\phi(B_i)\prod_{j=1,j\neq i}^{k}\phi(B_j\setminus v)\Big)+(k-1)(\lambda-\alpha)\prod_{i=1}^{k}\phi(B_i\setminus v).$$

Proof: The proof follows from Theorem 5.1. On the right hand side, the first term corresponds to the ϕ-summands of $\mathscr{B}$-partition when cut-vertex v is not removed. The second term corresponds to the ϕ-summands of $\mathscr{B}$-partition when cut-vertex v is removed. The cut-index of v is k. Hence, $k-1$ is multiplied to the weight $(\lambda-\alpha)$.

Example 5.4. *We have constructed the digraphs of matrices M_1 and M_2 in Figure 5.1. Now, we calculate the characteristic polynomial of these matrices in terms of the characteristic polynomial of the induced subdigraphs in the blocks in it, applying Theorem 5.1. We express the characteristic polynomial in terms of the ϕ-summands. Let X be a set of indices. The principal submatrix whose rows and column are indexed with elements in X is denoted by $[X]$.*

1. *First we calculate the characteristic polynomial of M_1. Parts of the characteristic polynomial are listed below in terms of the ϕ-summands of the digraph $G(M_1)$.*

   a) *Without removing any cut-vertex we get the following part of $\phi(M_1)$.*

   $$\phi[1,2,3]\phi[4,5,6]\phi[7]+\phi[1,2,3]\phi[4,5]\phi[6,7]+\phi[1,3]\phi[2,4,5,6]\phi[7]$$
   $$+\phi[1,3]\phi[2,4,5]\phi[6,7].$$

   b) *Recall that, the loop-weight of v_2 is $(5-\lambda)$ and its cut-index is 2. Removing the cut-vertex v_2 we get the following part,*

   $$(\lambda-5)\big(\phi[1,3]\phi[4,5,6]\phi[7]+\phi[1,3]\phi[4,5]\phi[6,7]\big).\tag{5.7}$$

   c) *The loop-weight of v_6 is $(-4-\lambda)$ and its cut-index is 2. Removing the cut-vertex v_6 we get the following part,*

   $$(\lambda+4)\big(\phi[1,2,3]\phi[4,5]\phi[7]+\phi[1,3]\phi[2,4,5]\phi[7]\big).\tag{5.8}$$

   d) *Removing the cut-vertices v_2 and v_6 we get the following part,*

   $$(\lambda-5)(\lambda+4)\phi[1,3]\phi[4,5]\phi[7].\tag{5.9}$$

   Adding (a),(b),(c) and (d) give $\phi(M_1)$.

   Now, we calculate the characteristic polynomial of M_2. Parts of the characteristic polynomial are listed below in terms of the ϕ-summands of the digraph $G(M_2)$.

a) *Without removing any cut-vertex we get the following part of* $\phi(M_2)$.

$$\phi[1,2,3]\phi[4,5,6]\phi[7]\phi[8]+\phi[1,2,3]\phi[4,5]\phi[6,7]\phi[8]+\phi[1,3]\phi[2,4,5,6]\phi[7]\phi[8]+\phi[1,3]\phi[2,4,5]\phi[6,7]\phi[$$
$$+\phi[1,2,3]\phi[4,5]\phi[7]\phi[6,8]+\phi[1,3]\phi[2,4,5]\phi[7]\phi[4,8$$

b) *The loop-weight of* v_2 *is* $(5-\lambda)$ *and its cut-index is 2. Removing the cut-vertex* v_2 *we get the following part,*

$$(\lambda-5)\phi[1,3]\phi[4,5,6]\phi[7]\phi[8]+(\lambda-5)\phi[1,3]\phi[4,5]\phi[6,7]\phi[8]$$
$$+(\lambda-5)\phi[1,3]\phi[4,5]\phi[7]\phi[6,8].$$

c) *The loop-weight of* v_6 *is* $(-4-\lambda)$ *and its cut-index is 3. Removing the cut-vertex* v_6 *we get the following part,*

$$2(\lambda+4)\phi[1,2,3]\phi[4,5]\phi[7]\phi[8]+2(\lambda+4)\phi[1,3]\phi[2,4,5]\phi[7]\phi[8]. \tag{5.10}$$

d) *Removing the cut-vertices* v_2 *and* v_6 *we get the following part,*

$$2(\lambda-5)(\lambda+4)\phi[1,3]\phi[4,5]\phi[7]\phi[8]. \tag{5.11}$$

Adding (a),(b),(c) and (d) give $\phi(M_2)$.

5.4 THE DETERMINANT OF DIGRAPHS

Note that, the determinant of a matrix can be calculated from its characteristic polynomial by setting $\lambda=0$. Thus, Theorem 5.1 is applicable for calculating determinant of a matrix corresponding to the digraph G_k by setting $\lambda=0$ and all the ϕ-summands replaced by the det-summands [see Definition 5.2].

Corollary 5.6. *Let* G_k *be a digraph having no loops on its cut-vertices, then the determinant of* G_k *is given by sum of the* det-*summands of all possible* $\mathscr{B}$*-partition of* G_k.

Proof: As all the cut-vertices has no loops, all the det-summands of resulting induced diagraph after removing any cut-vertices get multiplied by zero. Thus, only those det-summands contribute which are corresponding to $q=0$. Hence, the result follows from Theorem 5.1.

A digraph G whose determinant is zero is called a singular digraph. The next corollary provides a number of singular simple graphs. For the definition of the complete graph, cycle graph, tree and forest we refer [West et al., 2001; Harary et al., 1969].

Corollary 5.7. *A simple graph* G *is singular, if any of these conditions holds:*

1. *There is a pendant block* C_n *with cut-vertex* v *of* G. *Here,* C_n *is a cyclic graph with* $n=4r$, r *is a positive integer.*

2. *There are two pendant blocks* C_n *and* C_m *sharing a cut-vertex* v *of* G. *Here,* C_n *and* C_m *are cycle graphs, where* n *and* m *are even positive integers.*

3. *There is a singular tree with even number of vertices, containing a cut-vertex* v *of* G.

4. *There are two trees with* n_1 *and* n_2 *vertices which share a common cut-vertex* v *of* G. *Here, both* n_1,*and* n_2 *are either even or both of them odd positive integers.*

Proof: Below, we prove all these conditions separately.

1. There are two types of det-summands of G: one consists of determinant of C_n and another consists of determinant of $C_n \setminus v$, that is a path of odd length. Now, determinants of C_n and $C_n \setminus v$ are zero [Germina and Shahul Hameed, 2010]. Hence, the first condition follows.

2. In the det-summands of G, each term will either have determinants of C_n and $C_m \setminus v$ or determinants of $C_n \setminus v$ and C_m. Note that, $C_m \setminus v$ and $C_n \setminus v$ are paths of odd length which are singular [Germina and Shahul Hameed, 2010]. Hence, result follows.

3. Applying the Lemma 6.4, we expand the determinant of G, with respect to the cut-vertex v. In this expression, every term has either determinant of tree or determinant of a forest with an odd number of vertices. A forest with an odd number of vertices is singular [Bapat, 2010]. Hence, the result follows.

4. In the determinant expansion of G with respect to cut-vertex v, each term will have at least one term having the determinant of forest having an odd number of vertices. Hence, the result follows.

5.4.1 The determinant of the Block Graph

Complete graph on n vertices is denoted by K_n and its determinant is $(-1)^{n-1}(n-1)$. For a simple graph G, when each of its blocks is a complete graph then G is called block graph [Bapat and Roy, 2014]. The determinant of a block graph can be calculated from the theorem below, quoted from [Bapat and Roy, 2014]. Here, we present a simplified form of this theorem using Corollary 5.6.

Theorem 5.2. *Let G_k be a simple block graph with n vertices. Let $B_1, B_2, \ldots, B_k$ be its blocks of size $b_1, b_2, \ldots, b_k$, respectively. Let A be the adjacency matrix of G. Then*

$$\det(A) = (-1)^{n-k} \sum \prod_{i=1}^{k} (\alpha_i - 1),$$

where the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;

2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denote the subgraph of G induced by the blocks B_i, $i \in S$.

Proof: From Corollary 5.6 determinant of a block graph is equal to

$$\sum \prod_{i=1}^{k} \det(\hat{B}_i),$$

where, $\hat{B}_i$ is subgraph of B_i and summation is over all $\mathscr{B}$-partition of G_k.

Contribution of the det-summand of a $\mathscr{B}$-partition with induced subgraphs $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ of sizes $\beta_1, \beta_2, \ldots, \beta_k$, respectively, is the following,

$$\prod_{i=1}^{k} (-1)^{\beta_i - 1} (\beta_i - 1).$$

As $(\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k)$ are vertex disjoint induced subgraphs which partition G_k, $\sum_{i=1}^{k} \beta_i = n$. Thus, contribution of the above tuple can be written as,

$$(-1)^{n-k} \prod_{i=1}^{k} (\beta_i - 1).$$

Also, for any nonempty set $S \subset \{1, 2, \ldots, k\}$,

$$\sum_{i \in S} |V(\hat{B}_i)| = \sum_{i \in S} \beta_i \leq |V(G_S)|,$$

where G_S denotes the subgraph of G induced by the blocks B_i, $i \in S$. Thus, k-tuples $(\beta_1, \beta_2, \ldots, \beta_k)$ resulted from $\mathscr{B}$-partitions of G_k satisfy both the conditions of the theorem.

Conversely, consider a k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying both the condition of theorem. We will prove by induction that each such k-tuple corresponds to a unique $\mathscr{B}$-partition of G_k.

If G_1 has only one block B_1 of size b_1, then G_1 is a complete graph, K_{b_1}. The only possible choice for 1-tuple is $\alpha_1 = b_1$. Clearly, α_1 corresponds to a $\mathscr{B}$-partition which consists of K_{b_1} only. Let G_2 have block B_1 and B_2 of sizes b_1 and b_2, respectively. The possible 2-tuples are $(\alpha_1 = b_1,\ \alpha_2 = b_2 - 1)$ and $(\alpha_1 = b_1 - 1,\ \alpha_2 = b_2)$. Therefore, both the 2-tuple induce possible two $\mathscr{B}$-partitions in G_2. One $\mathscr{B}$-partition consists of induced subgraphs K_{b_1}, K_{b_2-1}. Another $\mathscr{B}$-partition consists of induced subgraphs K_{b_1-1}, K_{b_2}.

Now we discuss the proof for G_3, which will clarify the reasoning for the general case. In G_3 block B_3 can occur in two ways.

1. Let B_3 be added to a non cut-vertex of G_2. Without loss of generality, let B_3 is attached to a non-cut-vertex of B_2. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are the following:

   a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1;$

   b) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 2,\ \alpha_3 = b_3;$

   c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1;$

   d) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3.$

   Note that, in this case, each 2-tuple of G_2 give rise to two 3-tuple in G_3 where α_1 is unchanged. Clearly, all the tuples in G_3 can induce its all the possible $\mathscr{B}$-partitions.

2. Let B_3 be added to cut-vertex v of G_2. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are the following:

   a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1;$

   b) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1;$

   c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3.$

   Here, each 2-tuple of G_2 give rise to a 3-tuple of G_3 where α_1, α_2 are unchanged and $\alpha_3 = b_3 - 1$. Other than these there is one more 3-tuple where $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1, \alpha_3 = b_3$. Clearly, all the tuples in G_3 can induce its all the possible $\mathscr{B}$-partitions.

Now let us assume that all possible m-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ in G_m can induce all possible $\mathscr{B}$-partitions in it. We need to prove that all possible $(m+1)$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1})$ in G_{m+1} can induce its all possible $\mathscr{B}$-partitions in it. In G_{m+1} block B_{m+1} can occur in two ways.

1. Let B_{m+1} be added to non cut-vertex of G_m. Each m-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ of G_k give rise to two $(m+1)$-tuple of G_{m+1} where, $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ are unchanged. In one such tuple α_m is also unchanged and $\alpha_{m+1} = b_{m+1} - 1$. In other tuple, α_m is one less than the value it had earlier and $\alpha_{m+1} = b_{m+1}$. Thus, $(m+1)$-tuples in G_{m+1} can induce its all the $\mathscr{B}$-partitions in G_{k+1}.

2. Let B_{k+1} be added to a cut-vertex v of G_k. Each k-tuple of G_k give rise to one $(k+1)$-tuple of G_{k+1} where $\alpha_{k+1} = b_{k+1} - 1$. Beside these there are also $(k+1)$-tuples where $\alpha_{k+1} = b_{k+1}$, along with k-tuples of $(G_k \setminus v)$. Clearly, all the tuples in G_{k+1} can induce its all the $\mathscr{B}$-partitions.

Hence, there is one to one correspondence between partitions and k-tuples.

5.5 CONCLUSION

In this chapter we exploited blocks in digraph corresponding a square matrix to find characteristic and permanent polynomials. First we derived an recursive expression for these polynomials with respect to a pendant block. On solving the recursive expression we find the emergence of $\mathscr{B}$-partitions in digraphs. The $\mathscr{B}$-partitions are handy tool to calculate the determinant and permanent of matrices.