6

$\mathscr{B}$-partitions, its Application to Determinant and Permanent of Graphs

In the previous chapter, we have introduced the $\mathscr{B}$-partitions. We have seen that determinant (permanent) of any graph without loops can be written as the summation of det-summands (per-summands) corresponding to the $\mathscr{B}$-partitions. In this chapter will carry the study forward, and find out determinant and permanent of various signed, directed graphs, in particular, unicyclic graphs, block graph with negative cliques, mix complete graph.

6.1 BASIC TERMINOLOGY

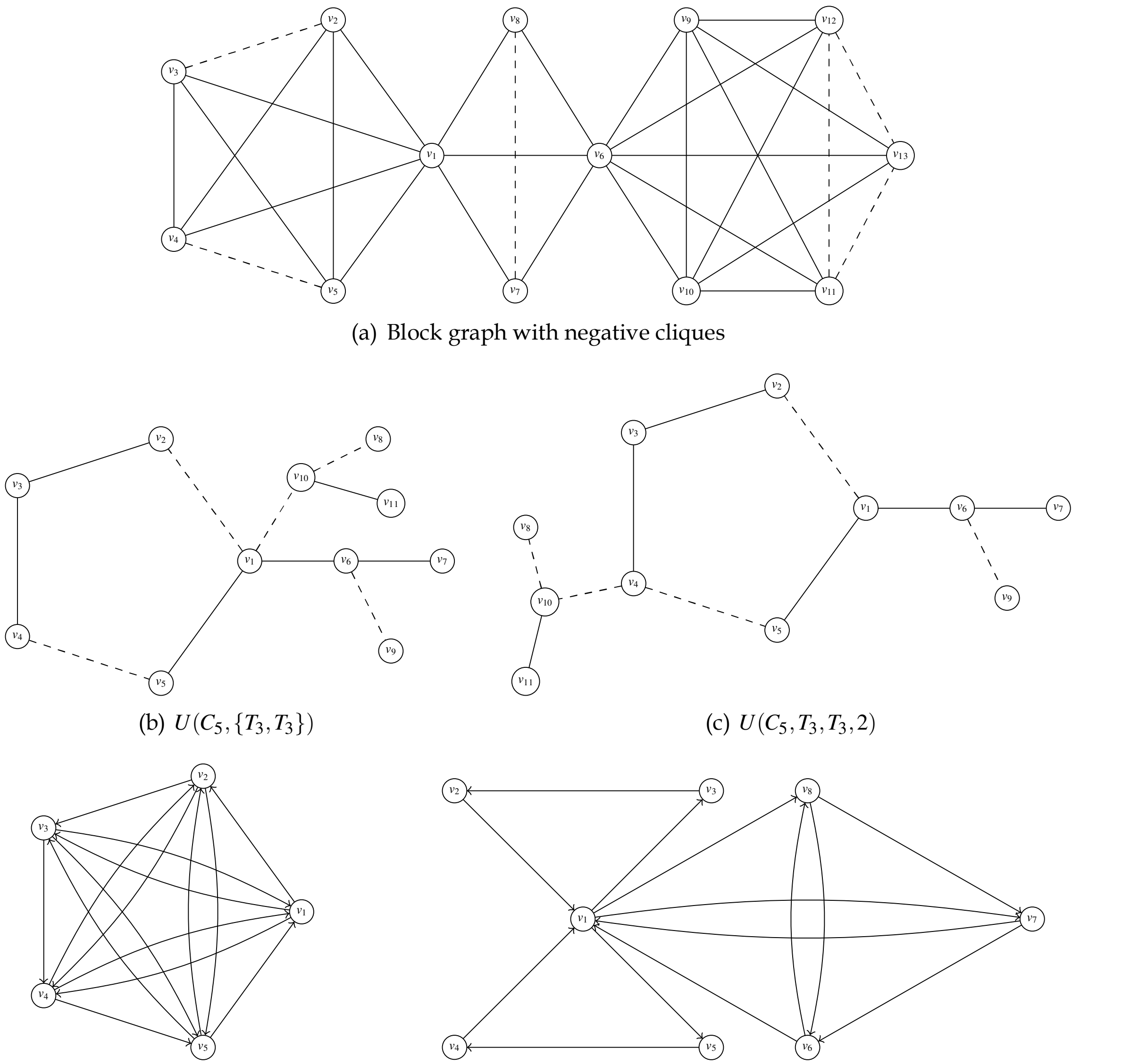
A signed unicyclic graph is a connected signed graph in which number of edges equals the number of vertices. Thus, a signed unicyclic graph is either a cycle or a cycle with trees attached to the vertices of the cycle. If the cycle is balanced then the signed unicyclic graph is balanced otherwise unbalanced. Let T_m denotes a signed tree graph having m vertices. Then, $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})$ denotes a signed unicyclic graph having a signed cycle C_n and k signed trees $T_{m_1}, T_{m_2}, \ldots, T_{m_k}$ such that the root of each $T_{m_i}, i = 1, 2, \ldots, k$ is linked to a fix vertex of C_n. An example of a balanced $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})$ is given in Figure 6.1(b). By, $U(C_n, T_{m_1}, T_{m_2}, l)$ we denote a unicyclic graph having a signed cycle C_n and roots of trees T_{m_1}, T_{m_2} are attached to two vertices v_1 and v_2 of C_n, respectively, at a distance l. An example of a balanced $U(C_n, T_{m_1}, T_{m_2}, l)$ is given in Figure 6.1(c).

A directed cycle dC_n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{(v_i, v_{i+1})\}$ for $i = 1, 2, \ldots, n-1$, and $(v_n, v_1) \in E$. A graph whose all the blocks are directed cycles is called cactoid graph. Adding all the possible arcs (directed edges) between any non adjacent vertices of the cycle dC_n $(n > 3)$ we get a mixed complete graph mK_n, see Figure 6.1(d) [Zhou, 2017]. A mix star block graph G is a graph in which complete mixed graph are connected by one cut vertex. An example of mix star block graph is shown in Figure 6.1(e).

Determinant of K_n is equal to $(-1)^{n-1}(n-1)$ and permanent of K_n is given by

$$\text{per}(K_n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

The rest of the chapter is organized as follows: in Section 6.2, we give some preliminary results on permanent and determinant of weighted signed graph, and in particular, signed block graph. In Section 6.3, we show that how the $\mathscr{B}$-partitions are used to calculate determinant and permanent of block graphs. In Subsection 6.3.1, we calculate the determinant of a block graph having negative vertex disjoint cliques. In the Section 6.4, we find the determinant and permanent of signed unicyclic graphs. In Section 6.5, first, we find eigenvalues of the mixed complete graph and negative mixed complete graph. Thus, we give their determinant expressions. Then we calculate the determinant of mixed star block graph as well as the determinant of negative mixed star block graph.



(a) Block graph with negative cliques

(b) $U(C_5, \{T_3, T_3\})$

(c) $U(C_5, T_3, T_3, 2)$

(d) Mix complete graph

(e) Mix star block graph

Figure 6.1 : Examples. (a) Block graph with negative cliques. (b) $U(C_5, \{T_3, T_3\})$. (c) $U(C_5, T_3, T_3, 2)$. (d) Mix complete graph. (e) Mix star block graph.

6.2 PRELIMINARY RESULTS

We give some preliminary results on determinant and permanent of signed graphs depending upon whether they are balanced or not.

Theorem 6.1. *In a weighted signed block graph G if all the triangles are balanced then, G and $|G|$ have the same determinant.*

Proof: From the definition, each block of G is a complete graph. As all the triangles are balanced, every block is a balanced graph [Harary et al., 1953]. Which implies all the cycles in all the blocks of G are balanced. There can not be any common cycle between any two blocks thus, all the cycles of G are balanced hence, G is balanced. As G and $|G|$ have same eigenvalues, hence G and $|G|$ have the same determinant.

Theorem 6.2. *If a weighted signed graph G is balanced then, G and $|G|$ have same permanent.*

Proof: From [Harary et al., 1953], a balanced graph can be partitioned into two vertex sets such that all the edges between vertices of the same set are positive while all the edges between vertices of different sets are negative. Let X, Y are two such sets for balanced graph G. Let S be the diagonal matrix, whose diagonal elements corresponding to the vertices in X are 1 while elements corresponding to the vertices in Y are -1. Then, $|A| = SAS$. Hence, per($|A|$)=per(A)$(\pm 1)^2$=per(A).

Theorem 6.3. *Let G be a weighted signed block graph; if all the triangles in G are balanced then, G and $|G|$ have same permanent.*

Proof: From the proof of Theorem 6.1, if all the triangles in signed block graph G are balanced then G is balanced. Now, theorem directly follows from Theorem 6.2.

6.3 $\mathscr{B}$-PARTITIONS, DETERMINANT AND PERMANENT OF BLOCK GRAPHS

We first state a theorem for determinant of simple block graphs [Bapat and Roy, 2014]. We see that, in theorem, conditions on k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ can induce $\mathscr{B}$-partitions and vice versa. Then, we can write the determinant and the permanent of a matrix using these conditions. The theorem is as follows

Theorem 6.4. *[Bapat and Roy, 2014] Let G be a block graph with n vertices and having all the edges of weight 1. Let $B_1, B_2, \ldots, B_k$ be its blocks. Let A be the adjacency matrix of G. Then*

$$\det(A) = (-1)^{n-k} \sum \prod_{i=1}^{k} (\alpha_i - 1), \tag{6.1}$$

where, the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of non negative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;
2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denote the subgraph of G induced by the blocks B_i, $i \in S$.

Note that, with the same conditions on k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ the Equation 6.1 of Theorem 6.4 can be written as

$$\det(A) = \sum \prod_{i=1}^{k} \det(K_{\alpha_i}) \tag{6.2}$$

assuming that $\det(K_0) = 1$.

Let wG be a weighted digraph having no loops on cut-vertices. In [Singh and Bapat, 2017b](corollary 5.1) a combinatorial expression for determinant and permanent of wG given in terms of determinant and permanent of subdigraphs of blocks respectively. Statement for determinant, permanent is as follows.

Lemma 6.1. *Let wG be a weighted digraph having no loops on its cut-vertices. Let $B_1, B_2, \ldots, B_k$ are blocks in it. Then, the determinant, permanent of wG is given by*

$$\sum \prod_{i=1}^{k} \det(\hat{B}_i),\ \sum \prod_{i=1}^{k} \text{per }(\hat{B}_i),$$

respectively, where, if $\hat{B}_i$ is a null graph then $\det(\hat{B}_i) = 1$, $\text{per }(\hat{B}_i) = 1$. And the summation is over all possible k-combinations of induced subgraphs $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ such that for $i, j = 1, 2, \ldots, k$,

1. $\hat{B}_i \subseteq B_i$.
2. $\bigcup_{i=1}^{k} V(\hat{B}_i) = V(wG)$.
3. $V(\hat{B}_i) \bigcap V(\hat{B}_j) = \phi$, *for* $i \neq j$.

Thus, the summation is over all k-combinations $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k$ of induced subgraphs which partition wG. These partitions are called as $\mathscr{B}$-partitions, and corresponding terms $\prod_{i=1}^{k} \det(\hat{B}_i)$, $\prod_{i=1}^{k} \text{per}(\hat{B}_i)$ are called det-summands, per-summands , respectively. We will now prove that each k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ in Theorem 6.4 produces a unique $\mathscr{B}$-partition of any weighted graph and vice versa.

Lemma 6.2. *Let G be a graph with n vertices and k blocks. Let $B_1, B_2, \ldots, B_k$ be its blocks having $b_1, b_2, \ldots, b_k$, number of vertices, respectively. Then, each $\mathscr{B}$-partition produce a unique k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of non negative integers satisfying the following conditions:*

1. $\sum_{i=1}^{k} \alpha_i = n$;
2. *for any nonempty set* $S \subseteq \{1, 2, \ldots, k\}$

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denote the subgraph of G induced by the blocks B_i, $i \in S$.

Proof: From the Lemma 6.1 determinant and permanent of G is equal to

$$\sum \prod_{i=1}^{k} \det(\hat{B}_i),\ \sum \prod_{i=1}^{k} \text{per}(\hat{B}_i),$$

respectively, where, $\hat{B}_i$ is subgraph of B_i and summation is over all $\mathscr{B}$-partition of G.

$\{\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_k\}$ are vertex disjoint induced subgraphs which create a $\mathscr{B}$-partition of G, thus, $\sum_{i=1}^{k} |V(\hat{B}_i)| = n$.

Also, for any nonempty set $S \subset \{1,2,\dots,k\}$,

$$\sum_{i\in S} |V(\hat{B_i})| \leq |V(G_S)|,$$

where, G_S denote the subgraph of G induced by the blocks B_i, $i \in S$. Assume, the number of vertices in a given $\mathscr{B}$-partition $\hat{B_1}, \hat{B_2}, \dots, \hat{B_k}$ be $\alpha_1, \alpha_2, \dots, \alpha_k$. Thus, k-tuples $(\alpha_1, \alpha_2, \dots, \alpha_k)$ resulted from $\mathscr{B}$-partitions of G satisfy both the conditions of theorem.

Conversely, consider a k-tuple $(\alpha_1, \alpha_2, \dots, \alpha_k)$ satisfying both the condition of theorem. We will prove by induction that each such k-tuple corresponds to a unique $\mathscr{B}$-partition of G.

If G has only one block B_1 of order b_1, then the only possible choice for 1-tuple is $\alpha_1 = b_1$. Clearly, α_1 corresponds to a $\mathscr{B}$-partition which consists of B_1 only. Let G has two blocks B_1, and B_2 of order b_1, and b_2, respectively, and a cut-vertex v. The possible 2-tuples are $(\alpha_1 = b_1,\ \alpha_2 = b_2 - 1)$, and $(\alpha_1 = b_1 - 1,\ \alpha_2 = b_2)$. Both the 2-tuple induce possible two $\mathscr{B}$-partitions in G. One $\mathscr{B}$-partition consists of induced subgraphs $B_1, B_2 \setminus v$. Another $\mathscr{B}$-partition consists of induced subgraphs $B_1 \setminus v, B_2$.

Now we discuss the proof for G consisting of three blocks, which will clarify the reasoning for the general case. For the time being let us denote the graph having k blocks by G_k. Let the blocks are $B_1, B_2, \dots, B_k$ of order $b_1, b_2, \dots, b_k$, respectively. Formation of a G_k can be seen as k-step process. At any intermediate i-th step a block B_i is added to G_{i-1} and then B_i becomes a pendant block for G_i. In G_3, block B_3 can occur in two ways.

1. Let B_3 be added to a non cut-vertex of G_2. Without loss of generality, let B_3 get attached to a non-cut-vertex of B_2 in G_2. In resulting G_3, let v_1 be the cut-vertex in B_1, B_2, and v_2 be the cut-vertex in B_2, B_3. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are following:

   a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1;$

   b) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 2,\ \alpha_3 = b_3;$

   c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1;$

   d) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3.$

   Note that, in this case, each 2-tuple of G_2 give rise to two 3-tuple in G_3 where α_1 is unchanged. Clearly, all the tuples in G_3 can induce the following all possible $\mathscr{B}$-partitions.

   a) $B_1,\ B_2 \setminus v_1,\ B_3 \setminus v_2;$

   b) $B_1,\ B_2 \setminus (v_1, v_2),\ B_3;$

   c) $B_1 \setminus v_1,\ B_2,\ B_3 \setminus v_2;$

   d) $B_1 \setminus v_1,\ B_2 \setminus v_2,\ B_3.$

2. Let B_3 be added to cut-vertex v of G_2. Choices for 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ are following:

   a) $\alpha_1 = b_1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3 - 1;$

   b) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2,\ \alpha_3 = b_3 - 1;$

   c) $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1,\ \alpha_3 = b_3.$

Here, each 2-tuple of G_2 give rise to a 3-tuple of G_3 where α_1, α_2 are unchanged and $\alpha_3 = b_3 - 1$. Beside these there is one more 3-tuple where $\alpha_1 = b_1 - 1,\ \alpha_2 = b_2 - 1, \alpha_3 = b_3$. Clearly, all the tuples in G_3 can induce the following all possible $\mathscr{B}$-partitions.

a) $B_1,\ B_2 \setminus v,\ B_3 \setminus v$;

b) $B_1 \setminus v,\ B_2,\ B_3 \setminus v$;

c) $B_1 \setminus v,\ B_2 \setminus v,\ B_3$.

Now, let us assume that all possible k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ in G_k can induce all possible $\mathscr{B}$-partitions in it. We need to prove that all possible $(k+1)$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1})$ in G_{k+1} can induce its all possible $\mathscr{B}$-partitions in it. In G_{k+1} block B_{k+1} can occur in two ways.

1. Let B_{k+1} be added to non cut-vertex of G_k. Each k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of G_k give rise to two $(k+1)$-tuple of G_{k+1} where, $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ are unchanged. In one such tuple α_k is also unchanged and $\alpha_{k+1} = b_{k+1} - 1$. In other tuple α_k is one less than the value it had earlier and $\alpha_{k+1} = b_{k+1}$. Thus, $(k+1)$-tuples in G_{k+1} can induce all its $\mathscr{B}$-partitions in G_{k+1}.

2. Let B_{k+1} be added to a cut-vertex v of G_k. Each k-tuple of G_k give rise to one $(k+1)$-tuple of G_{k+1} where $\alpha_{k+1} = b_{k+1} - 1$. Beside these there are also $(k+1)$-tuples where $\alpha_{k+1} = b_{k+1}$, along with k-tuples of $(G_k \setminus v)$. Clearly, all the tuples in G_{k+1} can induce its $\mathscr{B}$-partitions.

Hence, there is one to one correspondence between $\mathscr{B}$- partitions and the k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_3)$.

Now we give a formula for the permanent of balanced signed block graphs.

Theorem 6.5. *Let G be a balanced signed block graph with n vertices and having all the edges of weight 1. Let $B_1, B_2, \ldots, B_k$ be its blocks. Let A be the adjacency matrix of G. Then,*

$$\operatorname{per}(A) = \sum \prod_{i=1}^{k} \alpha_i! \sum_{j=0}^{\alpha_i} \frac{(-1)^j}{j!}, \tag{6.3}$$

where, the summation is over all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of non negative integers satisfying the following conditions:

1. $\sum_{i=1}^{k} \alpha_i = n$;

2. *for any nonempty set $S \subseteq \{1, 2, \ldots, k\}$*

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S denote the subgraph of G induced by the blocks B_i, $i \in S$.

Proof: The proof directly follows from Lemma 6.1, 6.2 and the fact that

$$\operatorname{per}(K_{\alpha_i}) = \alpha_i! \sum_{j=0}^{\alpha_i} \frac{(-1)^j}{j!}.$$

6.3.1 Block graph with negative cliques.

First, we give the determinant of a complete graph with negative cliques, $K_n^{m,r}$. Subsequently, the determinant of block graph with negative cliques is given.

Lemma 6.3. *[Singh and Bapat, 2017a](Corollary 3.6) Determinant of $A(K_n^{m,r})$ is given by*

$$(1-2r)^{m-1}(-1)^{n-mr-1}\left(n(1-2r)+2r\Big(1+m(r-1)\Big)-1\right).$$

Theorem 6.6. *Let G be a signed block graph of order n having k blocks $B_1,B_2,\ldots,B_k$. Let all the edges connecting cut-vertices are positive. For $i=1,2,\ldots,k$, let B_i has m_i number of vertex-disjoint negative cliques each of size r_i, such that $0\le m_ir_i\le(n_i-1)$. Then,*

$$\det(G)=(-1)^{n-k}\sum\prod_{i=1}^{k}(1-2r_i)^{m_i-1}(-1)^{-m_ir_i}\left(\alpha_i(1-2r_i)+2r_i\Big(1+m_i(r_i-1)\Big)-1\right). \tag{6.4}$$

where, the summation is over all k-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_k)$ of non negative integers satisfying the following conditions:

1. $\sum_{i=1}^{k}\alpha_i=n$;
2. *for any nonempty set $S\subseteq\{1,2,\ldots,k\}$*

$$\sum_{i\in S}\alpha_i\le|V(G_S)|,$$

where G_S denote the subgraph of G induced by the blocks B_i, $i\in S$.

Proof: The result directly follows from Lemma 6.2, 6.1, and 6.3.

6.4 DETERMINANT AND PERMANENT OF SIGNED UNICYCLIC GRAPHS

Let U be a unicyclic graph which contains a signed cycle C_n as a subgraph with vertices $v_1,v_2,\ldots,v_n$. Let the vertex v_i is linked with m_i number of signed trees say $T_1^i,T_2^i,\ldots,T_{m_i}^i$, such that the root vertex of each $T_j^i, j=1,2,\ldots,m_i$ is linked with v_i by an edge. Note that the vertex v_i then becomes a cut-vertex. As trees are acyclic graph, determinant and permanent of any signed tree is equal to the determinant and the permanent of its underlying tree with positive edges. Let $\{T_1^i,T_2^i,\ldots,T_{m_i}^i\}$ denote the subgraph of U induced by the trees $T_j^i, j=1,\ldots,m_i$. Let $U\setminus\{T_1^i,T_2^i,\ldots,T_{m_i}^i\}$ denotes the induced subgraph of U after $\{T_1^i,T_2^i,\ldots,T_{m_i}^i\}$ is removed from U, and $\{T_1^i,T_2^i,\ldots,T_{m_i}^i,v_i\}$ denotes the subgraph of U induced by trees $T_1^i,T_2^i,\ldots,T_{m_i}^i$ and vertex v_i. From [Singh and Bapat, 2017b], Lemma 2.3 and Corollary 2.4, can be re-written for determinant and permanent, respectively for graphs with no loop on cut-vertices.

Lemma 6.4. *Let G be a digraph with at least one cut-vertex. Let H be a non empty subdigraph of G having cut-vertex v, such that $H\setminus v$ is union of connected components. The determinant of G,*

$$\det(G)=\det(H)\times\det(G\setminus H)+\det(H\setminus v)\times\det\Big(G\setminus(H\setminus v)\Big). \tag{6.5}$$

Corollary 6.1. *Let G be a digraph with at least one cut-vertex. Let H be a non empty subdigraph of G having cut-vertex v, such that $H\setminus v$ is union of connected components. The permanent of G,*

$$\mathrm{per}(G)=\mathrm{per}(H)\times\mathrm{per}(G\setminus H)+\mathrm{per}(H\setminus v)\times\mathrm{per}\Big(G\setminus(H\setminus v)\Big). \tag{6.6}$$

Applying Lemma 6.4 on U at v_i we get

$$\begin{aligned}\det(U) &= \det\Big(U\setminus\{T_1^i,T_2^i,\dots,T_{m_i}^i\}\Big)\det\Big(\{T_1^i,T_2^i,\dots,T_{m_i}^i\}\Big)\\ &\quad+\det\Big(U\setminus\{T_1^i,T_2^i,\dots,T_{m_i}^i,v_i\}\Big)\det\Big(\{T_1^i,T_2^i,\dots,T_{m_i}^i,v_i\}\Big).\end{aligned} \tag{6.7}$$

Applying Corollary 6.1 on U at v_i we get

$$\begin{aligned}\mathrm{per}(U) &= \mathrm{per}\Big(U\setminus\{T_1^i,T_2^i,\dots,T_{m_i}^i\}\Big)\mathrm{per}\Big(\{T_1^i,T_2^i,\dots,T_{m_i}^i\}\Big)\\ &\quad+\mathrm{per}\Big(U\setminus\{T_1^i,T_2^i,\dots,T_{m_i}^i,v_i\}\Big)\mathrm{per}\Big(\{T_1^i,T_2^i,\dots,T_{m_i}^i,v_i\}\Big).\end{aligned} \tag{6.8}$$

Then we have the following theorems.

Theorem 6.7. *Consider a unicyclic signed graph $U(C_n,T_m)$ where a signed tree T_m is linked with the signed cycle C_n by an edge between the root vertex of T_m and a vertex v of C_n. Then,*

$$\det\Big(U(C_n,T_m)\Big)=\begin{cases}0, & \text{if } n \text{ is even and } T_m \text{ has no perfect matching}\\ (-1)^{\frac{m}{2}}\Big(-2\delta+2(-1)^{\frac{n}{2}}\Big), & \text{if } n \text{ is even and } T_m \text{ has a perfect matching}\\ (-1)^{\frac{m+n}{2}}, & \text{if } n \text{ is odd and } \{T_m,v\} \text{ has a perfect matching}\\ 2\delta(-1)^{\frac{m}{2}}, & \text{if } n \text{ is odd and } T_m \text{ has a perfect matching}\end{cases}$$

where $\delta=1$ if C_n is balanced, otherwise $\delta=-1$.

Proof: Let the tree T_m be attached to C_n via an edge between the vertices u_1 of T_m and v of C_n. Applying Lemma 6.4 the determinant of $U(C_n,T_m)$ can be written as

$$\begin{aligned}\det\Big(U(C_n,T_m)\Big) &= \det(C_n)\times\det(T_m)+\det(C_n\setminus v)\times\det(\{T_m,v\})\\ &= \det(C_n)\times\det(T_m)+\det(P_{n-1})\times\det(\{T_m,v\}),\end{aligned} \tag{6.9}$$

where, $C_n\setminus v$ is the subgraph in which vertex v is removed from C_n and hence it becomes P_{n-1}. Also, a signed tree without a perfect matching has determinant zero. From [Singh and Bapat, 2017a], Corollary 2.3

Determinant of signed cycle C_n, having weight $\delta\in\{-1,1\}$ is given by

$$\det(C_n)=\begin{cases}2-2\delta & \text{if } n \text{ is even and even multiple of 2}\\ -2-2\delta & \text{if } n \text{ is even and odd multiple of 2}\\ 2\delta & \text{if } n \text{ is odd}\end{cases}$$

Now we consider the following cases.

Case I n is even and T_m has no perfect matching: As in this case $\det(T_m)=0,\ \det(P_{n-1})=0$. From Equation 6.9 $\det(U(C_n,T_m))=0$.

Case II n is even and T_m has a perfect matching: Consider $n = 2k, m = 2k'$, where $k \geq 2$ and $k' \geq 1$ are positive integers. As $\det(P_{n-1}) = 0$ from Equation 6.9

$$\det(U(C_n, T_m)) = \det(C_n) \times \det(T_m) = (-2\delta + 2(-1)^k)(-1)^{k'},$$

where, for balanced C_n, $\delta = 1$ and for unbalanced C_n, $\delta = -1$.

Case III n is odd and as m is odd, T_m has no perfect matching: In this case $\det(T_m) = 0$. Thus, from Equation 6.9

$$\det(U(C_n, T_m)) = \det(P_{n-1}) \times \det(\{T_m, v\}).$$

If $\{T_m, v\}$ has no perfect matching then $\det(U(C_n, T_m)) = 0$. Otherwise

$$\det(U(C_n, T_m)) = (-1)^{\frac{n-1}{2}}(-1)^{\frac{m+1}{2}} = (-1)^{\frac{n+m}{2}}.$$

Case IV n is odd and T_m has a perfect matching: In this case $m+1$ is an odd number so, $\det(\{T_m, v\}) = 0$. Thus, from Equation 6.9

$$\det(U(C_n, T_m)) = \det(C_n) \times \det(T_m) = 2\delta(-1)^{\frac{m}{2}}$$

where for balanced C_n, $\delta = 1$ and for unbalanced C_n, $\delta = -1$.

Corollary 6.2. *Consider a unicyclic signed graph $U(C_n, T_m)$ as in Theorem 6.7. Then,*

$$\operatorname{per}\Big(U(C_n, T_m)\Big) = \begin{cases} 0, & \text{if } n \text{ is even and } T_m \text{ has no perfect matching} \\ -2\delta + 2, & \text{if } n \text{ is even and } T_m \text{ has a perfect matching} \\ 1, & \text{if } n \text{ is odd and } \{T_m, v\} \text{ has a perfect matching} \\ 2\delta, & \text{if } n \text{ is odd and } T_m \text{ has a perfect matching} \end{cases}$$

where $\delta = 1$ if C_n is balanced, otherwise $\delta = -1$.

Proof: Using Equation 2.5

$$\operatorname{per}(C_n) = \begin{cases} 2 - 2\delta & \text{if } n \text{ is even} \\ 2\delta & \text{if } n \text{ is odd.} \end{cases}$$

Rest of the steps are similar to Theorem 6.7.

Theorem 6.8. *Let $U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})$ denotes a unicyclic graph having a signed cycle C_n and k signed trees $T_{m_1}, T_{m_2}, \ldots, T_{m_k}$ and root of each $T_{m_i}, i = 1, \ldots, k$ is linked with vertex v of C_n by an edge for all i. Then*

$$\det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})\Big) = \det(C_n)\prod_{i=1}^{k}\det(T_{m_i}) + \det(P_{n-1})\sum_{i=1}^{k}\Big(\det(\{T_{m_i}, v\})\prod_{j=1, j\neq i}^{k}\det(T_{m_j})\Big).$$

Proof: From Equation 6.7 observe that

$$\begin{aligned}\det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\})\Big) = {} & \det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}\Big) \\ & \times \det(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \\ & + \det\Big(U(C_n, \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}\}) \setminus \{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}\Big) \\ & \times \det(\{T_{m_1}, T_{m_2}, \ldots, T_{m_k}, v\}).\end{aligned} \tag{6.10}$$

where, $U(C_n,\{T_{m_1},T_{m_2},\dots,T_{m_k}\})\setminus\{T_{m_1},T_{m_2},\dots,T_{m_k}\}=C_n$. Also,

$$\det\Big(\{T_{m_1},T_{m_2},\dots,T_{m_k}\}\Big)=\prod_{i=1}^{k}\det(T_{m_i})$$

since $\{T_{m_1},T_{m_2},\dots,T_{m_k}\}$ is the induced subgraph of the unicyclic graph having k connected components $T_{m_i}, i=1,\dots,k$. Next, $U(C_n,\{T_{m_1},T_{m_2},\dots,T_{m_k}\})\setminus\{T_{m_1},T_{m_2},\dots,T_{m_k},v\}=P_{n-1}$. The only thing that is left to know is $\det(\{T_{m_1},T_{m_2},\dots,T_{m_k},v\})$. Again applying Lemma 6.4 on $\{T_{m_1},T_{m_2},\dots,T_{m_k},v\}$ at v

$$\det\Big(\{T_{m_1},T_{m_2},\dots,T_{m_k},v\}\Big)=\sum_{i=1}^{k}\Big(\det(\{T_{m_i,v}\})\prod_{j=1,j\neq i}^{k}\det(T_{m_j})\Big).$$

Thus, the desired result follows.

Corollary 6.3. *Let $U(C_n,\{T_{m_1},T_{m_2},\dots,T_{m_k}\})$ denote a unicyclic graph as considered in Theorem 6.8. Then*

$$\begin{aligned}\operatorname{per}(U(C_n,\{T_{m_1},T_{m_2},\dots,T_{m_k}\}))=&\operatorname{per}(C_n)\prod_{i=1}^{k}\operatorname{per}(T_{m_i})\\&+\operatorname{per}(P_{n-1})\sum_{i=1}^{k}\Big(\operatorname{per}(\{T_{m_i},v\})\prod_{j=1,j\neq i}^{k}\operatorname{per}(T_{m_j})\Big).\end{aligned}$$

Theorem 6.9. *Let $U(C_n,T_{m_1},T_{m_2},l)$ denote a signed unicyclic graph having a signed cycle C_n and two trees T_{m_1},T_{m_2} are attached by additional edges to two vertices v_1 and v_2 of C_n respectively at a distance l. Then,*

$$\begin{aligned}\det\Big(U(C_n,T_{m_1},T_{m_2},l)\Big)=&\det\Big(U(C_n,T_{m_2})\Big)\det(T_{m_1})\\&+\det(\{T_{m_1},v_1\})\det(\{T_{m_2},v_{l+1}\})\det(P_{l-1})\det(P_{n-l-1})\\&+\det(\{T_{m_1},v_1\})\det(\{T_{m_2}\})\det(P_{n-1}).\end{aligned}$$

Proof: By Equation 6.7 it follows that

$$\begin{aligned}\det\Big(U(C_n,T_{m_1},T_{m_2},l)\Big)\;=\;&\det\Big(U(C_n,T_{m_1},T_{m_2},l)\setminus\{T_{m_1}\}\Big)\det(T_{m_1})+\\&\det\Big(U(C_n,T_{m_1},T_{m_2},l)\setminus\{T_{m_1},v_1\}\Big)\det(\{T_{m_1},v_1\}).\end{aligned}\tag{6.11}$$

Note that, $\det\Big(U(C_n,T_{m_1},T_{m_2},l)\setminus\{T_{m_1}\}\Big)=\det\Big(U(C_n,T_{m_2})\Big)$ and $\{T_{m_1},v_1\}$ is a tree with m_1+1 vertices. The only thing remains to figure out is $\det\Big(U(C_n,T_{m_1},T_{m_2},l)\setminus\{T_{m_1},v_1\}\Big)$. Let for the time being denote $U(C_n,T_{m_1},T_{m_2},l)$ by U. Applying Lemma 6.4 on $U\setminus\{T_{m_1},v_1\}$ at v_2

$$\begin{aligned}\det\Big(U\setminus\{T_{m_1},v_1\}\Big)=&\det(\{T_{m_2},v_2\})\det\Bigg(\Big(U\setminus\{T_{m_1},v_1\}\Big)\setminus\{T_{m_2},v_2\}\Bigg)\\&+\det(\{T_{m_2}\})\det\Bigg(\Big(U\setminus\{T_{m_1},v_1\}\Big)\setminus\{T_{m_2}\}\Bigg).\end{aligned}$$

Further observe that $\Big(U\setminus\{T_{m_1},v_1\}\Big)\setminus\{T_{m_2},v_2\}$ is a disconnected subgraph with two connected components P_{l-1} and $P_{n-(l+1)}$, and hence

$$\det\Bigg(\Big(U\setminus\{T_{m_1},v_1\}\Big)\setminus\{T_{m_2},v_2\}\Bigg)=\det(P_{l-1})\det(P_{n-l-1}),$$

and $\Big(U \setminus \{T_{m_1}, v_1\}\Big) \setminus \{T_{m_2}\} = P_{n-1}$. Thus the desired result follows.

Corollary 6.4. *Let $U(C_n, T_{m_1}, T_{m_2}, l)$ be a signed unicyclic as considered in Theorem 6.9. Then*

$$\begin{aligned}
\operatorname{per}(U(C_n, T_{m_1}, T_{m_2}, l)) = \operatorname{per}\Big(U(C_n, T_{m_2})\Big)&\operatorname{per}(T_{m_1}) \\
&+ \operatorname{per}(\{T_{m_1}, v_1\})\operatorname{per}(\{T_{m_2}, v_2\})\operatorname{per}(P_{l-1})\operatorname{per}(P_{n-l-1}) \\
&+ \operatorname{per}(\{T_{m_1}, v_1\})\operatorname{per}(\{T_{m_2}\})\operatorname{per}(P_{n-1}).
\end{aligned}$$

6.5 MIXED COMPLETE GRAPH, MIXED STAR BLOCK GRAPH.

The adjacency matrix $A(mK_n)$, of mix complete graph mK_n can be written as:

$$A(mK_n) = J_n I_n - Q_n,$$

where, J_n is all one matrix , I_n is an identity matrix, and Q_n is the full-cycle permutation matrix of order n. Thus, the $(i, i+1)$-element of Q_n is 1, $i = 1, 2, \ldots, n-1$, the $(n, 1)$-element of Q_n is 1, and the remaining elements of Q_n are zero [Bapat, 2010].

The eigenvalues of Q_n are $w^i (0 \le i \le n-1)$, and the corresponding eigenvectors are

$$v_i = [1, w^i, w^{2i}, \ldots, w^{(n-1)i}]^T$$

for $0 \le i \le n-1$, where, w is an n-th primitive root of 1. The eigenvectors are orthogonal to each other, i.e. $v_i^T v_j = 0$ for $0 \le i, j \le n-1$. Note that v_0 is all one column vector. Then the eigenvalues of $A(mK_n)$ are $\lambda_0 = n-2$ and $\lambda_i = -1 - w^i (1 \le i \le n-1)$.

Lemma 6.5.

$$\prod_{i=1}^{n-1}(-1 - w^i) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof: As

$$x^n - 1 = (x-1)\prod_{i=1}^{n-1}(x - w^i),$$

$$\implies \sum_{i=1}^{n} x^{n-i} = \prod_{i=1}^{n-1}(x - w^i).$$

Hence, the result follows.

Theorem 6.10. *Determinant of $A(mK_n)$ is given by*

$$\det\Big(A(mK_n)\Big) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (n-2) & \text{if } n \text{ is odd} \end{cases} \tag{6.12}$$

Proof: As the eigenvalues of $A(mK_n)$ are $\lambda_0 = n-2$ and $\lambda_i = -1 - w^i$ for $(1 \le i \le n-1)$.

$$\det\Big(A(mK_n)\Big) = (n-2)\prod_{i=1}^{n-1}(-1 - w^i).$$

Now, proof directly follows from Lemma 6.5.

6.5.1 Mixed star block graph

A mixed block graph is a strongly connected directed graph whose blocks are mixed complete graphs. A mixed block graph having maximum one cut vertex is called mixed star block graph, see Figure 6.1(e). In other words, a mixed star block graph is obtained from a star cactoid graph after adding all possible directed edges between any two non adjacent vertices in each block. As a star cactoid graph cannot have cycle cover it is evident that it is singular. Let $mK_n \setminus v_i$ denotes a induced subgraph resulting after vertex v_i is removed from mK_n.

Lemma 6.6. *The determinant of $mK_n \setminus v_i (i = 1,2,\dots n)$ is given by*

$$(-1)^n \left(\lfloor \frac{n-2}{2} \rfloor \right).$$

Proof: Without loss of generality let us remove the first vertex v_1 of mK_n. Adjacency matrix of $mK_n \setminus v_1$ can be written as

$$A\left(mK_n \setminus v_1\right) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

In other words, $A\left(mK_n \setminus v_1\right)$ is a square matrix of size $n-1$ whose diagonal and sub-diagonal elements are zero and rest of the elements are 1. Let R_i denotes the i-th row of $A\left(mK_n \setminus v_1\right)$. In order to calculate its determinant let us first make following elementary row operations.

1. $R_i = R_i - R_{i+1}$ for $i = 1,2,\dots(n-2)$.
2. Add all the resulting $n-2$ rows in 1. to $(n-1)$-th row.

These elementary row operations produce following matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \ddots & 0 \\ 0 & -1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Digraph corresponding to above matrix is shown in Figure 6.2. Using cycle covers of digraph we calculate the determinant as follows.

1. n is odd: In this case cycle covers are following. For $i = 1,2,\dots,\frac{n-3}{2}$, in a cycle cover there are directed 2-cycles, each having weight -1, on vertices $\{v_{2j-1}, v_{2j}\}$ $(j = 1,2\dots,i)$, and a directed $(n-1-2i)$-cycle of weight 1 on vertices $\{v_{n-1}, v_{2i+1}, v_{2i+2},\dots,v_{n-1}\}$. Hence,

$$\begin{aligned} \det\left(A\left(mK_n \setminus v_1\right)\right) &= (-1)^{n-1} \sum_{i=1}^{\frac{n-3}{2}} (-1)^{i+1} \times (-1)^i \times 1 \\ &= \frac{3-n}{2}. \end{aligned} \tag{6.13}$$

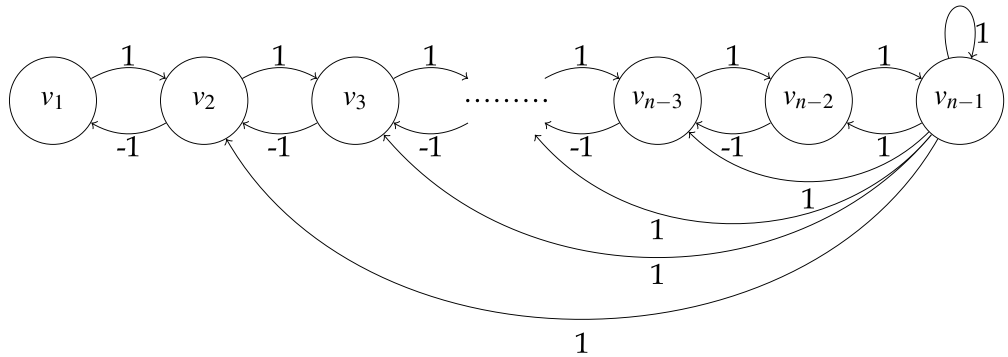


Figure 6.2 : Digraph of matrix $A\left(mK_n \setminus v_1\right)$ after elementary operations.

2. n is even: In this case cycle covers are following. For $i = 1, 2, \ldots, \frac{n-4}{2}$, in a cycle cover there are directed 2-cycles, each having weight -1, on vertices $\{v_{2j-1}, v_{2j}\}$ $(j = 1, 2 \ldots, i)$, and a directed $(n-1-2i)$-cycle of weight 1 on vertices $\{v_{n-1}, v_{2i+1}, v_{2i+2}, \ldots, v_{n-1}\}$. Other than these there is one more cycle cover having loop at vertex v_{n-1}, and $\frac{n-2}{2}$ directed 2-cycles on $\{v_{2i-1}, v_{2i}\}(i = 1, 2, \ldots, \frac{n-2}{2})$ each of weight -1. Hence,

$$
\begin{aligned}
det\left(A\left(mK_n \setminus v_1\right)\right) &= (-1)^{n-1}\left(\sum_{i=1}^{\frac{n-4}{2}}(-1)^{i+1} \times (-1)^i \times 1 + (-1)^{1+\frac{n-2}{2}} \times (-1)^{\frac{n-2}{2}} \times 1\right) \\
&= \frac{n-2}{2}.
\end{aligned}
\tag{6.14}
$$

Combining the Equations (6.13) and (6.14) the result follows.

Theorem 6.11. *Let mG be mixed star block graph having k blocks $B_1, B_2, \ldots, B_k$ of order $n_1, n_2, \ldots, n_k$, respectively, then*

$$\det(mG) = \sum \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} (-1)^{n_j}\left(\lfloor \frac{n_j - 2}{2} \rfloor\right),$$

where summation is over all i such that n_i is odd.

Proof: Let v be the cut-vertex of mG. From Lemma 6.6 and 6.1

$$
\begin{aligned}
\det(mG) &= \sum_{i=1}^{k} \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} \det\left(mK_{n_i} \setminus v\right) \\
&= \sum_{i=1}^{k} \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} (-1)^{n_j}\left(\lfloor \frac{n_j - 2}{2} \rfloor\right).
\end{aligned}
\tag{6.15}
$$

from Lemma 6.5, for even n_i, $\det(mK_{n_i}) = 0$. Hence,

$$\det(mG) = \sum \det(mK_{n_i}) \prod_{j=1, j\neq i}^{k} (-1)^{n_j}\left(\lfloor \frac{n_j - 2}{2} \rfloor\right),$$

where, summation is over all i such that n_i is odd.

6.5.2 Negative mix complete graph

A negative directed cycle dC_n is cycle graph whose each directed edge is negative that is each of its edges have weight -1. A negative mixed complete graph $\overline{m}K_n$ is obtained from a negative directed cycle dC_n of length $n > 3$ by adding all the possible positive arcs between any non-adjacent vertices of the underlying cycle C_n. Adjacency matrix $A(\overline{m}K_n)$ can be written as:

$$A(\overline{m}K_n) = J_nI_n - 2Q_n - Q^{n-1},$$

where, J_n is all one matrix , I_n is an identity matrix, and Q_n is the full-cycle permutation matrix of order n. Then the eigenvalues of $A(\overline{m}K_n)$ are $\lambda_0 = n-4$ and $\lambda_i = -1-2w^i - w^{i(n-1)}(1 \leq i \leq n-1)$, where $w = e^{\frac{2\pi\imath}{n}}$.

Lemma 6.7. *The determinant of $A(\overline{m}K_n)$ is given by*

$$\det\left(A(\overline{m}K_n)\right) = \begin{cases} 2(n-4)\prod_{i=1}^{\frac{(n-2)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right), & \text{if } n \text{ is even} \\ (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right), & \text{if } n \text{ is odd.} \end{cases} \tag{6.16}$$

Proof: For $i = 1,2,\ldots,(n-1), w^i = \cos\left(\frac{2\pi i}{n}\right) + \imath\sin\left(\frac{2\pi i}{n}\right)$, and

$$\begin{aligned} \lambda_i &= -1-2w^i - w^{i(n-1)} \\ &= -1-2w^i - w^{-i} \\ &= -1-3\cos\left(\frac{2\pi i}{n}\right) - \imath\sin\left(\frac{2\pi i}{n}\right). \end{aligned}$$

Now, $3\cos\left(\frac{2\pi(n-i)}{n}\right) - \imath\sin\left(\frac{2\pi(n-i)}{n}\right) = 3\cos\left(\frac{2\pi i}{n}\right) + \imath\sin\left(\frac{2\pi i}{n}\right)$, if n is even then, $\lambda_{n/2} = 2$. Following are the determinant expressions for $A(\overline{m}K_n)$.

1. n is odd:

$$\begin{aligned} \det\left(A(\overline{m}K_n)\right) &= (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(\left(-1-3\cos\left(\frac{2\pi i}{n}\right)\right)^2 + \sin^2\left(\frac{2\pi i}{n}\right)\right) \\ &= (n-4)\prod_{i=1}^{\frac{(n-1)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right). \end{aligned}$$

2. n is even:

$$\det\left(A(\overline{m}K_n)\right) = 2(n-4)\prod_{i=1}^{\frac{(n-2)}{2}}\left(2+8\cos^2\left(\frac{2\pi i}{n}\right)+6\cos\left(\frac{2\pi i}{n}\right)\right).$$

6.5.3 The determinant of negative mixed star block graph

A negative mixed block graph is a strongly connected directed graph whose blocks are negative mixed complete graphs. A negative mixed block graph having maximum one cut vertex is called negative mixed star block graph. Let $\overline{m}K_n \setminus v_i$ denotes a induced subgraph resulting after vertex v_i is removed from $\overline{m}K_n$.

Lemma 6.8. *The determinant of* $\overline{m}K_n \setminus v_i (i = 1, 2, \ldots n)$ *is given by*

$$\left(1 + \frac{1}{g_{n-1}}\Big(\sum_{i\leq j} 2^{j-i} g_{i-1} h_{j+1} + \sum_{j<i} g_{j-1} h_{i+1}\Big)\right) g_{n-1},$$

where,

$$g_i = r_1 s_1^i + r_2 s_2^i, \text{ for } i = 2, 3 \ldots, n-1,$$

$$h_i = r_{h1} s_1^{n-1-i} + r_{h2} s_2^{n-1-i}, \text{ for } i = n-2, \ldots, 1,$$

$$r_1 = \frac{1}{2} + \frac{\iota}{2\sqrt{7}}, \quad r_2 = \frac{1}{2} - \frac{\iota}{2\sqrt{7}}, \quad r_{h1} = \frac{-1}{2} + \frac{3\iota}{2\sqrt{(7)}}, \quad r_{h2} = \frac{-1}{2} - \frac{3\iota}{2\sqrt{(7)}}, \text{ and}$$

$$s_1 = \frac{-1}{2} + \frac{\iota\sqrt{7}}{2}, \quad s_2 = \frac{-1}{2} - \frac{\iota\sqrt{7}}{2}.$$

Proof: Without loss of generality let us remove the first vertex v_1 of $\overline{m}K_n$. Adjacency matrix of $\overline{m}K_n \setminus v_1$ can be written as

$$A\left(\overline{m}K_n \setminus v_1\right) = \begin{bmatrix} 0 & -1 & 1 & \ldots & 1 \\ 0 & 0 & -1 & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

Let $m = n-1$. We can write, $A\left(\overline{m}K_n \setminus v_1\right) = uu^T + T$, where, u is a $m \times 1$ column vector having all entries equal to 1. And, T is the tridiagonal matrix of order m, having diagonal, subdiagonal entries equal to -1 and superdiagonal entries equal to -2. From matrix determinant lemma [Ding and Zhou, 2007]

$$\det(T + uu^T) = (1 + u^T T^{-1} u)\det(T).$$

From [Zhou, 2017], we need to solve some recursive expressions, in order to calculate the determinant and inverse of T. We solve these recursive expressions using roots of their characteristic equations. For the determinant of A, recursive expression is

$$f_m = -f_{m-1} - 2f_{m-2}, \quad f_0 = 1,\ f_{-1} = 0.$$

Roots of the resulting characteristic equation $x^2 + x + 2 = 0$, are

$$s_1 = \frac{-1}{2} + \frac{\iota\sqrt{7}}{2}, \quad s_2 = \frac{-1}{2} - \frac{\iota\sqrt{7}}{2}.$$

Hence,

$$\det(T) = f_m = r_1 s_1^m + r_2 s_2^m,$$

where, using initial conditions

$$r_1 = \frac{1}{2} + \frac{\iota}{2\sqrt{7}}, \quad r_2 = \frac{1}{2} - \frac{\iota}{2\sqrt{7}}.$$

Now, to calculate T^{-1} we need to solve following recursive expressions

$$g_i = -g_{i-1} - 2g_{i-1}, \text{ for } i = 2,3\ldots,m, \;\; g_0 = 1, \; g_1 = -1$$

$$h_i = -h_{i+1} - 2h_{i+2}, \text{ for } i = m-1,\ldots,1, \;\; h_{m+1} = 1, \; h_m = -1.$$

Similar to f_n, solving these recursive expressions we get

$$g_i = r_1 s_1^i + r_2 s_2^i, \;\; \text{for } \; i = 2,3\ldots,n,$$

and,

$$h_i = r_{h1} s_1^{m-i} + r_{h2} s_2^{m-i}, \;\; \text{for } \; i = m-1,\ldots,1,$$

where,

$$r_{h1} = \frac{-1}{2} + \frac{3\imath}{2\sqrt{(7)}}, \quad r_{h2} = \frac{-1}{2} - \frac{3\imath}{2\sqrt{(7)}}.$$

Entries of T^{-1} are clearly given by g_i, h_i [Ding and Zhou, 2007].

$$T_{ij}^{-1} = \begin{cases} \frac{2^{j-i} g_{i-1} h_{j+1}}{g_m} & \text{if } i \leq j \\ \frac{g_{j-1} h_{i+1}}{g_m} & \text{if } j < i \end{cases}.$$

As, $u^T T^{-1} u$ equals to sum of all the entries of T^{-1}. Thus,

$$u^T T^{-1} u = \frac{1}{g_m}\Big(\sum_{i\leq j} 2^{j-i} g_{i-1} h_{j+1} + \sum_{j<i} g_{j-1} h_{i+1}\Big) \tag{6.17}$$

Hence, the determinant of $\overline{m}K_n \setminus v_i (i = 1,2,\ldots,n)$ is given by

$$\left(1 + \frac{1}{g_m}\Big(\sum_{i\leq j} 2^{j-i} g_{i-1} h_{j+1} + \sum_{j<i} g_{j-1} h_{i+1}\Big)\right) g_{n-1}.$$

Theorem 6.12. *Let $\overline{m}G$ be mixed star negative block graph having k blocks $B_1, B_2, \ldots, B_k$ of order $n_1, n_2, \ldots, n_k$, respectively, then*

$$\det(\overline{m}G) = \sum_{i=1}^{k} \det(\overline{m}K_{n_i}) \prod_{j=1, j\neq i}^{k} D_n.$$

Proof: Proceeding as the proof of Theorem 6.11 the result directly follows from Lemma 6.6 and 6.1.

6.6 CONCLUSION

In this chapter, we find uses of $\mathscr{B}$-partitions to permanent and determinant of various graphs including some directed graphs, unicyclic graphs, block graph with negative cliques, mix complete graph.