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An Algorithm for $\mathscr{B}$-partitions, and Parameterized Complexity of Determinant and Permanent

In the Chapter 5, it was shown that the determinant (permanent) of a matrix can be calculated in terms of the determinant (permanent) of some subdigraphs of blocks in its digraph. The combination of these subdigraphs give the required $\mathscr{B}$-partitions. In this chapter, first, we give an algorithm for calculating the $\mathscr{B}$-partitions, later we analyzed the parameterized complexity of matrix determinant and permanent.

The chapter is organized as follows: In Section 7.1, an algorithm to find $\mathscr{B}$-partition and steps to find determinant and permanent of a matrix are given. In Section 7.2, we give the parameterized complexity analysis of the determinant and permanent.

7.1 ALGORITHM FOR $\mathscr{B}$-PARTITIONS

Let G be the digraph corresponding to any square matrix A of order n. Let us assume that G has k blocks $B_1, B_2, \ldots, B_k$. It is to note that, for G having k blocks with number of vertices in blocks equal to $n_1, n_2, \ldots, n_k$, respectively, following relation holds

$$n = \sum_{i=1}^{k} (n_i - 1) + 1.$$

Let G has t number of cut-vertices, assume them to be $v_1^c, v_2^c, \ldots, v_t^c$, where superscript c denotes cut-vertex. For $i = 1, 2, \ldots, t$, assume that cut-vertex v_i^c has cut-index equal to $T(i)$. And, let $S(i)$ is the array which contains indices of the blocks to which cut-vertex v_i^c belongs, and $S(i,j)$ denotes its j-th element.

Example 7.1. *For the digraph of matrix M_1, $t = 2$. Let $v_1^c = v_2, v_2^c = v_6$ then, $S(1) = [1,2], S(2) = [2,3], T(1) = 2, T(2) = 2$. Similarly, for the digraph of matrix M_2, $t = 2$ and assume $v_1^c = v_2, v_2^c = v_6$ then, $S(1) = [1,2], S(2) = [2,3,4], T(1) = 2, T(2) = 3$.*

The algorithm to find all $\mathscr{B}$-partitions of a digraph G is given in Algorithm 1. The algorithm associates each cut-vertex to exactly one block and remove from rest of the blocks it belongs, thus it recursively finds all the $\mathscr{B}$-partitions. Interested readers can see the Matlab codes for the determinant[1] and the permanent [2] using Algorithm 1.

As an example following are the steps using Algorithm 1 to calculate the $\mathscr{B}$-partitions of the digraph corresponding to matrix M_1.

Example 7.2. *We use the information in Example 7.1 of the digraph corresponding to matrix M_1 to calculate its $\mathscr{B}$-partitions. Algorithm 1 begins the execution by calling the function $\mathscr{B}$-part$(1, X, 2, S(1), T(1))$ in line*

[1]<https://in.mathworks.com/matlabcentral/fileexchange/62442-matrix-det--a-->
[2]<https://in.mathworks.com/matlabcentral/fileexchange/62443-matrix-per--a-->

Algorithm 1: Algorithm for $\mathscr{B}$-partitions.

Result: $\mathscr{B}$-partition of G

1 $it = 1$, X is zero column vector of order t;
2 $\mathscr{B}$-part$(it, X, t, S(it), T(it))$
3 **if** $(it > t)$ **then**
4   **for** $i = 1 : t$ **do**
5   **for** $j = 1 : T(i)$ **do**
6   Remove cut-vertex v_i^c from blocks $B_{S(i,j)}$ where $X(i) \neq S(i,j)$.
7   **end**
8   **end**
9   The resulting subgraphs of blocks will give a $\mathscr{B}$-partition. save it;
10   return
11 **else**
12   **for** $i = 1 : T(it)$ **do**
13         $X(it) = S(it, i)$;
14         $it = it + 1$;
15         $\mathscr{B}$-part$(it, X, t, S(it), T(it))$
16   **end**
17 **end**

2. As $1 \not> t$ execution will proceed from line 12. Line 13 will make $X(1) = 1$, that is $X = [1,0]$. In the line 14 the value of it will increase by 1. At line 15 the function $\mathscr{B}$-part $(2, X, 2, S(2), T(2))$ will be called, which again starts from line 12, as still $2 \not> t$. Line 13 makes $X(2) = 2$, that is $X = [1,2]$. Line 15 then call the function $\mathscr{B}$-part $(3, X, 2, S(3), T(3))$. This time $3 > t$ thus, now the execution starts from line 4. We need to remove vertex v_1^c, that is vertex v_2 from block B_2. Similarly, we need to remove vertex v_2^c, that is vertex v_6 from block B_3. The resulting subgraphs of blocks are $B_1, B_2 \setminus v_2, B_3 \setminus v_6$, which gives a $\mathscr{B}$-partition. In line 10, execution will return to line 12, where now $i = 2$, thus $X(2) = 3$, that is $X = [1,3]$. Line 15 then call the function $\mathscr{B}$-part $(3, X, 2, S(3), T(3))$. This time we need to remove vertex v_1^c, from block B_2 and vertex v_2^c from block B_2. The resulting subgraphs of blocks are $B_1, B_2 \setminus (v_2, v_6), B_3$, which gives the second $\mathscr{B}$-partition. In line 10 execution will return to execution where $it = 1$, and $i = 2$. Proceeding as before we get two more $\mathscr{B}$-partitions namely $B_1 \setminus v_2, B_2, B_3 \setminus v_6$ and $B_1 \setminus v_2, B_2 \setminus v_6, B_3$.

In Chapter 5, the procedure to calculate the characteristic and permanent polynomial, hence the determinant and permanent is given in Procedure 5.1. It is to be noted that for the determinant and permanent the cut-vertices without loops have no contribution in Procedure 5.1. Hence an alternative procedure can be given in term of only those cut-vertices which are having loops on them.

Procedure 7.1. *Let G be a digraph having k blocks $B_1, B_2, \ldots, B_k$. Let t_{nz} is a number of cut-vertices in G which have non-zero weight on their loops. Let the weights of loops at these vertices are $\alpha_1, \alpha_2, \ldots, \alpha_{t_{nz}}$, respectively. Also, assume that these cut-vertices have cut-indices $T(1), T(2), \ldots, T(t_{nz})$, respectively. Then, the following are the steps to calculate the determinant (permanent) of G.*

1. *For $q = 0, 1, 2, \ldots, t_{nz}$,*

    a) *Select any q cut-vertices at a time which have non-zero loop weights. In each $\mathscr{B}$-partition, remove these q cut-vertices from subdigraph to which they belong.*

        i. *For all the resulting partitions, sum their* det-summands *(per-summands). Multiply the* *sum by* $\prod_i \frac{(-\alpha_i)\binom{T(i)-1}{}}{T(i)}$ *where,* α_i*, and* $T(i)$ *are the weight and the cut-index of the removed* i*-th cut-vertex, respectively, for* $i = 1, 2, \ldots, q$*.*

ii. *For all possible* $\binom{t_{nz}}{q}$ *combinations of removed cut-vertices, sum all the terms in i.*

2. *Sum all the terms in 1.*

7.2 PARAMETRIZED COMPLEXITY OF DETERMINANT AND PERMANENT

From the last section, we see that how the subdigraphs of blocks are used to calculate the determinant and permanent of a matrix.

Before we proceed to find parametrized complexity of matrix determinant, let us observe the following. Let A be a square matrix of order r, as follows

$$A = \begin{bmatrix} A_1 & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}.$$

where, A_1 as the principle submatrix of order $r-1$, $\mathbf{b}$ is the column vector of order $r-1$, $\mathbf{c}$ is the row vector of order $r-1$, and $A(r,r) = d$.

If A_1 is invertible then from Schur' complement for determinant [Bapat and Roy, 2014],

$$\det(A) = \det(A_1)\det(d - \mathbf{c}A_1^{-1}\mathbf{b}).$$

If A_1 is not invertible, and d is non-zero then,

$$\det(A) = d \times \det(A_1 - \mathbf{b}\frac{1}{d}\mathbf{c}).$$

If A_1 is not invertible, and d is zero then,

$$\det(A) = \det\begin{bmatrix} A_1 & \mathbf{b} \\ \mathbf{c} & 0 \end{bmatrix} = \det\begin{bmatrix} A_1 & \mathbf{b} \\ \mathbf{c} & 1 \end{bmatrix} - 1 \times \det(A_1) = \det(A_1 - \mathbf{bc}).$$

Fortunately, complexity of inverse of a matrix of order r is also $O(r^{\varepsilon})$[Aho and Hopcroft, 1974]. In all the above cases, we observe that $\det(A)$ can be calculated in terms of the determinant of lower order matrices. For example in Section 5.3, in Example 5.4, $\det[4,5,6]$ can be calculated using $\det[4,5]$. Now, in G consider a block B_i which has t_i number of cut-vertices. We can first calculate the determinant of resulting subgraph after the removal of all the t_i cut-vertices from B_i. Then this determinant can be used to calculate the determinant of resulting after removal of any $t_i - 1$ cut-vertices. In this way, in general determinant of resulting subgraph after removal of i cut-vertices can be used to calculate the determinant of resulting subgraph after removal of $i-1$ cut-vertices.

Let digraph G have k blocks $B_1, B_2, \ldots, B_k$ with sizes, and number of cut-vertices $n_1, n_2, \ldots, n_k$, and $t_1, t_2, \ldots, t_k$, respectively. Then, during the calculation of determinant of G, for a particular block B_k, determinant of subdigraph of size $n_i - j$ is being calculted $\binom{t_i}{j}$ times. Also, in the view of above observation, in order to calculate determinant of subdigraph of order $n_i - j$ we can use determinant of subdigraph of order $n_i - j - 1$. Hence, the complexity of calculating the determinant is

$$O\Big(\sum_{i=1}^{k}\sum_{j=0}^{t_i}\binom{t_i}{j}(n_i - j - 1)^{\varepsilon}\Big).$$

One upper bound of the above complexity is

$$O\Big(\sum_{i=1}^{k} 2^{t_i} n_i^{\varepsilon}\Big) \tag{7.1}$$

The obvious pertinent question that follows is, for which combinations of n_i, t_i the above parameterized complexity beats $O(n^{\varepsilon})$. Thus, we need to solve the following inequality

$$\sum_{i=1}^{k} 2^{t_i} n_i^{\varepsilon} < n^{\varepsilon} \tag{7.2}$$

provided,

$$n = \sum_{i=1}^{k} (n_i - 1) + 1.$$

Similarly, for the permanent the parameterized complexity is

$$O\Big(\sum_{i=1}^{k} \sum_{j=0}^{t_i} \binom{t_i}{j} 2^{(n_i-j)} (n_i - j)^2\Big).$$

One upper bound of the above complexity is

$$O\Big(\sum_{i=1}^{k} 2^{t_i} 2^{n_i} n_i^2\Big) \tag{7.3}$$

Another pertinent question that follows is, for which combinations of n_i, t_i the above parametrized complexity beats $O(2^n n^2)$. Thus is, we need to solve the following inequality

$$\sum_{i=1}^{k} 2^{t_i} 2^{n_i} n_i^2 < 2^n n^2 \tag{7.4}$$

provided,

$$n = \sum_{i=1}^{k} (n_i - 1) + 1.$$

7.2.1 Parametric complexities

Let Γ be the largest of numbers of cut-vertices in any block that is $\Gamma = max\{t_1, t_2, \ldots t_k\}$. Let Δ be the size of the largest block that is $\Delta = max\{n_1, n_2, \ldots, n_k\}$. From expression 7.1

$$\sum_{i=1}^{k} 2^{t_i} n_i^{\varepsilon} \le 2^{\Gamma} \sum_{i=1}^{k} n_i^{\varepsilon} \le k 2^{\Gamma} \Delta^{\varepsilon} \tag{7.5}$$

In order to beat n^{ε} we have

$$k 2^{\Gamma} \Delta^{\varepsilon} \le n^{\varepsilon} \tag{7.6}$$

Taking logarithm on both the sides, we have

$$\Gamma \ln 2 \le \ln \left(\frac{1}{k} \Big(\frac{n}{\Delta} \Big)^{\varepsilon} \right), \tag{7.7}$$

that is

$$\Gamma = O\left(\ln\left(\frac{1}{k}\left(\frac{n}{\Delta}\right)^{\varepsilon} \right) \right). \tag{7.8}$$

Similarly for the permanent from the expression 7.3

$$\sum_{i=1}^{k} 2^{t_i} 2^{n_i} n_i^2 \leq k 2^{\Gamma} 2^{\Delta} \Delta^2 \tag{7.9}$$

in order to beat $2^n n^2$ we have

$$k 2^{\Gamma} 2^{\Delta} \Delta^2 \leq 2^n n^2. \tag{7.10}$$

Taking logarithm on both the sides

$$\Gamma \ln 2 \leq \ln\left(\frac{2^n n^2}{k 2^{\Delta} \Delta^2}\right), \tag{7.11}$$

that is

$$\Gamma = O\left(\ln\left(\frac{2^n n^2}{k 2^{\Delta} \Delta^2}\right) \right). \tag{7.12}$$

7.3 CONCLUSION

In this chapter, we first give an algorithm to find the $\mathscr{B}$-partitions in any digraph. Then we find the parameterized complexity of matrix determinant and permanent in terms of the size of blocks and number of cut-vertices in its digraph. For a class of matrices, the parametrized complexities beat the state of art complexities.