4

Nonlocal Quantum Correlations and Cumulant operators

4.1 INTRODUCTION

The discussion on nonlocality for bipartite and multiqubit systems in previous chapters confirm the fundamental importance associated with quantum correlations for the essential speed-up and efficiency of quantum information and computation, and hence is essential to identify, distinguish and manipulate classical and quantum correlations present in a system. Such nonlocal correlations between entangled qubits with no classical analogues not only distinguish the quantum world from its classical counterparts, but also play a key role in providing a physical insight into the fundamentals of quantum information theory and computation. Although the nonlocal properties of pure bipartite systems are well studied [Gisin, 1991], the characterization of nonlocal properties of entangled mixed bipartite systems still requires adequate attention [Werner, 1989; Munro *et al.*, 2001a; Ghosh *et al.*, 2001; Ma *et al.*, 2015; Singh and Kumar, 2018]. For example, there exists bipartite mixed entangled states such as Werner state which are entangled but do not violate the Bell inequality [Werner, 1989]. Moreover, the notion of quantum correlations was initially limited to entangled systems only- the existence of nonclassical correlations in some separable systems, however, raised serious questions on entanglement being solely responsible for quantum correlations. For example, the deterministic quantum computation model shows that highly mixed states can be used to achieve essential speed-up over the best known classical algorithms [Knill and Laflamme, 1998; Vedral, 2003; Datta *et al.*, 2008]. Therefore, the need to study correlations from a perspective different than the entanglement versus separability paradigm. Ollivier and Zurek proposed quantum discord [Ollivier and Zurek, 2001] as a prominent measure of nonclassical quantum correlations for an underlying bipartite state. It has received considerable attention for efficient, secure and optimal quantum information processing applications especially beyond the scope of entanglement [Henderson and Vedral, 2001; Badziag *et al.*, 2003; Koashi and Winter, 2004; Peuntinger *et al.*, 2013; Wu and Zhou, 2015; Gheorghiu *et al.*, 2015; Chanda *et al.*, 2015; Qiang *et al.*, 2016; Roga *et al.*, 2016; Zou and Fang, 2016; Jebaratnam *et al.*, 2017; Moreva *et al.*, 2017; Bera *et al.*, 2017; Christ and Hinrichsen, 2017; De Chiara and Sanpera, 2017; Braun *et al.*, 2017]. Alternately, a closely related attempt is provided by Henderson and Vedral to separate classical and quantum correlations in bipartite quantum states [Henderson and Vedral, 2001]. Quantum systems coupled to a heat bath give rise to another measure of quantum correlations originating from the "work (extracted from the heat bath) deficit" using the bipartite state shared between two parties in comparison to the case when the entire state is in possession with one party only [Oppenheim *et al.*, 2002]. Other significant measures of quantum correlations are measurement induced disturbance [Luo, 2008b] and dissonance [Modi *et al.*, 2010] where the first one is based on the idea that generic quantum measurements disturb the quantum system which can be used to quantify the quantumness of correlations therein; the latter exploits the concept of relative entropy to distinguish different correlations. In addition, one can also define a measure to detect quantum correlations in large classes of separable bipartite systems based on the existence of linear witness operators for quantum discord [Adhikari and Banerjee, 2012] .

Quantum discord, among all the measures to distinguish quantum and classical correlations, has been extensively studied, e.g., it has received substantial attention in studies involving fuzzy measurement [Vedral, 2003], broadcasting [Piani *et al.*, 2008], complementarity and monogamy relationship between classical and quantum correlations [Oppenheim *et al.*, 2003; Koashi and Winter, 2004], dynamics of discord [Maziero *et al.*, 2009; Mazzola *et al.*, 2010], operational interpretations of quantum discord in terms of state merging [Madhok and Datta, 2011] and teleportation fidelity [Yang *et al.*, 2005], and the relation between discord and entanglement [Cornelio *et al.*, 2011; Adhikari and Banerjee, 2012]. Quantum discord also finds applications in computing [Datta and Shaji, 2011], nuclear magnetic resonance [Katiyar *et al.*, 2012], spin chains [Nag *et al.*, 2011; Dhar *et al.*, 2012] and ground and thermal states of the clusters [Pal and Bose, 2011]. Although, an algorithm has been developed to calculate quantum discord for general two-qubit states [Girolami and Adesso, 2011], the analytical expression for the quantum discord has only been obtained for two-qubit states with maximally mixed marginals [Luo, 2008a], e.g., a certain class of X-structured states. Recent studies have shown that an analytical expression is difficult to obtain for arbitrary two-qubit states because of the optimization procedures involved. Considering this complexity, Dakic *et al.* introduced another measure -geometric discord [Dakić *et al.*, 2010], which provides a much simpler way to quantify the amount of nonclassical correlations in an arbitrary two-qubit state [Luo and Fu, 2010]. The formulation of geometric discord was further generalized to the case of $d \otimes d$ dimensional systems [Luo and Fu, 2010].

Therefore, the characterization of states, which are entangled but do not violate the Bell inequality, requires more attention to understand the distinction between quantum and classical worlds, and to ascertain the efficient success of quantum information processing protocols. Hence, the relationship between nonclassicality and correlations needs a deeper analysis to understand the importance and significance of quantum correlations in communication and computing. In this chapter, to quantify nonlocal correlations, we propose to modify the Bell-CHSH inequality [Clauser *et al.*, 1969] using statistical correlation coefficients- such coefficients provide the extent of correlations between the qubits, which is a direct measure of entanglement [Osborn, 1977; Fano, 1983; Huang, 1987; Schlienz and Mahler, 1995; Audenaert and Plenio, 2006; Altafini, 2004; Kumar and Krishnan, 2009]. In this chapter, we first define the modified Bell inequality as the Bell-Cumulant inequality. For this, we derive an expression to modify the Bell-CHSH inequality using correlation coefficients, indicating the degree of correlations between the qubits. We show that a two-qubit state will violate the Bell-Cumulant inequality if the maximum expectation value of the Bell-Cumulant operator is greater than zero. The analysis using correlation coefficients further allows us to establish a relation between the maximum expectation value of Bell-Cumulant operator and the state parameter of a two-qubit pure state. For arbitrary two-qubit mixed states, we derive a criterion for the violation of Bell-Cumulant inequality using the celebrated Horodecki criteria for the Bell inequality violation. Interestingly, our results show that the Bell-Cumulant operator identifies the nonlocal correlations in certain classes of two-qubit mixed states, where the Bell-CHSH operator fails to capture the nonclassical correlations. Furthermore, we also propose a way to define geometric discord in terms of correlations coefficients. This further allows us to establish a relation between geometric discord and the Bell-Cumulant operator. We finally illustrate the results obtained in this chapter using examples of certain two-qubit mixed entangled states. The analysis using correlation coefficients may prove to be an important study, as these correlation coefficients can be estimated experimentally as well. Furthermore, we also extend our study to analyse and characterize the nonlocal properties in three-qubit systems.

4.2 BELL-CUMULANT INEQUALITY

In order to facilitate the discussion, we first briefly define the correlation coefficients existing between the qubits [Fano, 1983; Kumar and Krishnan, 2009]. For this, we alternately represent an arbitrary two-qubit density operator as

$$\rho = \frac{1}{4}\left[(I^1 + \mathbf{r}^1 \cdot \sigma^1) \otimes (I^2 + \mathbf{s}^2 \cdot \sigma^2) + \sigma^1 \cdot C_\rho \cdot \sigma^2\right] \tag{4.1}$$

where $(1,2)$ represent the qubit index, I represents a 2×2 identity operator, $\mathbf{r}$ and $\mathbf{s}$ are polarization vectors of two spins, σ's stands for standard Pauli matrices, and C_ρ represents a second rank tensor, i.e., $\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$ whose coefficients c_{nm} represent correlations between the qubits defined by $c_{nm} = \langle \sigma_n \sigma_m \rangle - \langle \sigma_n \rangle \langle \sigma_m \rangle$, where $n,m \in \{1,2,3\}$. Further, from Eq. (1.3) correlation matrix C_ρ can also be defined as $C_\rho = T_\rho - rs^\dagger$, where $s^\dagger$ represents conjugate transpose of s and T_ρ is a real matrix whose coefficients are given by $t_{nm} = Tr\left(\rho.\sigma_n \otimes \sigma_m\right)$. Therefore, coefficients c_{nm} of correlation matrix C_ρ are defined by $c_{nm} = t_{nm} - \mathbf{r}_n.\mathbf{s}_m$, where $n,m \in \{1,2,3\}$. The approach presented here can be applied to qubits or other spin-1/2 systems as well as photons. Interactions between qubits do result in correlations of the type described here.

The Bell-CHSH [Clauser *et al.*, 1969] inequality revealed the distinction between classical correlations allowed by local hidden variable models and quantum correlations which are less intuitive and are explained by quantum theory. As discussed in the second chapter, the Bell-CHSH inequality in its generalized form can be represented as

$$\left|E(AB) + E(AB') + E(A'B) - E(A'B')\right| \leq 2$$

where the operators A, A', B, and B' have standard representations as discussed in previous chapters, and $E(AB)$ represents average value of product of measurement outcomes of Alice and Bob and likewise for similarly defined terms. The violation of Bell-CHSH inequality in pure two-qubit states acts as a signature to confirm the existence of nonlocal correlations. The advent of discord, however, suggested that the potential of nonlocal correlations in bipartite mixed states may not be only related to entanglement, i.e., there are separable states exhibiting nonlocal correlations, which may be useful for quantum information processing. Moreover, the Bell-CHSH inequality fails to identify correlations in bipartite mixed entangled states as well for a certain range of state parameters [Werner, 1989; Horodecki, 1996; Ma *et al.*, 2015; Singh and Kumar, 2018]. The characterization of such systems, therefore, is essential for understanding the nature of nonlocal correlations and the fundamentals of quantum theory. We, therefore, proceed to modify the Bell-CHSH inequality using correlation coefficients to analyse nonlocal correlations between qubits. Using the formal definition of correlation coefficients, the modified Bell-CHSH operator is given by

$$B_C = E(AB)_c + E(AB')_c + E(A'B)_c - E(A'B')_c \tag{4.2}$$

where $E(AB)_c = E(AB) - E(A)E(B)$. The other terms in Eq. (4.2) can be defined in a similar fashion. We further assume that Alice and Bob always choose their measurements with an equal probability of $\frac{1}{2}$. Using the extremal strategy, e.g., a strategy where the outcomes of all measurements for both Alice and Bob are +1, the maximum classical value for the operator in Eq. (4.2) is zero. In general, we will get similar results for all other measurement outcomes of Alice and Bob. Hence, the Bell-Cumulant inequality is given by

$$\left|E(AB)_c + E(AB')_c + E(A'B)_c - E(A'B')_c\right| \leq 0 \tag{4.3}$$

where the equality can be achieved for product states. Further, to evaluate the maximum expectation value of Bell-Cumulant operator using quantum strategy, we consider a two-qubit pure state $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$, shared between Alice and Bob. The Bell-Cumulant operator, in terms of spin projection operators, can be re-expressed as

$$\begin{aligned} B_C(|\Psi\rangle) &= \langle \hat{a}\cdot\sigma_1\sigma_2\cdot\hat{b}\rangle - \langle \hat{a}\cdot\sigma_1\rangle\langle\sigma_2\cdot\hat{b}\rangle + \langle \hat{a}\cdot\sigma_1\sigma_2\cdot\hat{b}'\rangle - \langle \hat{a}\cdot\sigma_1\rangle\langle\sigma_2\cdot\hat{b}'\rangle \\ &+ \langle \hat{a}'\cdot\sigma_1\sigma_2\cdot\hat{b}\rangle - \langle \hat{a}'\cdot\sigma_1\rangle\langle\sigma_2\cdot\hat{b}\rangle - \langle \hat{a}'\cdot\sigma_1\sigma_2\cdot\hat{b}'\rangle + \langle \hat{a}'\cdot\sigma_1\rangle\langle\sigma_2\cdot\hat{b}'\rangle \end{aligned} \tag{4.4}$$

Similar to previous chapters, for evaluating the maximum expectation value of $B_C(|\Psi\rangle)$, we need to consider a pair of two mutually orthogonal unit vectors $\hat{d}$ and $\hat{d}'$ such that

$$\hat{b}+\hat{b}' = 2\cos\theta\hat{d}, \qquad \hat{b}-\hat{b}' = 2\sin\theta\hat{d}' \tag{4.5}$$

Therefore, Eq. (4.4) using Eq. (4.5) is further simplified as

$$B_C(|\Psi\rangle) = 2\left(\langle\hat{a}\cdot\sigma_1\hat{d}\cdot\sigma_2\rangle - \langle\hat{a}\cdot\sigma_1\rangle\cdot\langle\hat{d}\cdot\sigma_2\rangle\right)\cos\theta + 2\left(\langle\hat{a}'\cdot\sigma_1\hat{d}'\cdot\sigma_2\rangle - \langle\hat{a}'\cdot\sigma_1\rangle\cdot\langle\hat{d}'\cdot\sigma_2\rangle\right)\sin\theta \tag{4.6}$$

where unit vectors $\hat{a}$ and $\hat{d}$ are defined as

$$\begin{aligned}\hat{a} &= (\sin\phi\cos\delta, \sin\phi\sin\delta, \cos\phi)\\ \hat{d} &= (\sin\chi\cos\varepsilon, \sin\chi\sin\varepsilon, \cos\chi)\end{aligned} \tag{4.7}$$

The unit vectors $\hat{a}'$ and $\hat{d}'$ can also be defined as unit vectors $\hat{a}$ and $\hat{d}$ with primes on angles. In order to evaluate the maximum expectation value of Bell-Cumulant operator, we first evaluate the expectation value of the operator $\langle\hat{a}\cdot\sigma_1\hat{d}\cdot\sigma_2\rangle - \langle\hat{a}\cdot\sigma_1\rangle\cdot\langle\hat{d}\cdot\sigma_2\rangle$ in Eq. (4.6), such that

$$\langle\hat{a}\cdot\sigma_1\hat{d}\cdot\sigma_2\rangle - \langle\hat{a}\cdot\sigma_1\rangle\cdot\langle\hat{d}\cdot\sigma_2\rangle = 4\alpha^2\beta^2\cos\phi\cos\chi + 2\alpha\beta\sin\phi\sin\chi\cos(\delta+\varepsilon) \tag{4.8}$$

Eq. (4.8) when maximized with respect to ϕ, using the fact that the maximum value of $P\cos\theta + Q\sin\theta$ is $\sqrt{(P^2)+(Q^2)}$ and considering $\cos^2(\delta+\varepsilon) = 1$, gives

$$\left[\langle\hat{a}\cdot\sigma_1\hat{d}\cdot\sigma_2\rangle - \langle\hat{a}\cdot\sigma_1\rangle\cdot\langle\hat{d}\cdot\sigma_2\rangle\right]_{max} = \left[16\alpha^4\beta^4\cos^2\chi + 4\alpha^2\beta^2\sin^2\chi\right]^{\frac{1}{2}} \tag{4.9}$$

Similarly, the maximum expectation value of the other operator in Eq. (4.6) is

$$\left[\langle\hat{a}'\cdot\sigma_1\hat{d}'\cdot\sigma_2\rangle - \langle\hat{a}'\cdot\sigma_1\rangle\cdot\langle\hat{d}'\cdot\sigma_2\rangle\right]_{max} = \left[16\alpha^4\beta^4\cos^2\chi' + 4\alpha^2\beta^2\sin^2\chi'\right]^{\frac{1}{2}} \tag{4.10}$$

Maximizing Eq. (4.6), with respect to θ and using Eqs. (4.9) and (4.10), we have

$$B_C(|\Psi\rangle)_{max} = 2[16\alpha^4\beta^4\cos^2\chi + 16\alpha^4\beta^4\cos^2\chi' + 4\alpha^2\beta^2\sin^2\chi + 4\alpha^2\beta^2\sin^2\chi']^{\frac{1}{2}} \tag{4.11}$$

Eq. (4.11) can be further simplified using $\cos^2\theta_1 = (1-\sin^2\theta_1)$, such that

$$B_C(|\Psi\rangle)_{max} = 2[32\alpha^4\beta^4 + (4\alpha^2\beta^2 - 16\alpha^4\beta^4)\left\{\sin^2\chi + \sin^2\chi'\right\}]^{\frac{1}{2}} \tag{4.12}$$

To evaluate the optimum value of Bell-Cumulant operator $B_C(|\Psi\rangle)$, we use the orthogonality relation between $\hat{d}$ and $\hat{d}'$, such as the maximum value of $\sin^2(\chi)+\sin^2(\chi')$ is 2 [Ghose *et al.*, 2009]. Hence, Eq. (4.12) gives

$$B_C(|\Psi\rangle)_{opt} = 4\sqrt{2}\alpha\beta \tag{4.13}$$

Since, the optimum value of Bell operator for a generalized two-qubit state is $2[1+4\alpha^2\beta^2]^{\frac{1}{2}}$ [Popescu and Rohrlich, 1992], therefore, from Eq. (4.13), we have

$$B(|\Psi\rangle)_{opt} = \frac{1}{\sqrt{2}}[8 + B_C(|\Psi\rangle)^2_{opt}]^{\frac{1}{2}} \tag{4.14}$$

Figure 4.1 demonstrates the relationship between $B_C(|\Psi\rangle)$ and the state parameter α. It shows that if α is varied from 0 to 1 then the optimum value of Bell-Cumulant operator first increases, attains a maximum value and then decreases to zero. Evidently, the extreme value of 0 can be obtained for $\alpha = 0$ and 1, and the maximum value of $2\sqrt{2}$ can be obtained for $\alpha = 1/\sqrt{2}$, i.e., for a maximally entangled Bell state. Clearly, every pure bipartite state violates the Bell-Cumulant inequality. The analytical result obtained here is in complete agreement with the numerical optimization of Bell-Cumulant operator for generalized pure two-qubit states.

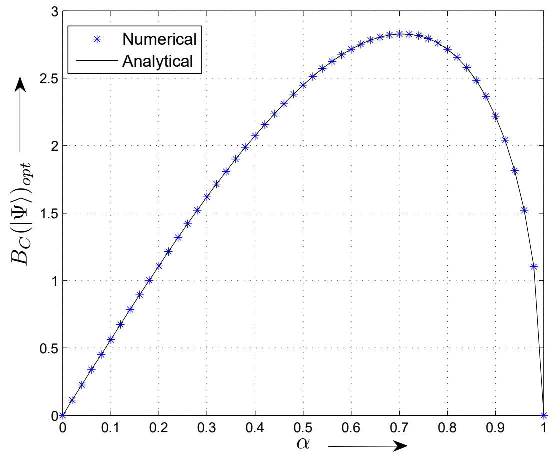

Figure 4.1 : Estimation of $B_C(|\Psi\rangle)_{opt}$ with respect to the state parameter α.

4.3 BELL-CUMULANT OPERATOR FOR AN ARBITRARY MIXED SPIN$-\frac{1}{2}$ STATE

The analysis and characterization of nonlocal correlations in an arbitrary bipartite mixed state is much more complex than the analysis of nonlocality in pure two-qubit states. In this section, we evaluate an effective criterion for an arbitrary mixed spin$-\frac{1}{2}$ state for violating the Bell-Cumulant inequality. For this, we use the similar optimization technique that was used by Horodecki *et al.* to propose a necessary and sufficient condition for violation of the Bell-CHSH inequality by an arbitrary mixed spin$-\frac{1}{2}$ state, represented in Eq. (1.3) [Horodecki, 1995]. The procedure further allows us to confirm that all pure two-qubit states violate the Bell-Cumulant inequality. We finally illustrate our results using some important two-qubit mixed entangled states where the Bell-CHSH inequality fails to detect nonlocal correlations but the Bell-Cumulant inequality detects the presence of nonlocality. In order to facilitate discussions, we define a diagonalizable symmetric matrix $\xi_\rho = C_\rho^\dagger.C_\rho$ where C_ρ stands for correlation matrix defined in Eq. (4.1) and $C_\rho^\dagger$ represents conjugate transpose of C_ρ. We further define a quantity $P(\rho) = (\vartheta_{max}^1 + \vartheta_{max}^2)$, where ϑ_{max}^1 and ϑ_{max}^2 are two positive, greatest, and real eigenvalues of ξ_ρ. Therefore, to evaluate the effective criterion for the violation of Bell-Cumulant inequality, we use general form of the Bell-Cumulant operator from Eq. (4.4) such that

$$B_C(\rho) = (\hat{a}, T_\rho(\hat{b}+\hat{b}')) - (\hat{a}, r_\rho s_\rho^T(\hat{b}+\hat{b}')) + (\hat{a}', T_\rho(\hat{b}-\hat{b}')) - (\hat{a}', r_\rho s_\rho^T(\hat{b}-\hat{b}')) \tag{4.15}$$

The linearity of inner products lead us to re-express Eq. (4.15) as

$$B_C(\rho) = \left(\hat{a}, (T_\rho - r_\rho s_\rho^T)(\hat{b}+\hat{b}')\right) + \left(\hat{a}', (T_\rho - r_\rho s_\rho^T)(\hat{b}-\hat{b}')\right) \tag{4.16}$$

therefore, we have

$$B_C(\rho) = \left(\hat{a}, C_\rho(\hat{b}+\hat{b}')\right) + \left(\hat{a}', C_\rho(\hat{b}-\hat{b}')\right) \tag{4.17}$$

In order to maximize $B_C(\rho)$, we use pair of mutually orthogonal vectors $\hat{d}$ and $\hat{d}'$ as defined in Eq. (4.5), such that

$$B_C(\rho) = 2\left[(\hat{a}, C_\rho\hat{d})\cos\theta + (\hat{a}', C_\rho\hat{d}')\sin\theta\right] \tag{4.18}$$

Therefore, the maximum expectation value of Bell-Cumulant operator is given as

$$
\begin{aligned}
B_C(\rho)_{max} &= \max_{\hat{a},\hat{a}',\hat{d},\hat{d}',\theta} 2[(\hat{a},C_\rho\hat{d})\cos\theta + (\hat{a}',C_\rho\hat{d}')\sin\theta] \\
&= \max_{\hat{d},\hat{d}',\theta} 2[(\|C_\rho\hat{d}\|)\cos\theta + (\|C_\rho\hat{d}'\|)\sin\theta] \\
&= \max_{\hat{d},\hat{d}'} 2\sqrt{(\|C_\rho\hat{d}\|)^2 + (\|C_\rho\hat{d}'\|)^2} \\
&= 2\sqrt{\vartheta^1_{max} + \vartheta^2_{max}} = 2\sqrt{P(\rho)} \qquad (4.19)
\end{aligned}
$$

where ϑ^1_{max} and ϑ^2_{max} are defined as $\vartheta^1_{max} = \|C_\rho\hat{d}\|^2_{max}$ and $\vartheta^2_{max} = \|C_\rho\hat{d}'\|^2_{max}$. Here, we have considered $\hat{d}_{max}$ and $\hat{d}'_{max}$ as eigenvectors related to the two largest eigenvalues of symmetric matrix ξ_ρ and maximum is taken over all unit and mutually orthogonal pair of vectors $\hat{d}$, and $\hat{d}'$. Therefore, we get

$$B_C(\rho)_{opt} \leq 2\sqrt{P(\rho)} \qquad (4.20)$$

Hence, if $P(\rho) > 0$ then the state ρ will violate the Bell-Cumulant inequality.

Theorem 4.3.1. *An arbitrary mixed spin$-\frac{1}{2}$ state represented by density operator ρ violates the Bell-Cumulant inequality iff $P(\rho) > 0$.*

We readdress the question whether all pure states violate the Bell-Cumulant inequality or not. For this, we represent a generalized pure bipartite entangled state in Hilbert-Schmidt basis as

$$|\psi\rangle = \lambda_1|00\rangle + \lambda_2|11\rangle \qquad (4.21)$$

where $|\lambda_1|^2 + |\lambda_2|^2 = 1$. For further discussion, Eq. (4.21) can be represented in terms of a density operator as

$$\rho' = \frac{1}{4}\left[I\otimes I + \left(\lambda_1^2 - \lambda_2^2\right)I\otimes\sigma_3 + \left(\lambda_1^2 - \lambda_2^2\right)\sigma_3\otimes I + \sigma_3\otimes\sigma_z + 2\lambda_1\lambda_2\sigma_1\otimes\sigma_1 - 2\lambda_1\lambda_2\sigma_2\otimes\sigma_2\right] \qquad (4.22)$$

From Eq. (4.22), we evaluate the two largest eigenvalues of real symmetric matrix $\xi_{\rho'}$ as $\vartheta^1_{max} = \vartheta^2_{max} = 4\lambda_1^2\lambda_2^2$, such that $P(\rho) = 8\lambda_1^2\lambda_2^2$. Therefore, for all pure two-qubit entangled states, $P(\rho) > 0$, and hence every pure bipartite entangled state violates the Bell-Cumulant inequality. Moreover, the maximum expectation value of Bell-Cumulant operator is

$$B_C(\rho')_{max} = 4\sqrt{2}\lambda_1\lambda_2 \qquad (4.23)$$

The result obtained in Eq. (4.23) completely agrees with the result evaluated in Eq. (4.13).

We now proceed to analyse nonlocal properties of some of the mixed states where the Bell inequality fails to confirm the presence of nonlocal correlations.

(1) For mixed states, we first consider two-qubit Werner states [Werner, 1989], represented in Eq. (2.41). Although Werner states are entangled for $x > \frac{1}{3}$, they only violate the Bell-CHSH inequality for $x > \frac{1}{\sqrt{2}}$, i.e., Werner states, though entangled, do not violate the Bell-CHSH inequality for $\frac{1}{3} < x \leq \frac{1}{\sqrt{2}}$. Hence, one can observe that the original Bell-CHSH inequality fails to detect nonlocal correlations in Werner states for a certain range of state parameter x. Using the analysis presented in this chapter, the eigenvalues of real symmetric matrix ξ_{ρ_w} of

Werner states are $\vartheta^1_{max} = \vartheta^2_{max} = x^2$. Thus, $P(\rho_w)$ is $2x^2$ and the maximum expectation value of Bell-Cumulant operator is

$$B_C\left(\rho_w\right)_{opt} = 2\sqrt{2}x \tag{4.24}$$

Therefore, if $x > 0$ then $P(\rho_w) > 0$, and hence, Werner states always violate the Bell-Cumulant inequality for the whole range of x, i.e., for $0 < x \leq 1$.

(2) As our next example, we consider an important set of two-qubit mixed states, namely Horodecki states [Horodecki, 1996], represented in Eq. (2.44). Horodecki states are entangled for $0 < a \leq 1$, but violate the Bell-CHSH inequality only for $\frac{1}{\sqrt{2}} < a \leq 1$. Hence, in this case, the original Bell-CHSH inequality again fails to detect nonlocal correlations for $0 < a \leq \frac{1}{\sqrt{2}}$.

The eigenvalues of ξ_{ρ_h} for Horodecki states are $\vartheta^1_{max} = \vartheta^2_{max} = a^2$, and thus, the maximum expectation value of Bell-Cumulant operator can be computed as

$$B_C\left(\rho_h\right)_{max} = 2\sqrt{2}a \tag{4.25}$$

Clearly, $P(\rho_h)$ is greater than $0 \;\forall\; a > 0$, and hence Horodoki states violate the Bell-Cumulant inequality for the complete range of state parameter a.

(3) We also analyse another class of mixed bipartite states proposed by WenChao Ma *et al.* [Ma *et al.*, 2015], given by

$$\rho_{wc} = f\left(|00\rangle\langle 00|\right) + \frac{(1-f)}{2}\left(|01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|\right) \tag{4.26}$$

where f is a state parameter. The state is entangled for $0 < f \leq 1$, but violates the Bell-CHSH inequality only for $0 < f \leq \frac{1}{2}\left[2-\sqrt{2}\right]$. Therefore, we evaluate $P(\rho_{wc})$ for the given state, such that $P(\rho_{wc}) = 2(1-f)^2$. Hence, the optimum value of Bell-Cumulant operator for the given state is

$$B_C\left(\rho_{wc}\right)_{opt} = 2\sqrt{2}(1-f) \tag{4.27}$$

Again the Bell-Cumulant inequality is violated for the complete range of state parameter f, i.e., for $0 \leq f < 1$.

(4) We finally consider the new class of bipartite mixed states proposed in chapter 2. We have already demonstrated that the proposed states are always entangled but do not violate the the Bell-CHSH inequality for certain ranges of amplitude-damping parameter γ and weak measurement strength η. Therefore, we now advance to analyse nonlocality in the proposed class of states using the Bell-Cumulant operator. For this, we first re-express the proposed class of two-qubit mixed states, namely

$$\varrho = \frac{1}{N}\left[\frac{1}{2}\gamma(1-\eta)\left\{\gamma(1-\eta)|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|\right\} + \left|\phi^+\right\rangle\left\langle\phi^+\right|\right] \tag{4.28}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$ and $N = \frac{1}{2}(2+\gamma(1-\eta)(2+\gamma(1-\eta)))$. Clearly, for the proposed class, $P(\varrho) = \frac{(N^2+1)}{N^4}$; hence the optimum value of Bell-Cumulant operator for the proposed class of states is

$$B_C\left(\varrho\right)_{opt} = \frac{2}{N^2}\sqrt{N^2+1} \tag{4.29}$$

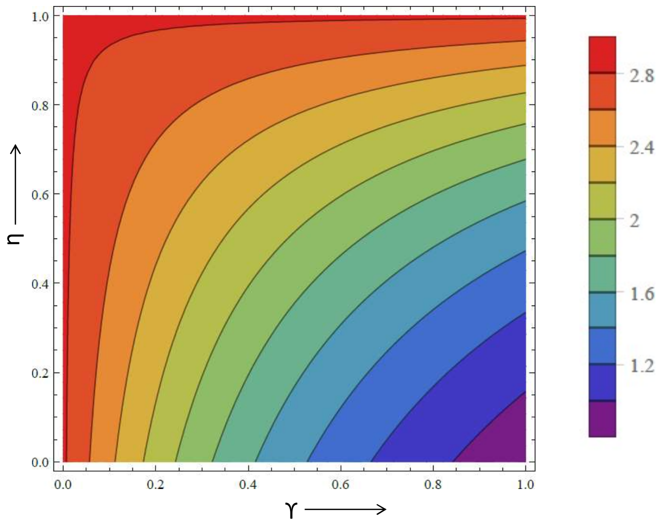

Figure 4.2 : Estimation of $B_C(\varrho)_{opt}$ with respect to state parameter η of the proposed state in chapter-2 and decoherence parameter γ.

Figure 4.2 shows the violation of Bell-Cumulant inequality with respect to state parameters η and γ. It suggests that the proposed class of states always violate the Bell-Cumulant inequality, further confirming the results shown in Figure 2.14 and Figure 2.15, respectively. Therefore, the Bell-Cumulant inequality detects nonlocal correlations in mixed entangled states, where even the Bell inequality fails to detect nonlocality for certain ranges of state parameters.

4.4 RELATIONSHIP BETWEEN THE BELL-CUMULANT INEQUALITY AND DISCORD

Quantum discord or discord captures nonlocal correlations in entangled as well as separable systems [Ollivier and Zurek, 2001; Henderson and Vedral, 2001]. Alternately, one can also evaluate geometric discord [Dakić *et al.*, 2010] to confirm the presence of nonlocal correlations in an underlying state. In this section, we evaluate the maximum and minimum values of geometric discord for an arbitrary two-qubit state in term of correlation matrix C_ρ. The analysis further allows us to establish an analytical relation between geometric discord and the Bell-Cumulant inequality. Finally, we illustrate the importance of our results for certain important classes of two-qubit states.

Geometric discord is a relatively simpler measure to quantify nonclassical correlations in an arbitrary bipartite state in comparison to discord, and is defined as

$$D_G(\rho) = \min_{\chi \in \Omega_0} \| \rho - \chi \|_2^2 \qquad (4.30)$$

where Ω_0 represents the set of zero-discord states and $\| A - B \|_2^2 = Tr(A-B)^2$ is the square norm. In order to simplify the calculations, we use the analytical expression for geometric measure of discord for an arbitrary two-qubit system ρ represented in Eq. (1.30), and given as

$$D_G(\rho) = \frac{1}{4}\left[\| T_\rho \|^2 + \| r \|^2 - \lambda_{max}\right]$$

where λ_{max} is the maximum eigenvalue of matrix $(r.r^\dagger + T_\rho.T_\rho^\dagger)$. In order to compute the maximum and minimum value of geometric discord, we use Weyl's theorem [Horn *et al.*, 1990; Knutson and

Tao, 2001] which states that if $A,B \in M_n$ where M_n denotes the set of $n \times n$ Hermitian matrices, and $\lambda_i(A), \lambda_i(B)$, and $\lambda_i(A+B)$ represent eigenvalues of A,B, and $(A+B)$ operators, respectively, arranged in an ascending order such that $\lambda_1(A)$ represents the minimum eigenvalue and $\lambda_n(A)$ represents the maximum eigenvalue of A, then for $j = 1,2....,n$

$$\lambda_1(A) + \lambda_j(B) \leq \lambda_j(A+B) \leq \lambda_n(A) + \lambda_j(B) \tag{4.31}$$

In terms of correlation matrix C_ρ, the geometric discord for an arbitrary two-qubit state, using a relation $T_\rho = C_\rho + r_\rho s_\rho^\dagger$, is defined as

$$\begin{aligned}
D_G(\rho) &= \frac{1}{4}[Tr[(C_\rho + r_\rho s_\rho^\dagger).(C_\rho + r_\rho s_\rho^\dagger)^\dagger)] + Tr[r.r^\dagger] - \lambda_{max}\left\{r.r^\dagger + (C_\rho + r_\rho s_\rho^\dagger).(C_\rho + r_\rho s_\rho^\dagger)^\dagger)\right\}] \\
&= \frac{1}{4}[Tr[(C_\rho + r_\rho s_\rho^\dagger).(C_\rho^\dagger + s_\rho.r_\rho^\dagger)] + Tr[r.r^\dagger] - \lambda_{max}\left\{r.r^\dagger + (C_\rho + r_\rho s_\rho^\dagger).(C_\rho^\dagger + s_\rho.r_\rho^\dagger)\right\}] \\
&= \frac{1}{4}\Big[Tr[C_\rho.C_\rho^\dagger + r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger] + Tr[r.r^\dagger] \\
&- \lambda_{max}\left\{r.r^\dagger + C_\rho.C_\rho^\dagger + r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger\right\}\Big] \\
&= \frac{1}{4}\Big[Tr[C_\rho.C_\rho^\dagger + r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger + r.r^\dagger] \\
&- \lambda_{max}\left\{r.r^\dagger + C_\rho.C_\rho^\dagger + r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger\right\}\Big] \\
&= \frac{1}{4}\Big[Tr[C_\rho.C_\rho^\dagger] + Tr[r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger + r.r^\dagger] \\
&- \lambda_{max}\left\{r.r^\dagger + C_\rho.C_\rho^\dagger + r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger\right\}\Big] \\
&= \frac{1}{4}\Big[Tr[C_\rho.C_\rho^\dagger] + Tr[K_\rho] - \lambda_{max}\left\{C_\rho.C_\rho^\dagger + K_\rho\right\}\Big]
\end{aligned} \tag{4.32}$$

where $K_\rho = (r_\rho s_\rho^\dagger C_\rho^\dagger + C_\rho s_\rho r_\rho^\dagger + r_\rho s_\rho^\dagger s_\rho r_\rho^\dagger + r.r^\dagger)$. Moreover, from the inequality in Eq. (4.31), and from Eq. (4.32), we have

$$\begin{aligned}
&\frac{1}{4}[Tr[C_\rho C_\rho^\dagger] + Tr[K_\rho] - \lambda_{max}(K_\rho) - \lambda_{max}(C_\rho C_\rho^\dagger)] \leq \\
&\quad D_G(\rho) \leq \frac{1}{4}[Tr[C_\rho C_\rho^\dagger] + Tr[K_\rho] - \lambda_{max}(K_\rho) - \lambda_{min}(C_\rho C_\rho^\dagger)]
\end{aligned} \tag{4.33}$$

From the inequalities in Eq. (4.33), one can define

$$D_G^{min}(\rho) = \frac{1}{4}[\| C_\rho \|^2 - \lambda_{max}(C_\rho.C_\rho^\dagger)] \tag{4.34}$$

$$D_G^{max}(\rho) = \frac{1}{4}[\| C_\rho \|^2 - \lambda_{min}(C_\rho.C_\rho^\dagger)] \tag{4.35}$$

Clearly, from Eqs. (4.34-4.35), if all eigenvalues of the matrix $\xi_\rho = C_\rho^\dagger.C_\rho$ are equal then $D_G^{max}(\rho) = D_G^{min}(\rho) = D_G(\rho)$. Using the above result, one can also establish an analytical relation between geometric discord and the Bell-Cumulant operator. For example, from Eq. (4.35), we have

$$D_G^{max}(\rho) = \frac{1}{4}[\| C_\rho \|^2 - \lambda_{min}(C_\rho.C_\rho^\dagger)] = \frac{1}{4}P(\rho) \tag{4.36}$$

Therefore, from Eq. (4.20) and Eq. (4.36), we get

$$B_C(\rho)_{max} = 4\sqrt{D_G^{max}(\rho)} \tag{4.37}$$

Hence, for an arbitrary two-qubit mixed entangled state ρ, we have

$$0 < B_C\left(\rho\right)_{opt} \leq 4\sqrt{D_G^{max}(\rho)} \tag{4.38}$$

Further, from Theorem 4.3.1, an arbitrary two-qubit state violates the Bell-Cumulant inequality, iff $P(\rho) > 0$, hence it follows the Bell-Cumulant inequality will be violated iff $D_G^{max}(\rho) > 0$. In order to demonstrate the importance of our results, we now illustrate few specific examples of entangled bipartite states.

(1) We first consider a pure two-qubit state $|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, and then represent it in term of a density operator as

$$\rho' = \frac{1}{4}(I\otimes I + \cos 2\theta\cdot\sigma_z\otimes I + \sum_{i=1}^{3} t_i\sigma_i\otimes\sigma_i) \tag{4.39}$$

The correlation matrix $C_{\rho'}$ and geometric discord for ρ' are given as

$$C_{\rho'} = \begin{pmatrix} \sin 2\theta & 0 & 0 \\ 0 & -\sin 2\theta & 0 \\ 0 & 0 & \sin^2 2\theta \end{pmatrix} \tag{4.40}$$

and

$$D_G(\rho') = \frac{1}{2}\sin^2 2\theta, \tag{4.41}$$

respectively. From Eq. (4.40), the eigenvalues of matrix $\xi_{\rho'}$ are $\sin^2 2\theta$, $\sin^2 2\theta$, and $\sin^4 2\theta$. Hence, $P(\rho') = 2\sin^2 2\theta$ and from Eq. (4.20), Eq. (4.34), and Eq. (4.35), it is straight forward to obtain

$$B_C\left(\rho'\right)_{opt} \leq 2\sqrt{2}\sin 2\theta, \tag{4.42}$$

$$\begin{aligned} D_G^{max}(\rho') &= \frac{1}{4}\left[\sin^2 2\theta + \sin^2 2\theta + \sin^4 2\theta - min\left\{\sin^2 2\theta, \sin^2 2\theta, \sin^4 2\theta\right\}\right] \\ &= \frac{1}{2}\sin^2 2\theta, \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} D_G^{min}(\rho') &= \frac{1}{4}\left[\sin^2 2\theta + \sin^2 2\theta + \sin^4 2\theta - max\left\{\sin^2 2\theta, \sin^2 2\theta, \sin^4 2\theta\right\}\right] \\ &= \frac{1}{4}\left[\sin^2 2\theta + \sin^4 2\theta\right] \end{aligned} \tag{4.44}$$

Therefore, from Eq. (4.42) and Eq. (4.43), one can see that $B_C\left(\rho'\right)_{opt} \leq 4\sqrt{D_G^{max}(\rho')}$. This clearly validates the results obtained in Eq. (4.38). In addition, it is also evident that the maximum value of geometric discord can be easily computed from the optimum value of Bell-Cumulant operator. Hence, the Bell-Cumulant inequality is a very useful tool to detect nonclassical correlations in pure bipartite systems. Further, from Eq. (4.41) and Eq. (4.43), it is also clear that $D_G(\rho') = D_G^{max}(\rho')$. Figure 4.3 demonstrates the maximum and minimum value of geometric discord for pure two-qubit entangled states using the results evaluated in this section. Figure 4.3 also exhibits the validity of our analytical results computed in this section.

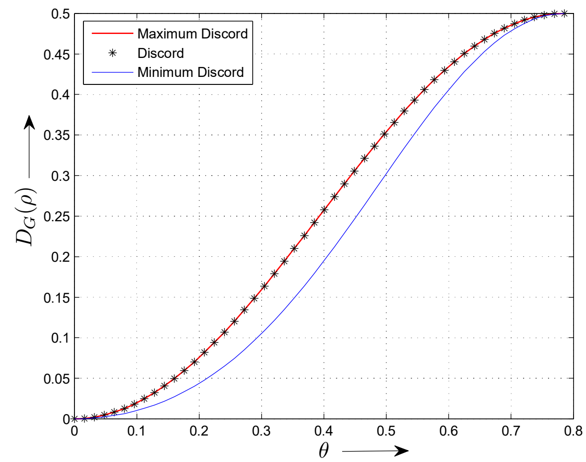

Figure 4.3 : A comparison between estimated values of geometric discord, and maximum and minimum value of geometric discord as evaluated using correlation coefficients with respect to the state parameter θ.

(2) For the next example, we consider two-qubit Werner states [Werner, 1989], defined in Eq. (2.41). The geometric discord of Werner states is evaluated as

$$D_G(\rho_w) = \frac{x^2}{2} \tag{4.45}$$

Similarly, the correlation matrix C_{ρ_w} for two-qubit Werner states ρ_w is given as

$$C_{\rho_w} = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -x \end{pmatrix} \tag{4.46}$$

Since all eigenvalues of matrix ξ_{ρ_w} are equal, i.e., $\vartheta^1 = \vartheta^2 = \vartheta^3 = x^2$, the expressions for maximum and minimum geometric discord are given as

$$D_G^{max}(\rho_w) = D_G^{min}(\rho_w) = \frac{x^2}{2} \tag{4.47}$$

Further, from Eq. (4.24) and Eq. (4.47), we also have $B_C(\rho_w)_{opt} \leq 4\sqrt{D_G^{max}(\rho_w)} = 2\sqrt{2}x$. Figure 4.4 further demonstrates that the maximum and minimum values of geometric discord are equal to geometric discord in case of Werner states due to the fact that eigenvalues of the matrix ξ_ρ are equal.

(3) As earlier, we now analyse Horodecki states [Horodecki, 1996] as defined in Eq. (2.44). Clearly, the correlation matrix C_{ρ_h} for Horodecki states is expressed as

$$C_{\rho_h} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a^2 \end{pmatrix}, \tag{4.48}$$

and from Eq. (4.34) and Eq. (4.35), the maximum and minimum value of geometric discord can be also be evaluated as

$$D_G^{max}(\rho_h) = \frac{a^2}{2} \tag{4.49}$$

$$D_G^{min}(\rho_h) = \frac{(a^2 + a^4)}{4} \tag{4.50}$$

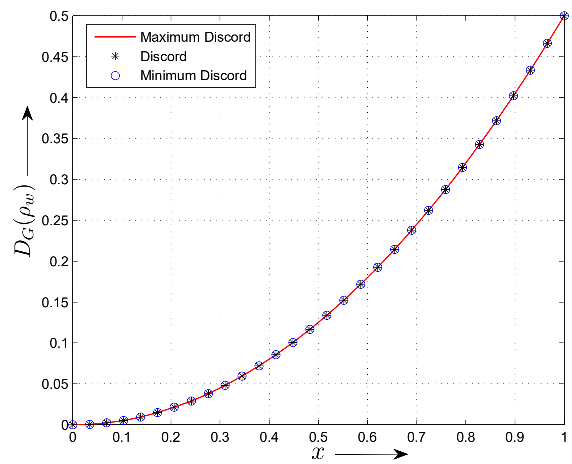

Figure 4.4 : A comparison between estimated values of geometric discord, and maximum and minimum value of geometric discord as evaluated using correlation coefficients with respect to the state parameter of the Werner state.

Interestingly, Figure 4.5 shows that the value of geometric discord always lies between the

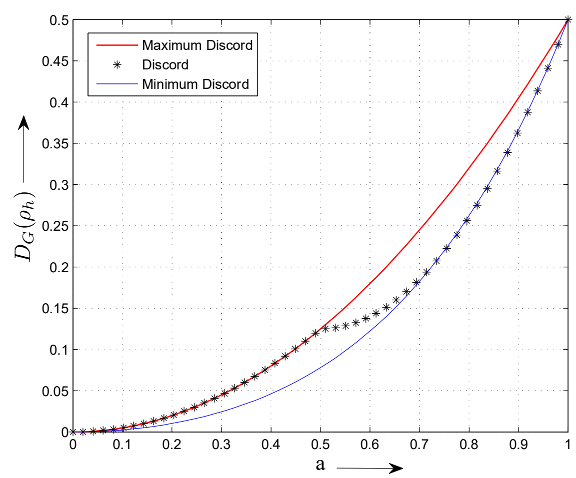

Figure 4.5 : A comparison between estimated values of geometric discord, and maximum and minimum value of geometric discord as evaluated using correlation coefficients with respect to the state parameter of Horodecki states.

maximum and minimum values of geometric discord for Horodecki states as evaluated using the correlation matrix C_ρ.

(4) We further consider another important class of two-qubit mixed states, as defined in Eq. (4.26), for which the correlation matrix $C_{\rho_{wc}}$ and expressions for maximum and minimum geometric discord can be given as

$$C_{\rho_{wc}} = \begin{pmatrix} (1-f) & 0 & 0 \\ 0 & (1-f) & 0 \\ 0 & 0 & -(1-f)^2 \end{pmatrix}, \tag{4.51}$$

$$D_G^{max}(\rho_{wc}) = \frac{(f-1)^2}{2}, \tag{4.52}$$

$$\text{and } D_G^{min}(\rho_{wc}) = \frac{(f-1)^2(f^2-2f+2)}{4}, \tag{4.53}$$

respectively. Figure 4.6 suggests that the value of geometric discord evaluated using correlation coefficients lies between the maximum and minimum value of geometric discord for the above class of states as well.

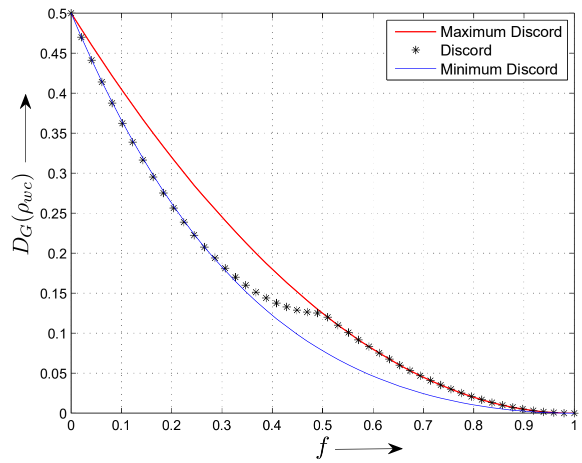

Figure 4.6 : A comparison between estimated values of geometric discord, and maximum and minimum value of geometric discord as evaluated using correlation coefficients with respect to the state parameter of the state proposed by WenChao Ma *et al.*

(5) As in the previous section, we finally analyse our proposed class of two-qubit mixed states, as defined in Eq. (4.28), such that the correlation matrix C_ϱ for ϱ is given as

$$C_\varrho = \begin{pmatrix} \frac{2}{N} & 0 & 0 \\ 0 & \frac{-2}{N} & 0 \\ 0 & 0 & \frac{4}{N^2} \end{pmatrix} \tag{4.54}$$

Moreover,

$$D_G^{max}(\rho_{wc}) = \frac{1}{2N^2} \tag{4.55}$$

$$D_G^{min}(\rho_{wc}) = \frac{(1+N^2)}{4N^4} \tag{4.56}$$

Similar to other cases, Figure 4.7 again confirms the validity of results evaluated in this section. Hence, from above analysis, one can conclude that analytical computation of the maximum and minimum values of geometric discord using statistical correlations coefficients can be used as an alternative to capture nonclassical correlations in an underlying two-qubit state.

4.5 MODIFIED SVETLICHNY INEQUALITY FOR THREE-QUBIT STATES

In the previous section, to quantify the nonlocal correlations in bipartite systems, we derived an analytical expression to modify the Bell-CHSH inequality using statistical correlation

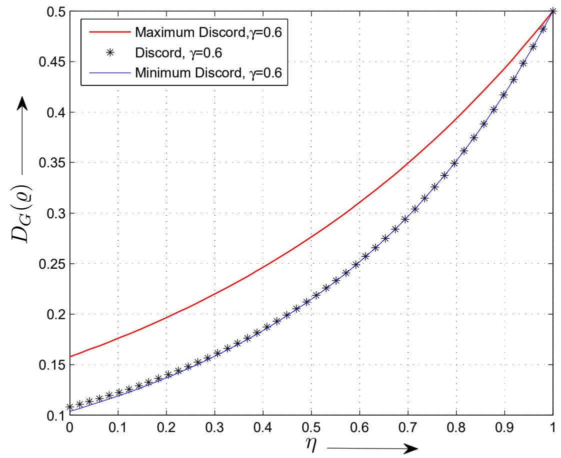

Figure 4.7 : A comparison between estimated values of geometric discord, and maximum and minimum value of geometric discord as evaluated using correlation coefficients with respect to the state parameter of the proposed class of states in chapter-2.

coefficients indicating the degree of correlations between the individual qubits. In this section, we modify the Svetlichny inequality using the statistical correlation coefficient for the characterization of nonlocality in three-qubit systems. Further, we derive an analytical relation between the modified Svetlichny inequality and three-tangle (τ), which measures the genuine tripartite entanglement for generalized GHZ states. As discussed earlier, due to the increased complexity in multiqubit systems, the characterization of nonlocality in three-qubit generalized GHZ states is much more complex in comparison to analysing nonlocal correlations in pure two-qubit states. For example, in case of three-qubit systems, to confirm the presence of genuine quantum correlations between three qubits, one needs to distinguish between bi-separable vs genuine tripartite nonlocality. In general, three-qubit entangled resources violating the Svetlichny inequality are considered to be useful resources for quantum information and computation.

The Svetlichny inequality for a three-qubit system as defined in Eq. (1.26), is represented as

$$\begin{aligned} |S| = \big|E(ABC) + E(ABC') &+ E(AB'C) - E(AB'C') \\ &+ E(A'BC) - E(A'BC') - E(A'B'C) - E(A'B'C')\big| \leq 4 \end{aligned}$$

where measurement operators A or A', B or B', and C or C' are defined in previous chapters. The above inequality is violated by three-qubit quantum systems exhibiting genuine tripartite nonlocality, and satisfied by all the separable and bi-separable systems. The three-qubit generalized GHZ states (ρ_g), represented in Eq. (3.1), violate the Svetlichny inequality only for $\tau(\rho_g) > \frac{1}{2}$, where $\tau(\rho_g) = \sin^2 2\theta$ measures the genuine tripartite entanglement in GHZ class. In general, the Svetlichny inequality cannot identify nonlocal correlations in a large set of GHZ class states, i.e., for states with $\tau(\rho_g) < \frac{1}{2}$. Hence, to characterize nonlocal correlations in tripartite systems, we modify the Svetlichny inequality using three-qubit correlation coefficients [Fano, 1983; Kumar and Krishnan, 2009]. Similar to the case of two-qubit states, we redefine the original

Svetlichny operator as

$$S_C = E\left(ABC\right)_c + E\left(ABC'\right)_c + E\left(AB'C\right)_c - E\left(AB'C'\right)_c + E\left(A'BC\right)_c - E\left(A'BC'\right)_c - E\left(A'B'C\right)_c - E\left(A'B'C'\right)_c \tag{4.57}$$

where

$$E(ABC)_c = E(ABC) - E(A)E(BC) - E(B)E(AC) - E(C)E(AB) + 2E(A)E(B)E(C) \tag{4.58}$$

and likewise for similarly defined terms. Analogous to the Bell-Cumulant inequality, the maximum classical value of modified operator in Eq. (4.57) using the extremal strategy is achieved as zero. Hence, the modified Svetlichny inequality can be expressed as

$$|S_C| = \left|E\left(ABC\right)_c + E\left(ABC'\right)_c + E\left(AB'C\right)_c - E\left(AB'C'\right)_c + E\left(A'BC\right)_c - E\left(A'BC'\right)_c - E\left(A'B'C\right)_c - E\left(A'B'C'\right)_c\right| \leq 0 \tag{4.59}$$

Clearly, the equality can be achieved by separable and bi-separable systems and the inequality must be violated by all entangled pure three-qubit states.

In order to evaluate the maximum expectation value of the modified Svetlichny operator S_C using quantum strategy, we start with three-qubit generalized GHZ states $|\Psi_g\rangle$ shared between Alice, Bob, and Charlie. As in the previous case, for optimizing the expectation value of modified Svetlichny operator, we first need to define S_C in terms of spin projection operators and unit vectors as defined in Eq. (3.8). Moreover, for evaluating the maximum expectation value of S_C, we further need to consider a pair of two mutually orthogonal unit vectors $R = \hat{r}.\sigma_2$ and $R' = \hat{r}'.\sigma_2$ such that $\hat{b} + \hat{b}' = 2\cos\chi.\hat{r}$, $\hat{b} - \hat{b}' = 2\sin\chi.\hat{r}'$, and

$$\hat{r}.\hat{r}' = \cos\theta_r \cos\theta_{r'} + \sin\theta_r \sin\theta_{r'} \cos\left(\phi_r - \phi_{r'}\right) = 0 \tag{4.60}$$

Following the above discussion, Eq. (4.57) can be re-expressed as

$$|S_C| = 2|E\left(ARC\right)_c \cos\chi + E\left(AR'C'\right)_c \sin\chi + E\left(A'R'C\right)_c \sin\chi - E\left(A'RC'\right)_c \cos\chi| \tag{4.61}$$

Eq. (4.61) when maximized with respect to χ, gives

$$|S_C| \leq 2\left|\{E\left(ARC\right)_c^2 + E\left(AR'C'\right)_c^2\}^{1/2} + \{E\left(A'R'C\right)_c^2 + E\left(A'RC'\right)_c^2\}^{1/2}\right| \tag{4.62}$$

Here, similar to the previous cases, we used the fact that

$$x\cos\theta_1 + y\sin\theta_1 \leq (x^2 + y^2)^{1/2} \tag{4.63}$$

In order to evaluate the maximum expectation value of S_C with respect to generalized GHZ states, we now consider calculating the first term $E\left(ARC\right)_c$ in Eq. (4.62), such that

$$E(ARC)_c = \sin 2\theta\left[-\sin 4\theta \cos\theta_a \cos\theta_r \cos\theta_c + \sin\theta_a \sin\theta_r \sin\theta_c \cos\phi_{arc}\right] \tag{4.64}$$

where $\cos\phi_{arc} = \cos(\phi_a + \phi_r + \phi_c)$. The expectation value $E\left(ARC\right)_c$ when maximized with respect to θ_c gives

$$[E(ARC)_c]_{max} = \sin 2\theta\left\{\sin^2 4\theta \cos^2\theta_a \cos^2\theta_r + \sin^2\theta_a \sin^2\theta_r \cos^2\phi_{arc}\right\}^{\frac{1}{2}} \tag{4.65}$$

In Eq. (4.65), we further assume $\cos^2\phi_{arc} = 1$, such that

$$[E(ARC)_c]_{max} = \sin 2\theta \left\{\sin^2 4\theta \cos^2\theta_a \cos^2\theta_r + \sin^2\theta_a \sin^2\theta_r\right\}^{\frac{1}{2}} \tag{4.66}$$

The maximum values of the operators $E(AR'C')$, $E(A'R'C)$ and $E(A'RC')$ can also be evaluated in a similar fashion with primes on required angles. Therefore, from Eq. (4.62), we get

$$\begin{aligned} S_C(\rho_g)_{max} &= 2\sin 2\theta \left\{\sin^2 4\theta \left(\cos^2\theta_r + \cos^2\theta_{r'}\right)\cos^2\theta_a + \left(\sin^2\theta_r + \sin^2\theta_{r'}\right)\sin^2\theta_a\right\}^{\frac{1}{2}} \\ &+ 2\sin 2\theta \left\{\sin^2 4\theta \left(\cos^2\theta_r + \cos^2\theta_{r'}\right)\cos^2\theta_{a'} + \left(\sin^2\theta_r + \sin^2\theta_{r'}\right)\sin^2\theta_{a'}\right\}^{\frac{1}{2}} \end{aligned} \tag{4.67}$$

Considering the orthogonality relation between unit vectors r and r' in Eq. (4.60), the maximum of $\cos^2\theta_r + \cos^2\theta_{r'}$ is 1, while the maximum of $\sin^2\theta_r + \sin^2\theta_{r'}$ is 2. Therefore, inserting these values in Eq. (4.67), it can be re-expressed as

$$S_C(\rho_g)_{max} = 2\sin 2\theta \left[\left\{\sin^2 4\theta \cos^2\theta_a + 2\sin^2\theta_a\right\}^{\frac{1}{2}} + \left\{\sin^2 4\theta \cos^2\theta_{a'} + 2\sin^2\theta_{a'}\right\}^{\frac{1}{2}}\right] \tag{4.68}$$

Further, we know that

$$x\cos^2\theta_1 + y\sin^2\theta_1 \leq \begin{cases} x, & x \geq y \\ y, & x \leq y \end{cases} \tag{4.69}$$

where the first equality results when θ_1=0 or π and the second equality results when θ_1=$\frac{\pi}{2}$. Using the above fact, Eq. (4.68) can be maximized with respect to θ_a and $\theta_{a'}$. Furthermore, the maximum value of $\sin^2 4\theta$ is 1, therefore, $\sin^2 4\theta$ is always less than 2, i.e., $(\sin^2 4\theta < 2)$, and hence the optimum expectation value of modified Svetlichny operator for generalized GHZ states, considering $\theta_a = \theta_{a'} = \frac{\pi}{2}$ is

$$S_C(\rho_g)_{opt} \leq 4\sqrt{2}\sin 2\theta = 4\sqrt{2\tau(|\Psi_g\rangle)} \tag{4.70}$$

Figure 4.8 clearly demonstrates the relationship between $S_C(\rho_g)_{opt}$ and three tangle $\tau(\rho_g)$ for

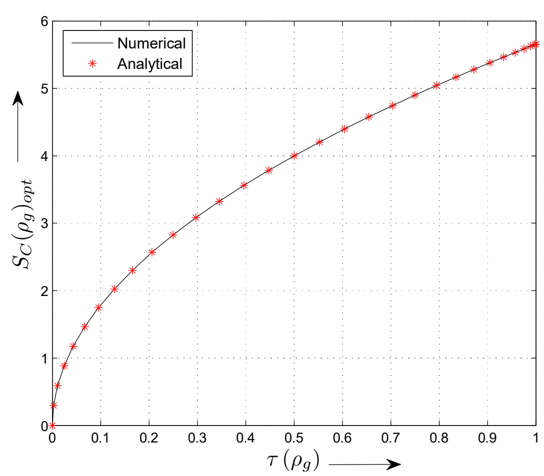

Figure 4.8 : Estimation of maximum expectation value of $S_C(\rho_g)_{opt}$ with respect to 3-tangle $\tau(\rho_g)$ of three-qubit generalized GHZ states.

generalized GHZ states. It shows that if $\tau(\rho_g)$ is varied from 0 to 1 then the optimum value of modified Svetlichny operator first increases, and then obtains a maximum value at $\tau(\rho_g) = 1$. The maximum value of modified Svetlichny operator is $4\sqrt{2}$, achieved for a maximally entangled three-qubit pure GHZ state. Our analysis shows an excellent agreement between the analytical and numerical results. Unlike the original Svetlichny inequality, the modified Svetlichny inequality identifies nonlocal correlations in all the generalized GHZ states for the complete range of the three-qubit entanglement measure τ.

4.6 SUMMARY

We have analysed the nonlocal correlations in bipartite and tripartite systems, where the original Bell-type inequalities fail to capture nonlocality, while the correlation coefficient based Bell-type Cumulant inequalities give credible results. In this chapter, we estimated the optimum expectation value of Bell-Cumulant operator for classical and quantum strategy. We also evaluated a necessary and sufficient condition for the violation of Bell-Cumulant inequality by an arbitrary two-qubit state. Moreover, we derived an analytical expression to calculate the maximum and minimum value of geometric discord using properties of correlations coefficients. For further analysing the nonclassical correlations, we demonstrated an analytical relation between the Bell-Cumulant inequality and geometric discord. Further, we also extended our analysis to characterize nonlocality in three-qubit GGHZ class using a modified Svetlichny inequality. Our results show that correlation coefficients can be used as quantifiers to characterize and analyse the usefulness of mixed bipartite and tripartite systems for quantum communication and information processing. Our study presents an important contribution for deeper analysis and understanding the importance of nonlocal correlations in an underlying two- or three-qubit state for quantum information and computation.

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