5
Bell's Inequality With Biased Experimental Settings

5.1 INTRODUCTION

The classification and quantification of entanglement and nonlocality have received a significant attention due to its usefulness as a resource in quantum information and computation. In order to understand and characterize the exact nature of nonlocal correlations derived from entanglement, one need to analyse as to when nonlocality is useful in context of nonlocal tasks in comparison to classical correlations. For example, there are instances, where nonlocality does not provide much advantage or perform better than classical mechanics [Linden *et al.*, 2007; Almeida *et al.*, 2010] for computing nonlocal tasks or where quantum theory may succeed against classical theory only under certain scenario [Allcock *et al.*, 2009]. In the context of quantum information, the uncertainty relations shed further light into the nature and applicability of nonlocal correlations [Deutsch, 1983; DiVincenzo *et al.*, 2004; Renes and Boileau, 2009; Berta *et al.*, 2010; Oppenheim and Wehner, 2010; Wehner and Winter, 2010]. A particularly useful relation that distinguishes between degree of nonlocality in classical, quantum and super-quantum correlations is fine-grained uncertainty relation proposed by Oppenheim and Wehner [Oppenheim and Wehner, 2010]. Another key insight to the degree of nonlocality of the underlying physical theory was provided by Lawson *et al.* [Lawson *et al.*, 2010] where they tried to understand the nature of quantum entanglement in context of a nonlocal game. They showed that for a class of Bell-CHSH inequality, if two players choose their measurements with a certain bias then for a certain range of biasing parameters, quantum correlations do not offer any advantage over classical ones.

In this chapter, we analyse the Bell-CHSH inequality in the settings of a biased nonlocal game to revisit the question of usefulness of quantum entanglement and nonlocal correlation as a resource for quantum information processing. Using standard methods with the help of Tsirelson's inequality [Tsirelson, 1980] and Horodecki's separability criterion [Horodecki, 1995], we optimize the expectation value of Bell-CHSH operator for classical and quantum theory to distinguish between classical, quantum and super-quantum correlations in a biased experimental scenario. This allows us to demonstrate that the quantum bound is always greater than the classical bound, for the complete range of biasing parameters. We also use fine-grained uncertainty [Oppenheim and Wehner, 2010; Dey *et al.*, 2013] relation to analyse the maximum winning probability of the nonlocal game using classical and quantum theory, which further confirms that quantum correlations are advantageous in comparison to classical correlations. We show this by evaluating the ranges of biasing parameters, where the fine-grained uncertainty relation can be a useful measure to detect nonlocal correlations in any bipartite system. We finally discuss the Bell-CHSH inequality for pure and mixed states under biased scenario. For this, we find a condition for the violation of Bell-CHSH inequality by an arbitrary spin-1/2 state under biased measurement settings. Our analysis shows that all the pure states violate the Bell-CHSH inequality [Clauser *et al.*, 1969] under biased scenario. For mixed states, the two-qubit states such as Werner state [Werner, 1989], Horodecki's state [Horodecki, 1996], a state proposed by WenChao Ma *et al.* [Ma *et al.*, 2015] and a class of states proposed in chapter-2 exhibit similar nonlocal behaviour in biased as well as unbiased experimental set-up.

5.2 CLASSICAL BOUND FOR THE CHSH OPERATOR WITH BIASED EXPERIMENTAL SET-UP

Bell type inequalities provide a way to understand the fundamental differences between nonlocality and local hidden variable theories. For bipartite systems the CHSH inequality [Clauser *et al.*, 1969] is given by

$$\frac{1}{4}\left[E\left(A_0B_0\right)+E\left(A_0B_1\right)+E\left(A_1B_0\right)-E\left(A_1B_1\right)\right]\leq\frac{1}{2} \tag{5.1}$$

In this case, Alice and Bob choose their measurements as A_0 or A_1, and B_0 or B_1, respectively, with equal probability of $\frac{1}{2}$ such that

$$A_0=\sigma_1.\hat{a},\ A_1=\sigma_1.\hat{a}',\ B_0=\sigma_2.\hat{b},\ and\ B_1=\sigma_2.\hat{b}' \tag{5.2}$$

where $\hat{a}$, $\hat{a}'$, $\hat{b}$, $\hat{b}'$ are unit vectors, σ_i's are spin projection operators, and $E(A_iB_j)$ represents average value of product of measurement outcomes of Alice and Bob with $i,j=0,1$. Since the measurement outcomes of operators A_0, A_1, B_0 or B_1 are ±1, Eq. (5.1) is valid for all different measurement outcomes. However, the above scenario is not that simple if Alice and Bob choose their measurement operators with a certain bias. For example, if Alice chooses to perform her measurement A_0 with probability p and A_1 with probability $1-p$, and Bob chooses to perform his measurement B_0 with probability q and B_1 with probability $1-q$ then the Bell operator can be expressed as

$$\begin{aligned} B^{p,q} &= pqE\left(A_0B_0\right)+p\left(1-q\right)E\left(A_0B_1\right)+q\left(1-p\right)E\left(A_1B_0\right)-\left(1-p\right)\left(1-q\right)E\left(A_1B_1\right)\\ &= pq(A_0B_0-A_0B_1-A_1B_0-A_1B_1)+p(A_0+A_1)B_1+qA_1(B_0+B_1)-A_1B_1 \end{aligned} \tag{5.3}$$

Interestingly, the introduction of biasing parameters p and q lead to four different regions to be considered based on the measurement outcomes of operators A_0, A_1, B_0, B_1 and on the values of p and q, such that the CHSH inequality modifies as

- **Case I:** If $p\leq\frac{1}{2}$ and $q\leq\frac{1}{2}$ then the CHSH inequality in Eq. (5.3) can be represented as

$$B_I^{p,q}\leq 1-2pq \tag{5.4}$$

- **Case II:** If $p\geq\frac{1}{2}$ and $q\leq\frac{1}{2}$ then the CHSH inequality in Eq. (5.3) can be represented as

$$B_{II}^{p,q}\leq 1-2q\left(1-p\right) \tag{5.5}$$

- **Case III:** If $p\leq\frac{1}{2}$ and $q\geq\frac{1}{2}$ then the CHSH inequality in Eq. (5.3) can be represented as

$$B_{III}^{p,q}\leq 1-2p\left(1-q\right) \tag{5.6}$$

- **Case IV:** If $p\geq\frac{1}{2}$ and $q\geq\frac{1}{2}$ then the CHSH inequality in Eq. (5.3) can be represented as

$$B_{IV}^{p,q}\leq 1-2\left(1-p\right)\left(1-q\right) \tag{5.7}$$

In this chapter, we show that quantum strategies are always better that classical ones in all the above four regions.

We begin our discussion with describing the CHSH inequality as a nonlocal quantum game. A key insight to the properties of nonlocal correlations was provided by Lawson *et al.* [Lawson *et al.*, 2010] where they introduced a biased nonlocal quantum game in which two parties decide to choose their measurements with some probability for a CHSH game [Clauser *et al.*, 1969]. In an unbiased scenario, the left side of CHSH inequality given in Eq. (5.1) can be considered as an average score which is calculated over all the rounds where the factor $\frac{1}{4}$ is considered as a probability to choose a particular pair of measurement A_i and B_j. Therefore, Eq. (5.1) represents the maximum possible score that can be achieved using a classical strategy. Clearly, if this game is played using quantum strategies then this inequality is violated, suggesting that one can perform better if the measurements are performed on entangled quantum systems. Alternately, this game can also be interpreted as an input-output problem in computer science and engineering [Dey *et al.*, 2013] where Alice and Bob have input binary variables s and t, and output binary variables x and y, respectively. If Alice chooses A_0 then s takes the value 0, and if Alice chooses A_1 then s takes the value 1. The output variable x takes values 0 or 1 depending on whether Alice's measurement outcome is $+1$ or -1. The values for Bob's input and output variables can be defined in a similar fashion. The CHSH game's winning conditions for Alice's and Bob's particles are

$$x \oplus y = st \tag{5.8}$$

where $\oplus$ denotes addition modulo 2. Therefore, the average score that can be achieved by Alice's and Bob's particles can then be represented as

$$\sum_{s,t=0}^{1} P(s,t) P(x \oplus y = st | st) \tag{5.9}$$

where $P(s,t)$ represents the probability that input pair is (s,t). For simplicity, we take $P(s,t) = P(s)P(t)$, where $P(s=0) = p, P(s=1) = (1-p), P(t=0) = q$, and $P(t=1) = (1-q)$. As we have shown above, we need to consider the average score in four different regions as given in Eq. (5.4) - Eq. (5.7). Lawson *et al.* [Lawson *et al.*, 2010] computed the maximum classical and quantum scores for this game, where they only considered $p, q \geq 1/2$. In subsequent sections, we analyse the whole $[p,q]$ space and compare the maximum classical and quantum score for the CHSH game.

5.3 QUANTUM BOUND FOR THE CHSH OPERATOR WITH BIASED EXPERIMENTAL SET-UP

In this section, we consider the nonlocal biased game with quantum strategy in which Alice and Bob initially share a bipartite entangled state. In general, when measurements are performed on entangled systems, inequality in Eq. (5.1) is violated due to the existence of nonlocal correlations between entangled particles. Quantum theory imposes a limit on such nonlocal correlations, and Tsirelson [Tsirelson, 1980] showed that the maximum limit of this violation is $\frac{1}{\sqrt{2}}$. Precisely, Tsirelson showed that

$$(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1)^2 = 4I + [A_0, A_1] \otimes [B_0, B_1] \tag{5.10}$$

Using Tsirelson's inequality, it is easy to show that

$$\langle A_0 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle + \langle A_1 \otimes B_0 \rangle - \langle A_1 \otimes B_1 \rangle \leq 2\sqrt{2} \tag{5.11}$$

We first analyse Tsirelson's inequality for a biased measurement set-up such that Eq. (5.10) can be expressed as

$$\begin{aligned} \left(B_{CHSH}^{(p,q)}\right)^2 &= [pq\,(A_0 \otimes B_0) + p\,(1-q)\,(A_0 \otimes B_1) + (1-p)\,q\,(A_1 \otimes B_0) \\ &- (1-p)\,(1-q)\,(A_1 \otimes B_1)]^2 \\ &= \left[\left(\left(p^2 + (1-p)^2\right)\left(q^2 + (1-q)^2\right)\right) I + q\,(1-q)\,(2p-1)\,\{B_0, B_1\}\right. \\ &+ \left. p\,(1-p)\,(2q-1)\,\{A_0, A_1\} + pq\,(1-p)\,(1-q)\,[A_0, A_1] \otimes [B_0, B_1]\right] \end{aligned} \tag{5.12}$$

where $\{A_0, A_1\}$ is an anti-commutator of operators A_0 and A_1, and $[A_0, A_1]$ is a commutator of operators A_0 and A_1. The commutator and anti-commutator for operators B_0 and B_1 are defined in a similar fashion. Thus, the maximum expectation value of Eq. (5.12) can be given as

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{2} = \max_{\hat{a},\hat{a}',\hat{b},\hat{b}'} 2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)\,\hat{a}\cdot\hat{a}'+q(1-q)(2p-1)\,\hat{b}\cdot\hat{b}'\right] \tag{5.13}$$

Since $\hat{a}\cdot\hat{a}' = \|a\|\|a'\|\cos\theta_1$ and $\hat{b}\cdot\hat{b}' = \|b\|\|b'\|\cos\theta_2$, hence Eq. (5.13) when maximized with respect to θ_1 and θ_2 gives the maximum violation of CHSH inequality for quantum measurements in biased scenario as

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} = \sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)+q(1-q)(2p-1)\right]} \tag{5.14}$$

Similar to the case described above, based on the biasing parameters p and q we need to consider four different cases as shown below

- **Case I:** If $p \leq \frac{1}{2}$ and $q \leq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} = \sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)-p(1-p)(2q-1)-q(1-q)(2p-1)\right]} \tag{5.15}$$

- **Case II:** If $p \geq \frac{1}{2}$ and $q \leq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} = \sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)-p(1-p)(2q-1)+q(1-q)(2p-1)\right]} \tag{5.16}$$

- **Case III:** If $p \leq \frac{1}{2}$ and $q \geq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} = \sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)-q(1-q)(2p-1)\right]} \tag{5.17}$$

- **Case IV:** If $p \geq \frac{1}{2}$ and $q \geq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} = \sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)+q(1-q)(2p-1)\right]} \tag{5.18}$$

We have computed the maximum possible score in a CHSH game using classical and quantum strategies in Eqs. (5.4-5.7) and Eqs. (5.15-5.18), respectively. In both the cases, the score depends on biasing parameters p and q and can be divided into four different regions. Figure 5.1 clearly

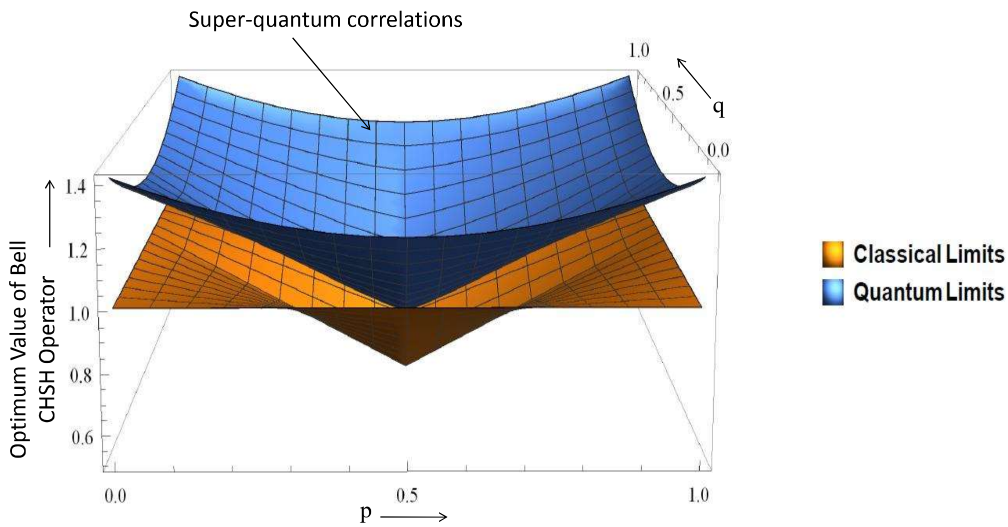


Figure 5.1 : Comparison of classical and quantum bounds of CHSH operator varying with parameters p,q in a biased experimental set-up.

demonstrates that for all $[p,q]$ space, the bound of the score of a CHSH game using quantum strategies is always greater than the score of CHSH game using classical strategies when we choose measurement operators with certain probability. Therefore, one can say that quantum correlations are better resources than classical correlations for whole range of biasing parameters p, q. Furthermore, Figure 5.1 also differentiates the classical, quantum and super-quantum correlations based on maximum possible value of CHSH function in biased experimental set-up.

5.4 FINE-GRAINED UNCERTAINTY RELATIONS

Alternately, classical, quantum and super-quantum correlations can be distinguished using fine-grained uncertainty relations as proposed by Oppenheim and Wehner [Oppenheim and Wehner, 2010]. For a CHSH nonlocal game, they established a relation between the upper bound of uncertainty relation, maximum winning probability, and degree of nonlocality associated with a physical theory. The upper bound of the CHSH nonlocal game with unbiased measurement settings in classical, quantum, and no-signalling theory was shown to be $\frac{3}{4}$, $\frac{1}{2}+\frac{1}{2\sqrt{2}}$, and 1 respectively. Moreover, if the winning probability of the game is less than 1, the outcome of the CHSH game is uncertain. The maximum winning probability for a nonlocal game with biased experimental set-up is given as [Oppenheim and Wehner, 2010; Dey *et al.*, 2013]

$$P^{game}\left(S,T,\rho_{ab}\right)=\frac{1}{2}\left[1+\left\langle B_{CHSH}^{(p,q)}\right\rangle_{max}^{\rho_{ab}}\right] \tag{5.19}$$

where $\left\langle B_{CHSH}^{(p,q)}\right\rangle_{max}^{\rho_{ab}}$ is the maximum expectation value of Bell-CHSH operator in Eq. (5.3) with respect to a bipartite system ρ_{ab} . Therefore, the fine-grained uncertainty can be used as a tool to distinguish between classical, quantum, and super-quantum correlations for the unbiased measurement settings. However, in biased measurement settings, fine-grained uncertainty succeeds only for certain range of biasing parameters. Ansuman *et al.* [Dey *et al.*, 2013] computed the maximum winning probability for this game in region IV where biasing parameters are $p \geq \frac{1}{2}$, and $q \geq \frac{1}{2}$. Although the other regions are symmetrical, here we consider the whole (p, q) space to compare our results in previous section and compute the maximum winning probability of the game in the underlying classical, and quantum theories using Eqs. (5.4-5.7) and Eqs. (5.15-5.18), such that

- **Case I:** For $0.19098 \leq p \leq \frac{1}{2}$, $0.19098 \leq q \leq \frac{1}{2}$, and $p \geq \left(\frac{1+2q^2-\sqrt{8q-16q^2+24q^3-12q^4-1}}{8q^2-4q+2}\right)$, the maximum winning probabilities with biased measurement settings in classical and quantum theories are

$$
\begin{aligned}
&P(S,T,\rho_{ab})|^{classical}_{max} = 1 - pq \\
&P(S,T,\rho_{ab})|^{quantum}_{max} = \\
&\quad \frac{1}{2}\left[1+\sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)-p(1-p)(2q-1)-q(1-q)(2p-1)\right]}\right]
\end{aligned}
\tag{5.20}
$$

- **Case II:** For $\frac{1}{2} \leq p < 0.80901$, $0.19098 < q \leq \frac{1}{2}$, and $p \leq \left(\frac{1+6q^2-4q+\sqrt{8q-16q^2+24q^3-12q^4-1}}{8q^2-4q+2}\right)$, the maximum winning probabilities with biased measurement settings in classical and quantum theories are

$$
\begin{aligned}
&P(S,T,\rho_{ab})|^{classical}_{max} = 1 - (1-p)q \\
&P(S,T,\rho_{ab})|^{quantum}_{max} = \\
&\quad \frac{1}{2}\left[1+\sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)-p(1-p)(2q-1)+q(1-q)(2p-1)\right]}\right]
\end{aligned}
\tag{5.21}
$$

- **Case III:** For $0.19098 < p \leq \frac{1}{2}$ and $\frac{1}{2} \leq q < 0.80901$, and $p \geq \left(\frac{3+2q^2-4q+\sqrt{3-16q^2+24q^3-12q^4}}{8q^2-12q+6}\right)$, the maximum winning probabilities with biased measurement settings in classical and quantum theories are

$$
\begin{aligned}
&P(S,T,\rho_{ab})|^{classical}_{max} = 1 - p(1-q) \\
&P(S,T,\rho_{ab})|^{quantum}_{max} = \\
&\quad \frac{1}{2}\left[1+\sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)-q(1-q)(2p-1)\right]}\right]
\end{aligned}
\tag{5.22}
$$

- **Case IV:** For $\frac{1}{2} \leq p < 0.80901, \frac{1}{2} \leq q < .80901$, and $p \leq \left(\frac{3+6q^2-8q+\sqrt{3-16q^2+24q^3-12q^4}}{8q^2-12q+6}\right)$, the maximum winning probabilities with biased measurement settings in classical and quantum theories are

$$P(S,T,\rho_{ab})|_{max}^{classical} = 1-(1-p)(1-q)$$
$$P(S,T,\rho_{ab})|_{max}^{quantum} = \frac{1}{2}\left[1+\sqrt{2\left[\left(p^2+(1-p)^2\right)\left(q^2+(1-q)^2\right)+p(1-p)(2q-1)+q(1-q)(2p-1)\right]}\right] \tag{5.23}$$

The winning probability reduces to the value $\frac{3}{4}$ in classical theory and to the value of $\frac{1}{2}+\frac{1}{2\sqrt{2}}$ in quantum theory for the unbiased case, when $p=\frac{1}{2}$ and $q=\frac{1}{2}$. Figure 5.2 illustrates a region, where

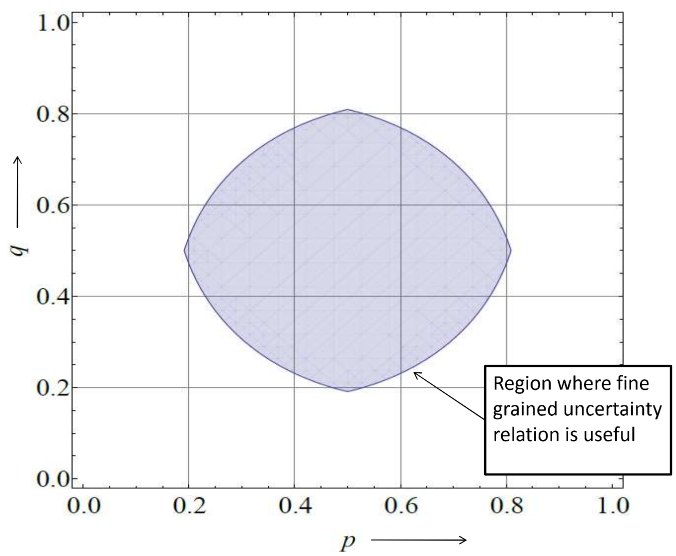


Figure 5.2 : Region in [p,q] space where fine-grained uncertainty can distinguish between quantum and classical correlations.

the upper bound of the fine grained uncertainty differentiates between the quantum and classical correlations.

The average score of this game for super quantum correlations or no-signalling theory [Barrett *et al.*, 2005] is

$$\begin{aligned} P^{game}|_{max}^{no-signalling} &= \sum_{s,t} p(s,t) \sum_{x,y} p(x,y|s,t) \\ &= pq+p(1-q)+q(1-p)+(1-p)(1-q) \\ &= 1 \end{aligned} \tag{5.24}$$

which is the same as upper bound of the fine-grained uncertainty relation. Here, one can easily observe that the winning probability of quantum theory is always greater than classical theory for the whole range of biasing parameters.

5.5 ARBITRARY SPIN$-\frac{1}{2}$ STATE AND THE CHSH INEQUALITY WITH BIASED EXPERIMENTAL SET-UP

The nonlocal properties of pure bipartite systems in an unbiased scenario is well defined. For example, all pure bipartite states violate the Bell-CHSH inequality [Gisin, 1991].However, due to complex nature of mixed entangled states, the nonlocal properties of such systems still surprise the research community [Werner, 1989; Horodecki, 1996; Munro *et al.*, 2001a; Ghosh *et al.*, 2001; Ma *et al.*, 2015]. Therefore, in this section, we analyse the nonlocal properties of pure and mixed states under biased scenario.

For an unbiased experimental set-up where Alice and Bob choose their measurements with equal probability, Horodecki *et al.* [Horodecki, 1995] proposed an effective criterion for violation of CHSH inequality by an arbitrary mixed spin-1/2 state. Any arbitrary spin-1/2 state can be represented using a density operator as

$$\rho = \frac{1}{4}(I\otimes I + \mathbf{r}\cdot\boldsymbol{\sigma}\otimes I + I\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \sum_{n,m=1}^{3} t_{nm}\sigma_n\otimes\sigma_m) \tag{5.25}$$

where I is a 2×2 identity operator, $\mathbf{r}$ and $\mathbf{s}$ represent polarization vectors of two spins, respectively, and coefficients t_{nm} form a real matrix which is denoted by M_ρ such that $t_{nm} = Tr(\rho.\sigma_n\otimes\sigma_m)$. In order to facilitate further discussions, we define a diagonalizable symmetric matrix $E_\rho = M_\rho^T M_\rho$ where M_ρ^T represents transpose of M_ρ. We further define e^1_{max} and e^2_{max} as two positive, greatest, and real eigenvalues of E_ρ. We now proceed to demonstrate the criterion for violation of the CHSH inequality under biased scenario. For this, we use the general form of Bell operators [Popescu and Rohrlich, 1992] associated with the CHSH inequality in the following form

$$B_{CHSH} = \quad \hat{a}\cdot\boldsymbol{\sigma}\otimes(\hat{b}+\hat{b}') + \hat{a}'\cdot\boldsymbol{\sigma}\otimes(\hat{b}-\hat{b}') \tag{5.26}$$

Eq. (5.26) can be re-expressed under biased scenario as

$$B^{(p,q)}_{CHSH} = p\hat{a}\cdot\boldsymbol{\sigma}\otimes\left(q\hat{b}+(1-q)\hat{b}'\right) + (1-p)\hat{a}'\cdot\boldsymbol{\sigma}\otimes\left(q\hat{b}-(1-q)\hat{b}'\right) \tag{5.27}$$

We now evaluate the mean value of $B^{(p,q)}_{CHSH}$ for an arbitrary mixed bipartite state ρ, such that

$$\left\langle B^{(p,q)}_{CHSH}\right\rangle^{\rho} = \left(p\hat{a}, M_\rho\left(q\hat{b}+(1-q)\hat{b}'\right)\right) + \left((1-p)\hat{a}', M_\rho\left(q\hat{b}-(1-q)\hat{b}'\right)\right) \tag{5.28}$$

For maximizing $\left\langle B^{(p,q)}_{CHSH}\right\rangle^{\rho}$, we introduce two orthogonal unit vectors $\hat{c},\hat{c}'$ such that

$$\hat{b} = \cos\theta\hat{c} + \sin\theta\hat{c}', \qquad \hat{b}' = \cos\theta\hat{c} - \sin\theta\hat{c}' \tag{5.29}$$

where $\theta \in \left[0, \frac{\Pi}{2}\right]$. Therefore, the maximum expectation value of CHSH operator is given by

$$\begin{aligned}
\left\langle B^{(p,q)}_{CHSH}\right\rangle^{\rho}_{max} &= \\
&\max_{\hat{a},\hat{a}',\hat{c},\hat{c}',\theta}\left(p\hat{a}, M_\rho\left(\cos\theta\hat{c}+(2q-1)\sin\theta\hat{c}'\right)\right) + \left((1-p)\hat{a}', M_\rho\left((2q-1)\cos\theta\hat{c}+\sin\theta\hat{c}'\right)\right) \\
&= \max_{\hat{c},\hat{c}',\theta}\left[\left(p+(1-p)(2q-1)\right)\|M_\rho\hat{c}\|\cos\theta + \left(p(2q-1)+(1-p)\right)\|M_\rho\hat{c}'\|\sin\theta\right] \\
&= \max_{\hat{c},\hat{c}'}\sqrt{\left(1-2(1-p)(1-q)\right)^2\|M_\rho\hat{c}\|^2 + \left(1-2p(1-q)\right)^2\|M_\rho\hat{c}'\|^2} \\
&= \sqrt{\left(1-2(1-p)(1-q)\right)^2 e^1_{max} + \left(1-2p(1-q)\right)^2 e^2_{max}}
\end{aligned} \tag{5.30}$$

The two largest eigenvalues of E_ρ can be defined as $e^1_{max} = \|M_\rho\hat{c}\|^2_{max}$ and $e^2_{max} = \|M_\rho\hat{c'}\|^2_{max}$ where $\hat{c}_{max}$ and $\hat{c}'_{max}$ are chosen as eigenvectors related to the two largest eigenvalues of real symmetric matrix

E_ρ, and maximum is taken over all unit and mutually orthogonal pair of vectors $(\hat{c}, \hat{c}')$. Similar to the arguments given above, we get four different regions based on the biasing parameters as

- **Case I:** If $p \leq \frac{1}{2}$ and $q \leq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ for arbitrary mixed spin-$\frac{1}{2}$ state is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho} = \sqrt{(1-2pq)^2 e_{max}^1 + (1-2q(1-p))^2 e_{max}^2} \tag{5.31}$$

- **Case II:** If $p \geq \frac{1}{2}$ and $q \leq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ for arbitrary mixed spin-$\frac{1}{2}$ state is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho} = \sqrt{(1-2q(1-p))^2 e_{max}^1 + (1-2pq)^2 e_{max}^2} \tag{5.32}$$

- **Case III:** If $p \leq \frac{1}{2}$ and $q \geq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ for arbitrary mixed spin-$\frac{1}{2}$ state is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho} = \sqrt{(1-2p(1-q))^2 e_{max}^1 + (1-2(1-p)(1-q))^2 e_{max}^2} \tag{5.33}$$

- **Case IV:** If $p \geq \frac{1}{2}$ and $q \geq \frac{1}{2}$, then the maximum expectation value of $\left\langle B_{CHSH}^{(p,q)} \right\rangle$ for arbitrary mixed spin-$\frac{1}{2}$ state is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho} = \sqrt{(1-2(1-p)(1-q))^2 e_{max}^1 + (1-2p(1-q))^2 e_{max}^2} \tag{5.34}$$

In order to address the nonlocality of pure and mixed states under biased scenario, we now illustrate few specific examples of entangled bipartite states.

(1) For evaluating the violation of Bell-CHSH inequality by pure states under biased scenario, we define a pure two-qubit state $|\psi\rangle = \lambda_1 |00\rangle + \lambda_2 |11\rangle$ in term of a density operator as given in Eq. (4.22). From Eq. (4.22), one can compute the two largest eigenvalues of real symmetric matrix $E_{\rho'}$ as $e_{max}^1 = 1$, and $e_{max}^2 = 4\lambda_1^2\lambda_2^2$. If we consider **Case I**, where $0 \leq p \leq \frac{1}{2}$ and $0 \leq q \leq \frac{1}{2}$, then the maximum classical bound of the CHSH function is $(1-2pq)$, and the maximum quantum bound of the CHSH function for any pure state is

$$\begin{aligned}\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho'} &= \sqrt{(1-2pq)^2 + (1-2q(1-p))^2 * 4\lambda_1^2\lambda_2^2} \\ &= (1-2pq)\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2 * 4\lambda_1^2\lambda_2^2}\end{aligned} \tag{5.35}$$

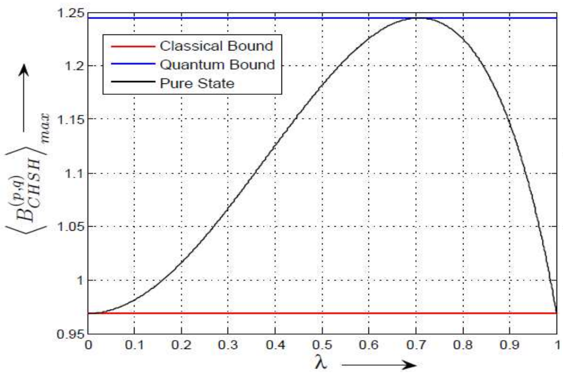
Further, it is well known that for any arbitrary pure bipartite entangled state $e_{max}^2 > 0$. Therefore,

$$\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2 * 4\lambda_1^2\lambda_2^2} > 1, \tag{5.36}$$

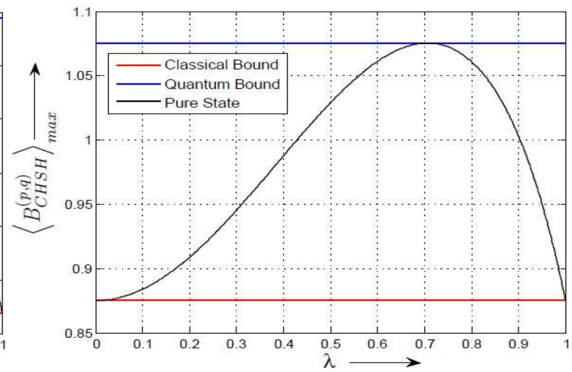
and, hence

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{quantum} > \left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{classical} \tag{5.37}$$

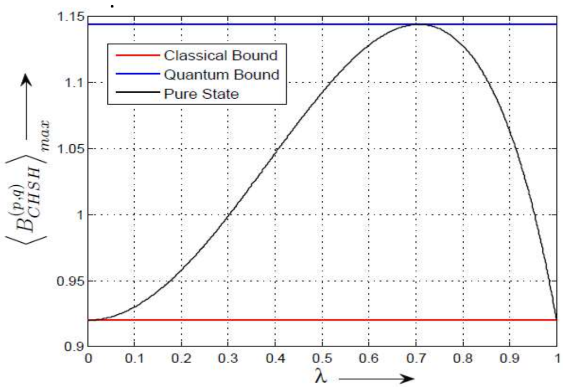
Similarly, for other regions of $[p, q]$ space, one can also find same results, and therefore every pure bipartite entangled state violates the CHSH inequality under biased experimental set-up. Figure 5.3 demonstrates the maximum classical and quantum score of the Bell-CHSH game for different values of biasing parameters p and q, which further confirms that when the state is maximally entangled, it achieves the maximum quantum bound of Bell inequality in biased experimental set-up.



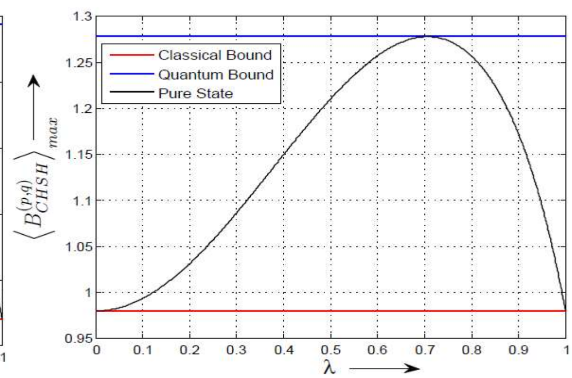
5.3 (a) : CHSH function for p=1/8 and q=1/8



5.3 (b) : CHSH function for p=3/4 and q=1/4



5.3 (c) : CHSH function for p=2/10 and q=8/10 .



5.3 (d) : CHSH function for p=9/10 and q=9/10

Figure 5.3 : Maximum possible classical and quantum scores of a CHSH game with an arbitrary pure bipartite entangled state for different values of biasing parameters p and q.

(2) For mixed states, we consider an example of Werner state [Werner, 1989] given in Eq. (2.41). The eigenvalues of the real symmetric matrix E_{ρ_w} of Werner state are $e_1 = e_2 = e_3 = x^2$. Considering $0 \leq p \leq \frac{1}{2}$ and $0 \leq q \leq \frac{1}{2}$, the maximum classical bound of the CHSH function is $(1-2pq)$, and the maximum quantum bound of the CHSH function is

$$\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho_w} = \sqrt{(1-2pq)^2 + (1-2q(1-p))^2} * x$$
$$= (1-2pq)\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2} * x \tag{5.38}$$

The $\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho_w}$ in Eq. (5.38) is greater than the classical bound iff

$$x > \frac{1}{\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2}} \tag{5.39}$$

The maximum value of $\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)$ is 1, therefore if $\frac{1}{\sqrt{2}} \leq x \leq 1$, then quantum bound is greater than the classical bound. Since in the unbiased scenario, the range of Bell inequality violation for a Werner state is also $\frac{1}{\sqrt{2}} \leq x \leq 1$ [Horodecki, 1995]; the range of violation of Bell-CHSH inequality is same in both biased and unbiased scenarios.

(3) As another example, we consider Horodecki's state [Horodecki, 1996], represented in Eq. (2.44). In an unbiased scenario, this state violates the Bell inequality iff $\frac{1}{\sqrt{2}} < a \leq 1$. In the present case, the eigenvalues of real symmetric matrix E_{ρ_h} are $e_1 = e_2 = a^2$ and $e_3 = (1-2a)^2$. Again, considering $0 \leq p \leq \frac{1}{2}$ and $0 \leq q \leq \frac{1}{2}$ for biased scenario, the maximum classical bound of CHSH function is $(1-2pq)$, and the maximum quantum bound of CHSH function is

$$\begin{aligned}\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\varrho_h} &= \sqrt{(1-2pq)^2 + (1-2q(1-p))^2} * a \\ &= (1-2pq)\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2} * a \end{aligned} \tag{5.40}$$

Clearly, if $\frac{1}{\sqrt{2}} < a \leq 1$ then quantum bound is greater than the classical bound. Hence, the range of violation of Bell-CHSH inequality in unbiased scenario for Horodecki's state is also the same as in biased scenario.

(4) We now consider a mixed state proposed by WenChao Ma *et al.* [Ma *et al.*, 2015] given in Eq. (4.26). The eigenvalues of the real symmetric matrix $E_{\rho_{wc}}$ are $e_1 = e_2 = (f-1)^2$ and $e_3 = (1-2f)^2$. Therefore, the given state violates the Bell-CHSH inequality in unbiased scenario for a range of state parameter f, i.e., for $0 \leq f < (1-\frac{1}{\sqrt{2}})$. Considering, $0 \leq p \leq \frac{1}{2}$ and $0 \leq q \leq \frac{1}{2}$, the maximum quantum bound in biased scenario is

$$\begin{aligned}\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho_{wc}} &= \sqrt{(1-2pq)^2 (f-1)^2 + (1-2q(1-p))^2 (f-1)^2} \\ &= (1-2pq)\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2 * (f-1)^2} \end{aligned} \tag{5.41}$$

Therefore, $\left\langle B_{CHSH}^{(p,q)} \right\rangle_{max}^{\rho_{wc}}$ is greater than the classical value iff

$$\pm(f-1) > \frac{1}{\sqrt{1+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2}}$$

Since the maximum value of $\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)$ is 1, one can see that if $0 \leq f < (1-\frac{1}{\sqrt{2}})$ then the quantum bound is greater than the classical bound. Moreover, range of state parameter f for violation of the Bell-CHSH inequality in biased scenario is same as in unbiased scenario.

(5) Finally, we analyse the behaviour of a new class of states proposed in chapter 2 under the biased scenario. The proposed class of states are defined in Eq. (2.35), where we have also demonstrated that the proposed set of states (ϱ) violate the Bell-CHSH inequality for a range of weak measurement strength, i.e., for $max\left\{0, \left(1-\frac{0.2428}{\gamma}\right)\right\} < \eta \leq 1$. Further, from Eq. (2.35), for **Case I**, where $0 \leq p \leq \frac{1}{2}$ and $0 \leq q \leq \frac{1}{2}$, the maximum quantum bound of the CHSH function for proposed state is

$$\begin{aligned}\left\langle B_{CHSH}^{(p,q)}\right\rangle_{max}^{\varrho_p} &= \sqrt{(1-2pq)^2\frac{(N+2\gamma(\eta-1))^2}{N^2}+(1-2q(1-p))^2\frac{1}{N^2}} \\ &= (1-2pq)\sqrt{\frac{(N+2\gamma(\eta-1))^2}{N^2}+\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)^2\frac{1}{N^2}}\end{aligned} \quad (5.42)$$

where $N=\frac{1}{2}(2-2\gamma(\eta-1)+\gamma^2(\eta-1)^2)$. From Eq. (5.42), if

$$(2\gamma^3(\eta-1)^3+4\gamma(\eta-1)+\frac{(1-2q(1-p))}{(1-2pq)})\geq 0 \quad (5.43)$$

then the proposed class of states violate the Bell-CHSH inequality. Since the maximum value of $\left(\frac{(1-2q(1-p))}{(1-2pq)}\right)$ is 1, one can deduce that the range of state parameter η for the violation of Bell-CHSH inequality is $max\left\{0,\left(1-\frac{0.2428}{\gamma}\right)\right\}<\eta\leq 1$, which is same as in the unbiased scenario.

5.6 SUMMARY

We analysed a CHSH game with biased experimental settings in which both Alice and Bob choose their measurements with certain probability. For this, we estimated maximum classical and quantum score for a biased nonlocal game and showed that quantum mechanics offers more powerful resources than classical mechanics for the whole range of biasing parameters. Our analysis using fine-grained uncertainty relations to distinguish classical, quantum and super-quantum correlations also confirmed the advantages the quantum theory holds over its classical counterpart. We further demonstrated the violation of Bell-CHSH inequality by all pure states under biased scenario. For bipartite mixed states such as Werner state, Horodecki's state, a state proposed by WenChao Ma *et al.*, and a set of states proposed in chapter-2, we found that the nonlocal behaviour in biased scenario is same as was reported for unbiased scenario.

It will be interesting to find a mixed entangled state whose behaviour can be differentiated in the two scenarios. Another problem of particular interest would be to analyse the behaviour of nonlocal correlations subjected to environmental effects under biased scenario.

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