6

Usefulness of Multiqubit W-Type States in Quantum Information Processing

6.1 INTRODUCTION

Quantum entanglement plays a key role in many potential applications in quantum information and computation [Bennett and Wiesner, 1992; Bennett *et al.*, 1993; Zukowski *et al.*, 1993; Boström and Felbinger, 2002; Gisin *et al.*, 2002]. The theoretical and experimental progress to characterize bipartite and multiqubit entanglement and nonlocality, and to use these entangled resources for quantum information and computation has received a lot of attention in last three decades [Svetlichny, 1987; Cirac and Zoller, 1994; Hagley *et al.*, 1997; Tittel *et al.*, 1998; Pan *et al.*, 2000; Seevinck and Svetlichny, 2002; Collins *et al.*, 2002a,b; Cereceda, 2002; Batle *et al.*, 2002; Pan *et al.*, 2003b; Zhao *et al.*, 2003; Eibl *et al.*, 2003; Di Giuseppe *et al.*, 2003; Eibl *et al.*, 2004a; Marcikic *et al.*, 2004; Zhao *et al.*, 2004; Peng *et al.*, 2005; Kiesel *et al.*, 2005; Bruß *et al.*, 2005; Leibfried *et al.*, 2005; Walther *et al.*, 2005b; de Oliveira *et al.*, 2006; Prevedel *et al.*, 2007; Lu *et al.*, 2007; Vallone *et al.*, 2007; Tokunaga *et al.*, 2008; Ghose *et al.*, 2009; Bancal *et al.*, 2010; Batle and Casas, 2011; Zhao *et al.*, 2012; Barrett *et al.*, 2013a; Batle *et al.*, 2016; Chaves and Budroni, 2016; Ringbauer *et al.*, 2016; Batle *et al.*, 2017; Brito *et al.*, 2018]. For example, in the simplest scenario, the direct product of Bell pairs can be used for theoretical description of multiqubit states, and can be efficiently used for several communication protocols [Lee *et al.*, 2002; Fang *et al.*, 2003; Rigolin, 2005; Deng *et al.*, 2005b]. On the other hand, multiqubit GHZ states, cluster states, and Brown states possess genuine and maximum multiqubit entanglement [Karlsson and Bourennane, 1998; Hein *et al.*, 2005; Muralidharan and Panigrahi, 2008]. Moreover, the experimental realization of these states further supports the analysis of entanglement, and strengthens the area of quantum information and computation [Zeilinger *et al.*, 1997; Sackett *et al.*, 2000; Weinfurter and Żukowski, 2001; Zhao *et al.*, 2004; Kiesel *et al.*, 2005; Tokunaga *et al.*, 2005; Browne and Rudolph, 2005; Vallone *et al.*, 2007; Tokunaga *et al.*, 2008; Gao *et al.*, 2010; Pan *et al.*, 2012]. The maximally entangled multiqubit states are used extensively in the literature to perform efficient computation that cannot be achieved using classical resources [Karlsson and Bourennane, 1998; Gottesman and Chuang, 1999; Briegel and Raussendorf, 2001; Hillery *et al.*, 1999; Raussendorf *et al.*, 2003; Lee *et al.*, 2006; Yeo and Chua, 2006; Man *et al.*, 2007; Wang *et al.*, 2007a; Muralidharan and Panigrahi, 2008]. In general, the optimal success of a quantum communication protocol can be ascertained by use of maximally entangled states as resources for information transfer. However, the use of non-maximally entangled resources largely leads to probabilistic protocols and the fidelity of information transfer is always less than the unity [Shi *et al.*, 2000; Li *et al.*, 2000; Agrawal and Pati, 2002; Albeverio *et al.*, 2002a; Yan and Wang, 2003; Gorbachev *et al.*, 2003; Cao and Song, 2005; Gordon and Rigolin, 2006; Wang *et al.*, 2007b; Jung *et al.*, 2008b; Kumar and Krishnan, 2009; Das *et al.*, 2014; Li and Jin, 2016; Kögler and Neves, 2017] . For example, quantum teleportation of a single-qubit using a three- and four-qubit W states is always probabilistic and teleportation fidelity depends on the unknown parameter of the teleported state [Shi and Tomita, 2002; Agrawal and Pati, 2002]. Although the maximally entangled four-qubit GHZ state is a quantum channel to teleport special cases of two-qubit states such as $|\phi\rangle_{ab} = [\alpha\,|00\rangle_{ab} + \beta\,|11\rangle_{ab}]$, the same cannot be accomplished for an arbitrary two-qubit state [Rigolin, 2005]. Similarly, teleportation of an arbitrary two-qubit state cannot be realized by four-qubit and five-qubit non-maximally entangled W states. Agrawal and Pati [Agrawal and Pati, 2006] proposed a new class of three-qubit W-type states for deterministic teleportation of a single-qubit by performing three-qubit joint measurements. The efficiency of these W-type states, however, decreases if one performs standard two-qubit and single-qubit measurements only [Adhikari and Gangopadhyay, 2009] instead of performing joint three-qubit measurements. Considering this, in this chapter, we discuss the issue of usefulness of partially entangled states for optimizing the information transfer between a sender and a receiver. For this, we demonstrate two separate analysis for information transfer; one including multiqubit joint measurements and another involving standard single- and two-qubit measurements only.

In the following sections, we propose new and efficient class of non-maximally entangled four-qubit W-type states for quantum information processing and demonstrate the possibility of deterministic teleportation of a single-qubit with unit fidelity. For practical purposes, we emphasize on a protocol to share optimal bipartite entanglement, e.g., we use partially entangled four-qubit W-type states as a starting resource between two users and achieve the optimal bipartite entanglement by performing standard single- and two-qubit measurements only. Our results show that the shared two-qubit entanglement can lead to a maximally entangled resource for certain state parameters. We further demonstrate the need to analyse four-qubit W-type states by comparing the efficacy of three- and four-qubit W-type states as resources in terms of concurrence [Wootters, 1998] of the finally shared entangled state between the two users. Interestingly, our results show that for certain ranges of parameters, four-qubit W-type states are more efficient resources in comparison to three-qubit W-type states for achieving optimal concurrence. For dense coding, we found that in principle a sender can transmit a 2-bit classical message to a receiver by locally manipulating his/her single qubit. Moreover, we also generalize teleportation and dense coding protocols using N-qubit W-type states as resources to achieve deterministic information transfer. In order to add another dimension and significance to the results obtained in this chapter, we finally demonstrate experimental preparation of four-qubit W-type states. The experimental generation of these states is achieved using standard single- and two-qubit unitary operations, and weak measurements.

6.2 TELEPORTATION USING FOUR-QUBIT W-TYPE STATES

Teleportation is a quantum mechanical process to transmit quantum information over arbitrary distances using a shared entangled resource. Although non-maximally entangled three-qubit and four-qubit W states can be used as resources for probabilistic teleportation of a single-qubit, one cannot achieve teleportation of a single-qubit state using the standard three- or four-qubit W state with certainty. In general, teleportation leads to probabilistic information transfer. Pati and Agrawal [Agrawal and Pati, 2006] , however, have shown that there exists a special class of W-type states, given by

$$|\Psi_k\rangle_{123} = \frac{1}{\sqrt{2k+2}}\left[|100\rangle + \sqrt{k}e^{i\gamma}|010\rangle + \sqrt{k+1}e^{i\delta}|001\rangle\right]_{123} \tag{6.1}$$

which can be used for perfect teleportation and dense coding. Here k is a positive integer and γ and δ are relative phases. Motivated from this, we propose a new class of four-qubit W states, namely

$$|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\left[|1000\rangle + \sqrt{k}e^{i\gamma}|0100\rangle + \sqrt{k+1}e^{i\delta}|0010\rangle + \sqrt{2k+2}e^{i\zeta}|0001\rangle\right]_{1234} \tag{6.2}$$

where k is a real number and γ, δ, ζ represent phases. The states proposed in Eq. (6.2) can be used as resources to achieve optimal and deterministic quantum teleportation. For example, if Alice wants to teleport an unknown state $|\phi\rangle_a = [\alpha|0\rangle + \beta|1\rangle]_a$ where $|\alpha|^2 + |\beta|^2 = 1$ to Bob then Alice and Bob need to share the four-qubit state $|\Psi_k\rangle_{1234}$ such that Alice has qubits 1, 2 and 3 and Bob has qubit 4.

Thus, the joint state of five qubits can be represented as

$$|\Phi\rangle_{a1234} = |\phi\rangle_a \otimes |\Psi_k\rangle_{1234} \tag{6.3}$$

In order to teleport the unknown state to Bob, Alice projects her four qubits onto the states

$$\begin{aligned}
|\eta_k\rangle^{\pm}_{a123} &= \frac{1}{2\sqrt{k+1}}\left[|0100\rangle + \sqrt{k}e^{i\gamma}|0010\rangle + \sqrt{k+1}e^{i\delta}|0001\rangle \pm \sqrt{2k+2}e^{i\zeta}|1000\rangle\right]_{a123} \\
|\xi_k\rangle^{\pm}_{a123} &= \frac{1}{2\sqrt{k+1}}\left[|1100\rangle + \sqrt{k}e^{i\gamma}|1010\rangle + \sqrt{k+1}e^{i\delta}|1001\rangle \pm \sqrt{2k+2}e^{i\zeta}|0000\rangle\right]_{a123}
\end{aligned} \tag{6.4}$$

Although the teleportation protocol works for all k, γ, δ and ζ, for simplicity, we assume $k = 1$ and $\gamma = \delta = \zeta = 0$. Therefore, the joint state of five qubits can be re-expressed using Alice's measurement basis as

$$|\Phi\rangle_{a1234} = \frac{1}{2}\left[|\eta_1\rangle^{+}_{a123}|\phi\rangle_4 + |\eta_1\rangle^{-}_{a123}\sigma_z|\phi\rangle_4 + |\xi_1\rangle^{+}_{a123}\sigma_x|\phi\rangle_4 + |\xi_1\rangle^{-}_{a123}\iota\sigma_y|\phi\rangle_4\right] \tag{6.5}$$

where $|\phi\rangle_4 = [\alpha|0\rangle + \beta|1\rangle]_4$. A four-qubit joint measurement on qubits $a, 1, 2$ and 3 will project the state of Bob's qubit onto one of the four possible states as shown in Eq. (6.5) with the equal probability of 1/4.

Hence, teleportation of a single-qubit using non-maximally entangled four-qubit W-type states is always successful. The use of proposed states as quantum channels also provides flexibility to the experimental set-ups by relaxing the requirement of a maximally entangled shared resource for a faithful teleportation. Since the teleportation is deterministic, the total probability and fidelity of teleporting a single-qubit using a partially entangled four-qubit W-type state is also unity.

6.3 TELEPORTATION USING N-QUBIT W-TYPE STATES

In the previous section, we have successfully demonstrated the efficient quantum teleportation of a single-qubit state using a new class of four-qubit W-type states. We now extend our method to generalize the optimal teleportation protocol using N-qubit W-type states as resources.

For successfully teleporting a single-qubit state $|\phi\rangle_a$ to Bob, Alice needs to share a N-qubit W-type state

$$\begin{aligned}
|\Psi_k\rangle_{12..N} &= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\left[|100...N\rangle_{12...N} + \sqrt{k}e^{i\gamma}|010...N\rangle_{12..N}\right. \\
&+ \sqrt{k+1}e^{i\delta}|001...N\rangle_{12..N} + ...\sqrt{k+(N-3)}e^{i\zeta}|000...10\rangle_{12..N} \\
&+ \left.\sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}e^{i\beta}|000...1\rangle_{12...N}\right]
\end{aligned} \tag{6.6}$$

with Bob such that qubits 1 to $(N-1)$ are with Alice and qubit N is with Bob. In this case, the projection basis used by Alice can be represented as

$$
\begin{aligned}
|\eta_k\rangle^{\pm}_{a,1,2...,N-1} &= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\Big[|010...N\rangle+\sqrt{k}e^{i\gamma}|001...N\rangle \\
&+ \sqrt{k+1}e^{i\delta}|0001...N\rangle+....\sqrt{k+(N-3)}e^{i\zeta}|000...1\rangle \\
&\pm \sqrt{(N-2)k+\frac{(N-2)(N-3)}{2}+1}e^{i\beta}|100...0\rangle\Big]_{a,1,2...,N-1} \\
|\xi_k\rangle^{\pm}_{a,1,2...,N-1} &= \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\Big[|110...N\rangle+\sqrt{k}e^{i\gamma}|101...N\rangle \\
&+ \sqrt{k+1}e^{i\delta}|1001...N\rangle+.....\sqrt{k+(N-3)}e^{i\zeta}|100...1\rangle \\
&\pm \sqrt{(N-2)k+\frac{(N-2)(N-3)}{2}+1}e^{i\beta}|000...0\rangle\Big]_{a,1,2...,N-1} \qquad (6.7)
\end{aligned}
$$

Similar to the teleportation protocol discussed in the previous section, we can express the joint state of $(N+1)$ qubits in terms of Alice's projection basis as

$$
\begin{aligned}
&|\Phi\rangle_{a12...N} = |\phi\rangle_a \otimes |\Psi_k\rangle_{123...N} \\
&\quad = \frac{1}{2}\left[|\eta_k\rangle^{+}_{a12...N-1}|\phi\rangle_N + |\eta_k\rangle^{-}_{a12...N-1}\sigma_z|\phi\rangle_N + |\xi_k\rangle^{+}_{a12...N-1}\sigma_x|\phi\rangle_N + |\xi_k\rangle^{-}_{a12..N-1}\imath\sigma_y|\phi\rangle_N\right] \quad (6.8)
\end{aligned}
$$

where $|\phi\rangle_N = [\alpha|0\rangle+\beta|1\rangle]_N$. Eq. (6.8) clearly shows that the teleportation protocol is always successful with equal probability of 1/4 for the four different measurement outcomes of Alice. Therefore, Bob can always recover the original state by performing single-qubit unitary transformations on the state of his qubit, once he receives the two-bit classical message from Alice regarding her measurement outcome.

6.4 ANALYSIS OF THE EFFICIENCY OF W-TYPE STATES IN TELEPORTATION PROCESS

We have shown that N-qubit W-type states can be successfully used as optimal resources for efficient teleportation. The successful completion of teleportation protocol depends on the availability of experimental set up to perform and distinguish multiqubit measurements. It is evident that with the present experimental techniques, one can only perform and distinguish different Bell measurements [Kim *et al.*, 2001b]. Therefore, we analyse the efficacy of our states for a protocol where two users want to create an efficient bipartite entangled channel between them using partially entangled four-qubit W-type states $|\Psi_k\rangle_{1234}$. For this, we assume that Alice initially has a two-qubit entangled state $|\phi\rangle_{ab} = [\alpha|00\rangle+\beta|11\rangle]_{ab}$ in addition to the shared W-type entangled state

$$
|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\Big[|1000\rangle+\sqrt{k}|0100\rangle+\sqrt{k+1}|0010\rangle+\sqrt{2k+2}|0001\rangle\Big]_{1234} \qquad (6.9)
$$

with Bob such that qubits 1, 2 and 3 are with Alice and qubit 4 is with Bob. In order to share a bipartite entanglement with Bob, Alice needs to perform Bell measurements

$$
\begin{aligned}
|\phi\rangle^{\pm} &= \frac{1}{\sqrt{2}}[|00\rangle \pm |11\rangle], \\
|\psi\rangle^{\pm} &= \frac{1}{\sqrt{2}}[|01\rangle \pm |10\rangle] \qquad (6.10)
\end{aligned}
$$

on her qubits. There are different combinations in which Alice can perform these Bell measurements to achieve the required two-qubit entanglement. We have examined all possible combinations and measurement outcomes, and here we will discuss only four optimal cases where the concurrence of finally shared two-qubit entangled state is optimal and efficient. We now proceed to analyse the efficacy of the protocol in terms of the concurrence of the finally shared entangled state.

- **Case I:** In the first case, we consider Alice's measurement outcomes to be $|\phi^+\rangle_{b1}$ and $|\phi^+\rangle_{23}$. Therefore, the joint state of two qubits shared between Alice and Bob can be represented as

$$|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2+\beta^2}}\left[\sqrt{2k+2}\alpha\,|01\rangle_{a4}+\beta\,|10\rangle_{a4}\right] \tag{6.11}$$

Consequently, the concurrence of $|\psi\rangle_{a4}$ is

$$C_4^{(1)} = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{2k+2}}{(2k+1)\alpha^2+1} \tag{6.12}$$

where the subscript represents number of qubits in initially shared W-type states and superscript represents different cases. Eq. (6.12) clearly demonstrates that for any given real positive number k, if $|\alpha|^2$ is varied from 0 to 1 then concurrence first increases and then decreases to a minimum value. Interestingly, for $\alpha^2 = \frac{1}{(2k+3)}$ concurrence of the shared entangled state is unity, i.e., Alice and Bob can share a maximally entangled state. The finally shared optimally entangled state, thus, can be used for various information processing protocols. This can be really useful in scenarios where the users in a communication protocol only have access to partially entangled multiqubit states. Further, the analysis presented here not only allows the users to create maximum entanglement but also releases the constraints on the experimental set up to perform and distinguish multiqubit measurements.

- **Case II:** In the second case, Alice's measurement outcomes are considered as $|\phi^+\rangle_{b2}$ and $|\phi^+\rangle_{13}$. Hence, the shared bipartite state and concurrence of this state can be given by

$$|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2+k\beta^2}}\left[\sqrt{2k+2}\alpha\,|01\rangle_{a4}+\sqrt{k}\beta\,|10\rangle_{a4}\right] \tag{6.13}$$

and

$$C_4^{(2)} = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{2k+2}\sqrt{k}}{(k+2)\alpha^2+k}, \tag{6.14}$$

respectively. Similar to the first case, the concurrence of the shared state first increases; attains the maximum and then decreases to 0 for any k and $0<\alpha<1$. Further, for $\alpha^2 = \frac{k}{(3k+2)}$, concurrence of the shared state is unity.

- **Case III:** The third case provides another interesting observation that for Alice's measurement outcomes $|\phi^+\rangle_{b3}$ and $|\phi^+\rangle_{12,}$ the concurrence of shared bipartite state is independent of the parameter k. In this scenario, the shared bipartite state and its concurrence are represented as

$$|\psi\rangle_{a4} = \frac{1}{\sqrt{(2k+2)\alpha^2+(k+1)\beta^2}}\left[\sqrt{2k+2}\alpha\,|01\rangle_{a4}+\sqrt{k+1}\beta\,|10\rangle_{a4}\right] \tag{6.15}$$

and

$$C_4^{(3)} = \frac{2\sqrt{2}\alpha\sqrt{1-\alpha^2}}{\alpha^2+1}, \tag{6.16}$$

respectively. Evidently, the concurrence given in Eq. (6.16) attains its maximum value for $\alpha^2 = \frac{1}{3}$.

- **Case IV:** The fourth case, i.e., when Alice's measurement outcomes are $|\phi^+\rangle_{a1}$ and $|\phi^+\rangle_{b2}$, is even more interesting as the concurrence of finally shared bipartite state is independent of both the parameters k and α. In this scenario, the shared bipartite state and its concurrence are represented as

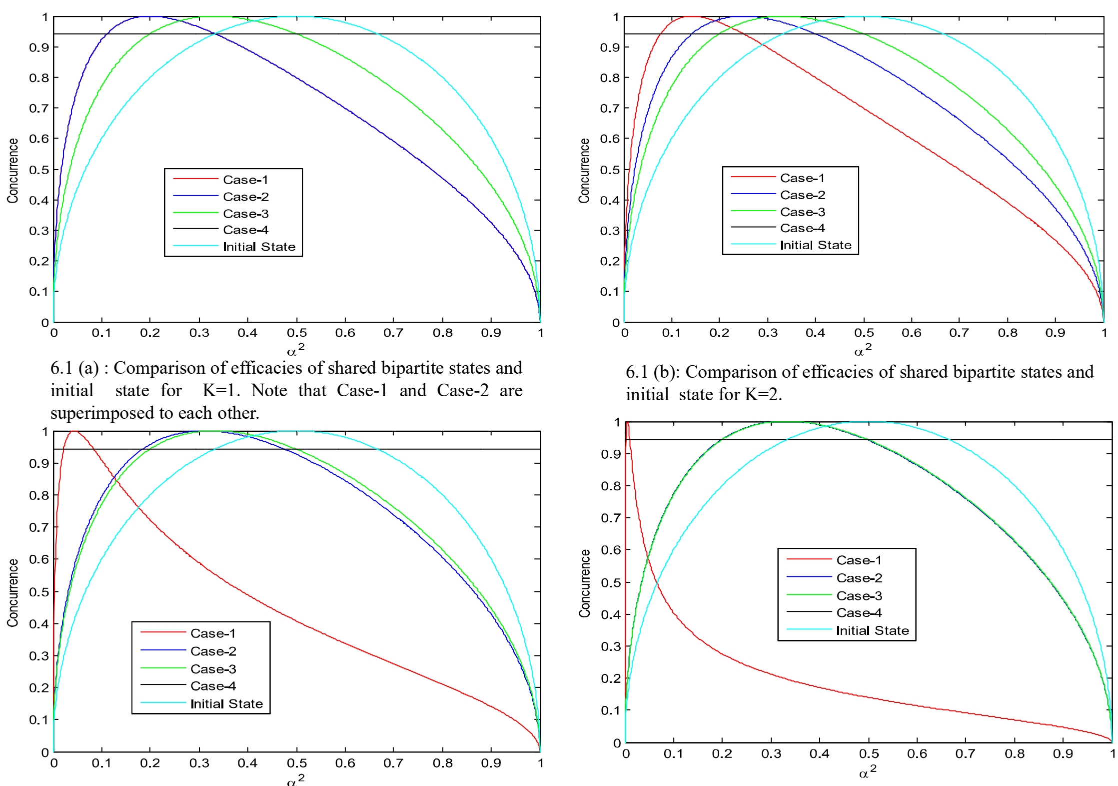
$$|\psi\rangle_{34} = \frac{1}{\sqrt{3k+3}}\left[\sqrt{2k+2}\,|01\rangle_{34} + \sqrt{k+1}\,|10\rangle_{34}\right] \tag{6.17}$$

and

$$C_4^{(4)} = \frac{2\sqrt{2}}{3}, \tag{6.18}$$

respectively. Clearly, the concurrence given in Eq. (6.18) does not depend on the parameters of input states.

Figure 6.1 compares the concurrence of initial state with that of above four cases to analyse the efficacy of finally shared bipartite states. For $0 < \alpha^2 \leq (\sqrt{2}-1)$ and $\frac{2}{3} \leq \alpha^2 < 1$, the efficiency of finally shared state is better than the efficiency of initial bipartite state in terms of concurrence. For $k = 1$, concurrence for cases 1 and 2 are the same. Similarly, for large k, case 2 and case 3 lead to identical results. Moreover, Figure 6.1 also shows a relation between α^2 and combination of Bell measurements to be performed to achieve the optimal concurrence.



6.1 (a) : Comparison of efficacies of shared bipartite states and initial state for K=1. Note that Case-1 and Case-2 are superimposed to each other.

6.1 (b): Comparison of efficacies of shared bipartite states and initial state for K=2.

6.1 (c) : Comparison of efficacies of shared bipartite states and initial state for K=10.

6.1 (d) : Comparison of efficacies of shared bipartite states and initial state for K=100. Note that Case-2 and Case-3 are approximately superimposed to each other.

Figure 6.1 : Comparison of efficacies of shared bipartite states in three optimal cases.

A similar calculation for shared N-qubit partially entangled states shows that the concurrence of finally shared states dependent on input parameters can be given as

$$C = \frac{2\alpha\beta\sqrt{k+r}\sqrt{(N-2)k+\frac{(N-2)(N-3)}{2}+1}}{((N-2)k+\frac{(N-2)(N-3)}{2}+1)\alpha^2+(k+r)\beta^2} \tag{6.19}$$

where r is a variable that takes values from 0 to $(N-3)$ or $(1-k)$. Eq. (6.19) suggests that for $r=(1-k)$, entanglement of the finally shared state between Alice and Bob depends on the input state parameters α and k. For $k\to\infty$, the concurrence is given by

$$C = \frac{2\alpha\sqrt{1-\alpha^2}\sqrt{N-2}}{(N-3)\alpha^2+1} \tag{6.20}$$

Hence, for a given range of α, if k is very large then the W-type state with smaller number of qubits is a better resource. Similarly the concurrence of finally shared states independent of input parameters can be expressed as

$$C = \frac{2\sqrt{k+r}\sqrt{(N-2)k+\frac{(N-2)(N-3)}{2}+1}}{((N-1)k+\frac{(N-2)(N-3)}{2}+1+r)}, \tag{6.21}$$

As above, if k is very large then the W-type states with smaller number of qubits can be considered as better resources. In order to analyse the usefulness of four-qubit W-type states for such a protocol, we further compare the efficacy of three- and four-qubit W-type states as resources in terms of concurrence of the finally shared entangled state. We found an interesting observation that for certain ranges of α^2, the four-qubit W-type states are more efficient resources in comparison to three-qubit W-type states for achieving optimal concurrence shared between two users. For this, let us first give the form of three-qubit W-type states as

$$|\Psi_k\rangle_{123} = \frac{1}{\sqrt{2k+2}}\left[|100\rangle+\sqrt{k}|010\rangle+\sqrt{k+1}|001\rangle\right]_{123} \tag{6.22}$$

Similar to the four-qubit case, there are optimal cases for which the concurrences of finally shared states can be given as

$$C_3^{(1)} = \frac{2\alpha\sqrt{(1-\alpha^2)}\sqrt{k+1}}{(k)\alpha^2+1} \tag{6.23}$$

and

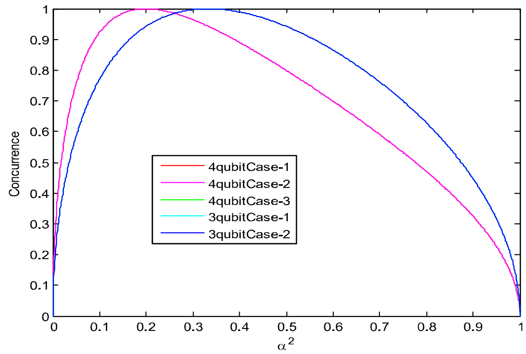
$$C_3^{(2)} = \frac{2\alpha\sqrt{k(k+1)(1-\alpha^2)}}{\alpha^2+k}. \tag{6.24}$$

In above two cases the optimal concurrence of finally shared entangled states is dependent on input state. However, similar to the four-qubit case, for three-qubit case also there exists an optimal scenario in which concurrence of finally shared state is independent of the input state, i.e.,

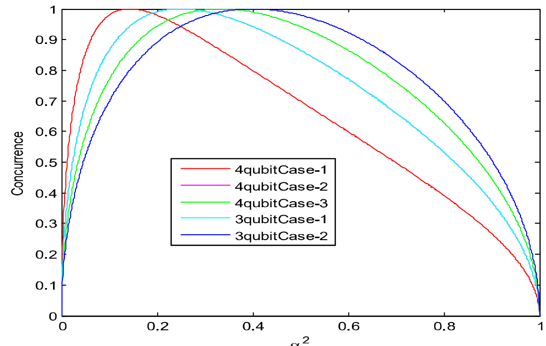
$$C_3^{(3)} = \frac{2\sqrt{k+1}}{(k+2)} \tag{6.25}$$

Figure 6.2 demonstrates the comparison between efficiencies of three- and four-qubit W-type states in terms of the concurrence of finally shared bipartite states. Depending on values of the parameter k, we identify four different cases;

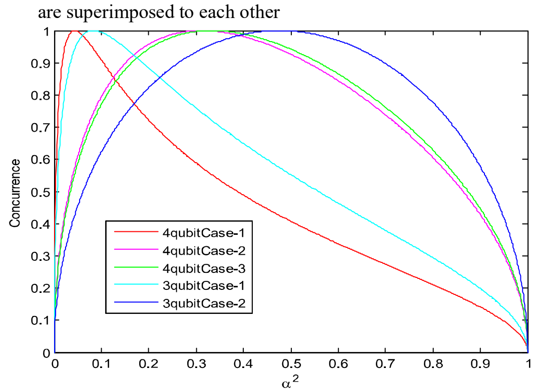
- **Case I:** For $k=1$ if $0<\alpha^2\leq\frac{k(\sqrt{2}-1)}{((k+2)-\sqrt{2})}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state else both are equally efficient.



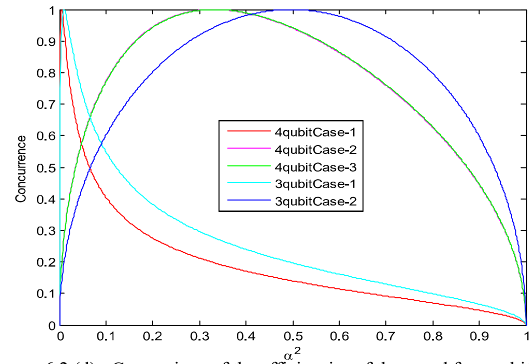
6.2 (a) : Comparison of the efficiencies of three and four-qubit W-type states for K=1. Note that Cases1 and 2 for 4-qubit , and Case-3 for 4-qubit & Cases-1 and 2 for 3-qubits are superimposed to each other



6.2 (b) : Comparison of the efficiencies of three and four-qubit W-type states for K=2. Note that Case-1 of 3-qubit and Case-2 of 4-qubit W-type state are superimposed to each other



6.2 (c) : Comparison of the efficiencies of three and four-qubit W-type states for K=10



6.2 (d) : Comparison of the efficiencies of three and four-qubit W-type states for K=100. Note that Case-2 and Case-3 of 4-qubit W-type state are approximately superimposed

Figure 6.2 : Comparison of the efficiencies of three and four-qubit W-type states as resources.

- **Case II:** For $k = 2$;
  - *Range* 1 : If $0 < \alpha^2 \leq \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
  - *Range* 2 : If $\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \leq \frac{\sqrt{k+1}-\sqrt{2}}{k\sqrt{2}-\sqrt{k+1}}$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.
  - *Range* 3 : If $\frac{\sqrt{k+1}-\sqrt{2}}{k\sqrt{2}-\sqrt{k+1}} < \alpha^2 \leq \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
  - *Range* 4 : If $\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.

- **Case III:** For $k > 2$;
  - *Range* 1 : If $0 < \alpha^2 \leq \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
  - *Range* 2 : If $\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \leq \frac{k-\sqrt{2k}}{\sqrt{2k}k-(k+2)}$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.
  - *Range* 3 : If $\frac{k-\sqrt{2k}}{\sqrt{2k}k-(k+2)} < \alpha^2 \leq \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
  - *Range* 4 : If $\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.

- **Case IV:** When k is very large

    - *Range* 1 : If $0 < \alpha^2 \leq \frac{\sqrt{2}-1}{(2-\sqrt{2})k+1}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
    - *Range* 2 : If $\frac{\sqrt{2}-1}{(2-\sqrt{2})k+1} < \alpha^2 \leq \frac{\sqrt{k+1}-\sqrt{2}}{\sqrt{2}k-\sqrt{k+1}}$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.
    - *Range* 3 : If $\frac{\sqrt{k+1}-\sqrt{2}}{\sqrt{2}k-\sqrt{k+1}} < \alpha^2 \leq \frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}}$ then the four-qubit W-type state is a better resource in comparison to the three-qubit W-type state.
    - *Range* 4 : If $\frac{\sqrt{2}k-\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}-\sqrt{2}} < \alpha^2 < 1$ then the three-qubit W-type state is a better resource in comparison to the four-qubit W-type state.

Hence, for practical implementation of an efficient bipartite state sharing protocol, one can choose W-type states as resources according to the range of parameters α^2 and k.

For numerical estimation of efficiencies of three- and four-qubit W_k states, one can analyse the ranges given above to understand the efficacy of such states in terms of concurrence of finally shared states. Alternately, Table 6.1 shows the range of parameters for $k = 1$, $k = 2$, $k = 5$, $k = 10$, and $k = 100$ to compare the efficiencies of three- and four-qubit W-type states. For example, for $k = 1$; four-qubit states are better resources that three-qubit states for $\alpha^2 \leq 0.261$, and for $\alpha^2 > 0.261$ one can choose either of the states as a starting shared resource. Thus, for $k = 1$, four-qubit W_1 states can always be used as a resource either for better efficiency or for same efficiency in comparison to three-qubit W_1 states. Similarly, one can find conclusions for other values of k as well.

6.5 SUPERDENSE CODING USING N-QUBIT W-TYPE STATES

Superdense coding deals with efficient information transfer between the users in a communication protocol using a shared entangled resource. We use

$$|\eta_1\rangle^+_{1234} = \frac{1}{2\sqrt{2}}\left[|0100\rangle + |0010\rangle + \sqrt{2}|0001\rangle + 2|1000\rangle\right]_{1234} \tag{6.26}$$

as a shared resource for superdense coding protocol between Alice and Bob such that the first qubit is with Alice and rest of the qubits are with Bob. In order to communicate the classical message to Bob, Alice first encodes her message using one of the four single qubit operations; $I, \sigma_x, \sigma_y, \sigma_z$ on her qubit 1. The four operations map the originally shared state between Alice and Bob to four orthogonal states

$$\begin{aligned}
(\sigma_x \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\xi_1\rangle^+_{1234} \\
(\sigma_z \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\eta_1\rangle^-_{1234} \\
(i\sigma_y \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\xi_1\rangle^-_{1234} \\
(I \otimes I \otimes I \otimes I)\,|\eta_1\rangle^+_{1234} &= |\eta_1\rangle^+_{1234}
\end{aligned} \tag{6.27}$$

Thus, in principle, Alice can prepare four distinct messages for Bob by locally manipulating her qubit. Once Alice encodes the message, she sends her qubit to Bob. In order to distinguish between the messages sent by Alice, Bob can always perform an appropriate joint measurement on the state of four qubits. Hence, Bob will always be able to distinguish between the four messages produced by Alice. The protocol is optimal as by locally manipulating her one qubit, Alice can transmit two bits of classical message to Bob.

We now proceed to demonstrate optimal dense coding protocol using our N-qubit W-type

Table 6.1 : Numerical estimation of comparison of concurrences of three and four-qubit W_k states

k	α^2	Concurrence of finally shared two-qubit state using initially shared three-qubit W_k state as a resource (Case-1)	Concurrence of finally shared two-qubit state using initially shared three-qubit W_k state as a resource (Case-2)	Concurrence of finally shared two-qubit state using initially shared three-qubit W_k state as a resource (Maximum)	Concurrence of finally shared two-qubit state using initially shared four-qubit W_k state as a resource (Case-1)	Concurrence of finally shared two-qubit state using initially shared four-qubit W_k state as a resource (Case-2)	Concurrence of finally shared two-qubit state using initially shared four-qubit W_k state as a resource (Case-3)	Concurrence of finally shared two-qubit state using initially shared four-qubit W_k state as a resource (Maximum)
k=1	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	0.771	0.771	0.771	0.923	0.923	0.771	0.923
	0.2	0.943	0.943	0.943	1.000	1.000	0.943	1.000
	0.3	0.997	0.997	0.997	0.965	0.965	0.997	0.997
	0.4	0.990	0.990	0.990	0.891	0.891	0.990	0.990
	0.5	0.943	0.943	0.943	0.800	0.800	0.943	0.943
	0.6	0.866	0.866	0.866	0.700	0.700	0.866	0.866
	0.7	0.762	0.762	0.762	0.591	0.591	0.762	0.762
	0.8	0.629	0.629	0.629	0.471	0.471	0.629	0.629
	0.9	0.447	0.447	0.447	0.324	0.324	0.447	0.447
	1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
k=2	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	0.866	0.700	0.866	0.980	0.866	0.771	0.980
	0.2	0.990	0.891	0.990	0.980	0.990	0.943	0.990
	0.3	0.992	0.976	0.992	0.898	0.992	0.997	0.997
	0.4	0.943	1.000	1.000	0.800	0.943	0.990	0.990
	0.5	0.866	0.980	0.980	0.700	0.866	0.943	0.943
	0.6	0.771	0.923	0.923	0.600	0.771	0.866	0.866
	0.7	0.661	0.831	0.831	0.499	0.661	0.762	0.762
	0.8	0.533	0.700	0.700	0.392	0.533	0.629	0.629
	0.9	0.371	0.507	0.507	0.267	0.371	0.447	0.447
	1.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
k=5	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	0.980	0.644	0.980	0.990	0.815	0.771	0.990
	0.2	0.980	0.843	0.980	0.866	0.968	0.943	0.968
	0.3	0.898	0.947	0.947	0.738	1.000	0.997	1.000
	0.4	0.800	0.994	0.994	0.629	0.973	0.990	0.990
	0.5	0.700	0.996	0.996	0.533	0.911	0.943	0.943
	0.6	0.600	0.958	0.958	0.447	0.825	0.866	0.866
	0.7	0.499	0.881	0.881	0.365	0.717	0.762	0.762
	0.8	0.392	0.755	0.755	0.283	0.585	0.629	0.629
	0.9	0.267	0.557	0.557	0.191	0.411	0.447	0.447
	1.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
k=10	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	0.995	0.623	0.995	0.908	0.795	0.771	0.908
	0.2	0.884	0.823	0.884	0.722	0.957	0.943	0.957
	0.3	0.760	0.933	0.933	0.589	1.000	0.997	1.000
	0.4	0.650	0.988	0.988	0.489	0.982	0.990	0.990
	0.5	0.553	0.999	0.999	0.408	0.927	0.943	0.943
	0.6	0.464	0.969	0.969	0.338	0.845	0.866	0.866
	0.7	0.380	0.898	0.898	0.274	0.739	0.762	0.762
	0.8	0.295	0.777	0.777	0.211	0.605	0.629	0.629
	0.9	0.199	0.577	0.577	0.141	0.428	0.447	0.447
	1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
k=100	0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	0.548	0.602	0.602	0.404	0.774	0.771	0.774
	0.2	0.383	0.802	0.802	0.276	0.944	0.943	0.944
	0.3	0.297	0.918	0.918	0.212	0.997	0.997	0.997
	0.4	0.240	0.981	0.981	0.171	0.989	0.990	0.990
	0.5	0.197	1.000	1.000	0.140	0.941	0.943	0.943
	0.6	0.161	0.979	0.979	0.115	0.864	0.866	0.866
	0.7	0.130	0.915	0.915	0.092	0.760	0.762	0.762
	0.8	0.099	0.798	0.798	0.070	0.626	0.629	0.629
	0.9	0.066	0.598	0.598	0.047	0.445	0.447	0.447
	1	0.000	0.000	0.000	0.000	0.000	0.000	0.000

states,

$$|\eta_k\rangle^+_{12...N} = \frac{1}{\sqrt{(N-2)(2k+N-3)+2}}\Big[|010...N\rangle + \sqrt{k}|001...N\rangle + \sqrt{k+1}|0001...N\rangle .....$$
$$+ \sqrt{k+(N-3)}|000...1\rangle + \sqrt{(N-2)k + \frac{(N-2)(N-3)}{2} + 1}\,|100...0\rangle\Big]_{12...N} \quad (6.28)$$

where qubit 1 is with Alice and rest of the qubits are with Bob. Similar to the four-qubit case, Alice can produce four distinct messages for Bob using single qubit unitary transformations $I, \sigma_x, \sigma_y, \sigma_Z$ such that

$$\begin{aligned}(I\otimes I\otimes I\otimes I)|\eta_k\rangle^+_{12...N} &= |\eta_k\rangle^+_{12...N}\\ (\sigma_x\otimes I\otimes I\otimes I)|\eta_k\rangle^+_{12...N} &= |\xi_k\rangle^+_{12...N}\\ (\sigma_z\otimes I\otimes I\otimes I)|\eta_k\rangle^+_{12...N} &= |\eta_k\rangle^-_{12...N}\\ (i\sigma_y\otimes I\otimes I\otimes I)|\eta_k\rangle^+_{12...N} &= |\xi_k\rangle^-_{12...N}\end{aligned} \quad (6.29)$$

Therefore, our N-qubit W-type states can also be used for optimal super dense coding protocol.

6.6 EXPERIMENTAL GENERATION OF W_K-TYPE STATES

In the previous sections, we have demonstrated the usefulness of four-qubit W-type states for quantum information processing. Considering the importance of four-qubit W-type states, it is imperative to propose a method for experimental realization of four-qubit W-type states. We now proceed to discuss a method for experimental realization of W-type states represented in Eq. (6.2).

The standard three-qubit W-type states have been experimentally realized using spontaneous parametric down-conversion [Eibl *et al.*, 2004a]. The fundamental and theoretical framework to analyse the properties of W class of states have allowed the experimental realization of W class of states to become an area of extensive research [Dogra *et al.*, 2015; Adhikari, 2015; Zang *et al.*, 2016]. Recently, Wu *et al* [Wu *et al.*, 2016] described the experimental generation of tripartite entangled polarization states using stokes operators and Dong *et al* [Dong *et al.*, 2016] proposed the experimental preparation of three-qubit W-type states originally proposed by Pati and Agrawal. In this section, we use three-qubit W-type states generated by Dong *et al* as a input to prepare four-qubit W-type states. Our procedure also involves the use of single- and two-qubit quantum gates [Nemoto and Munro, 2004; Fiorentino and Wong, 2004; Okamoto *et al.*, 2005; Djordjevic, 2010; Etesse *et al.*, 2015] along with weak measurements [Aharonov *et al.*, 1988; Kim *et al.*, 2012], if required. It is important to mention that the gates and measurements used here can be experimentally realized as well. We again define the three-qubit W state prepared by Dong *et al* as,

$$|\Psi_k\rangle_{123} = \frac{1}{\sqrt{2+2k}}\Big[|100\rangle + \sqrt{k}e^{i\gamma}|010\rangle + \sqrt{k+1}e^{i\delta}|001\rangle\Big]_{123} \quad (6.30)$$

In the simplest case, where all the phases are 0 and $k=1$, we have

$$|\Psi_1\rangle_{123} = \frac{1}{2}\Big[|100\rangle + |010\rangle + \sqrt{2}|001\rangle\Big]_{123} \quad (6.31)$$

We will use the state $|\Psi_1\rangle_{123}$ for preparation of the four-qubit W state $|\Psi_1\rangle_{1234}$ given by Eq. (6.9) for $k=1$. For this, we write

$$|\psi_1\rangle_{1234} = |0\rangle_1 \otimes \frac{1}{2}\Big[|100\rangle + |010\rangle + \sqrt{2}|001\rangle\Big]_{234} \quad (6.32)$$

In order to generate the four-qubit W-type state, we first perform a controlled-Hadamard operation on qubits 1 and 2, considering qubit 2 as the control qubit, such that

$$\left|\psi_1'\right\rangle_{1234} = \frac{1}{2\sqrt{2}}\left[|0100\rangle + |1100\rangle + \sqrt{2}\,|0010\rangle + 2\,|0001\rangle\right]_{1234} \tag{6.33}$$

We can now perform a C-NOT operation on qubits 1 and 2 by keeping qubit 1 as the control qubit to get the four-qubit W-type state as

$$|\Psi_1\rangle_{1234} = \frac{1}{2\sqrt{2}}\left[|1000\rangle + |0100\rangle + \sqrt{2}\,|0010\rangle + 2\,|0001\rangle\right]_{1234} \tag{6.34}$$

The experimental generation of the generalized four-qubit state requires the input state to be the direct product of a single-qubit state and a three-qubit W type state expressed in Eq. (6.30) so that the joint state of four qubits can be expressed as

$$\begin{aligned} |\phi_k\rangle_{1234} &= |\Psi_k\rangle_{123} \otimes |0\rangle_4 \\ &= \frac{1}{\sqrt{2+2k}}\left[|1000\rangle + \sqrt{k}e^{i\gamma}\,|0100\rangle + \sqrt{k+1}e^{i\delta}\,|0010\rangle\right]_{1234} \end{aligned} \tag{6.35}$$

On qubits 3 and 4, we perform controlled-Hadamard operation keeping qubit 3 as the control to get

$$\begin{aligned} \left|\phi_k'\right\rangle_{1234} = \frac{1}{2\sqrt{2+2k}}[2\,|1000\rangle &+ 2\sqrt{k}e^{i\gamma}\,|0100\rangle \\ &+ \sqrt{2}\sqrt{k+1}e^{i\delta}\,|0010\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}\,|0011\rangle\Big]_{1234} \end{aligned} \tag{6.36}$$

We now perform a C-NOT operation on qubits 3 and 4 by considering qubit 4 as the control qubit to get

$$\begin{aligned} \left|\phi_k''\right\rangle_{1234} = \frac{1}{2\sqrt{2+2k}}[2\,|1000\rangle &+ 2\sqrt{k}e^{i\gamma}\,|0100\rangle \\ &+ \sqrt{2}\sqrt{k+1}e^{i\delta}\,|0010\rangle + \sqrt{2}\sqrt{k+1}e^{i\delta}\,|0001\rangle\Big]_{1234} \end{aligned} \tag{6.37}$$

We further perform weak measurements $M = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1/4} \end{bmatrix}$ on qubits 1 and 2 and $M' = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1/2} \end{bmatrix}$ on qubit 3 such that $\left|\phi_k''\right\rangle_{1234}$ becomes

$$\begin{aligned} \left|\Psi_k'\right\rangle_{1234} = \frac{1}{2\sqrt{k+1}}[|1000\rangle &+ \sqrt{k}e^{i\gamma}\,|0100\rangle \\ &+ \sqrt{k+1}e^{i\delta}\,|0010\rangle + \sqrt{2k+2}e^{i\delta}\,|0001\rangle\Big]_{1234} \end{aligned} \tag{6.38}$$

The state in Eq. (6.38) is the same as the state in Eq. (6.2) except for the equal phase factors in the 3rd and the 4th terms. For this, we finally perform a simple unitary operation $U = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ on qubit 4. Therefore, we have

$$|\Psi_k\rangle_{1234} = \frac{1}{2\sqrt{k+1}}\left[|1000\rangle + \sqrt{k}e^{i\gamma}\,|0100\rangle + \sqrt{k+1}e^{i\delta}\,|0010\rangle + \sqrt{2k+2}e^{i\zeta}\,|0001\rangle\right]_{1234} \tag{6.39}$$

6.7 SUMMARY

We have analysed a class of partially entangled four-qubit W-type states for efficient quantum information processing tasks. Although performing and distinguishing multiqubit measurements is an uphill task, nevertheless, our states can be used for deterministic teleportation with unit fidelity. In order to demonstrate the practical utility of such states, we have discussed and compared the efficiencies of three- and four-qubit W-type states for sharing optimal bipartite entanglement between two users. Furthermore, we have also proposed experimental realization of four-qubit W_k states which increases the importance of results obtained in this study. Our results will be of high importance in situations where users only have access to partially entangled states and would like to establish optimal bipartite entanglement for efficient and deterministic information processing. The analytical relations between the range of state parameters, and optimal concurrence of the finally shared state is also obtained allowing one to decide when to use a three or four-qubit W-type state for a particular protocol. We have also shown that our states can be used for optimal dense coding as well. The protocols have also been generalized for the case of *N*-qubits.

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