5
Approximation for Conflict as Matching

In this chapter, we consider conflicts in approximation paradigm. We discuss the case when the conflict graph is a matching. For this, we take the problem SEMI-DISCRETE FRÉCHET DISTANCE introduced in Chapter 1.

SEMI-DISCRETE FRÉCHET DISTANCE **Input:** A set of points P and a line-segment ℓ in $\mathbb{R}^2$. **Question:** Find $d^F(P,\ell)$.

We consider the approximation on the conflict version of SEMI-DISCRETE FRÉCHET DISTANCE called CONFLICT-FREE FRÉCHET DISTANCE. Let us recall the problem.

CONFLICT-FREE FRÉCHET DISTANCE **Input:** A set $\mathscr{Q}$ of pairs of points, and a line-segment ℓ in $\mathbb{R}^2$. **Question:** Find a conflict-free subset of points $P^* \subset \bigcup_{Q\in\mathscr{Q}} Q$ which minimizes $d^F(P^*,\ell)$

In this chapter we provide 3-factor approximation algorithm for optimization version of CONFLICT-FREE FRÉCHET DISTANCE where we are trying to minimize semi-discrete Fréchet distance.

5.1 APPROXIMATION ALGORITHM FOR CONFLICT-FREE FRÉCHET DISTANCE

In this section we present an approximation algorithm for CONFLICT-FREE FRÉCHET DISTANCE. Let us first define some terminology. As before, assume that the line-segment ℓ coincides with the X-axis and has end points $(x_1,0)$ and $(x_2,0)$. For any point set A, denote the semi-discrete Fréchet distance between A and line-segment ℓ by $d^F(A,\ell)$. Also let $\Gamma(A)$ be the lower envelope of the functions $f_{p_i}(x) = \sqrt{(x-a)^2+b^2}$ for all $p_i = (a,b) \in A$ where $x_1 \le x \le x_2$. Now let us start our discussion with the following observation about the semi-discrete Fréchet distance.

Observation 5.1.1. *For any sets of points A and B where $A \subseteq B$, $d^F(A,\ell) \ge d^F(B,\ell)$.*

Proof. Let $C \subseteq A$ be the set of points that achieves $d^F(A,\ell) = d$. Since $A \subseteq B$, we have that $C \subset B$ and hence $d^F(B,\ell) \le d$. $\square$

Let $(\mathscr{Q} = \{Q_1, Q_2 \dots Q_n\}, \ell)$ be the input instance of CONFLICT-FREE FRÉCHET DISTANCE, where $Q_i = \{q_i, \overline{q_i}\}$. Let $Q = \bigcup_{i=1}^{n} Q_i$. By Theorem 3.1.2, we can find $d^F(Q,\ell)$ in $\mathscr{O}(n\log n)$ time. Assume P^{opt} is a conflict-free subset of Q that minimizes the semi-discrete Fréchet distance and let $d^{opt} = d^F(P^{opt},\ell)$ i.e. optimal conflict-free semi-discrete Fréchet distance. If $\Gamma(Q)$ contains at most one of f_{q_i} or $f_{\overline{q_i}}$ for each $Q_i = \{q_i, \overline{q_i}\}$, then $d^{opt} = d^F(Q,\ell)$. As $P^{opt} \subseteq Q$, from Observation 5.1.1 we have the following lemma.

Observation 5.1.2. *$d^{opt} \ge d^F(Q,\ell)$.*

Suppose the set of points for which the corresponding $f_{q_i}(x)$ are in $\Gamma(Q)$ be $P^{'}$. Observe that if $P^{'}$ does not contain points from the same pair i.e. if it is already conflict-free, then $d^F(P^{'},\ell)$ is the conflict-free semi-discrete Fréchet distance and we have $d^{opt} = d^F(Q,\ell) = d^F(P^{'},\ell)$. If not, then we want to choose a conflict-free subset $P^{''}$ of $P^{'}$ such that $d^F(P^{''},\ell) \leq 3d^F(P^{'},\ell)$.
Now let us construct the set $P^{''}$. First, for all the points $q_i \in P^{'}$ such that $\overline{q_i} \notin P^{'}$, we include q_i in $P^{''}$. For the rest of the points in $P^{'}$, let $P_{pair} = \{p_1, p_2, \dots p_{2k}\}$ be the sorted order of points along X-axis where each $p_i = q_j$ or $\overline{q_j}$ for some j. Now from P_{pair} , we create disjoint bags $B_1, B_2, \dots, B_k$ each containing two points i.e. $B_i = \{p_{2i-1}, p_{2i}\}$. We construct a bipartite graph $G = (U,V,E)$ where $U = \{B_1, B_2, \dots, B_k\}$ and V is the set of all k pairs $Q_i = \{q_i, \overline{q_i}\}$ such that both q_i and $\overline{q_i}$ are in P_{pair}. We add an edge $e_{ij} = (B_i, Q_j)$, if $B_i \cap Q_j \neq \emptyset$. For an example, see Figure 5.1.

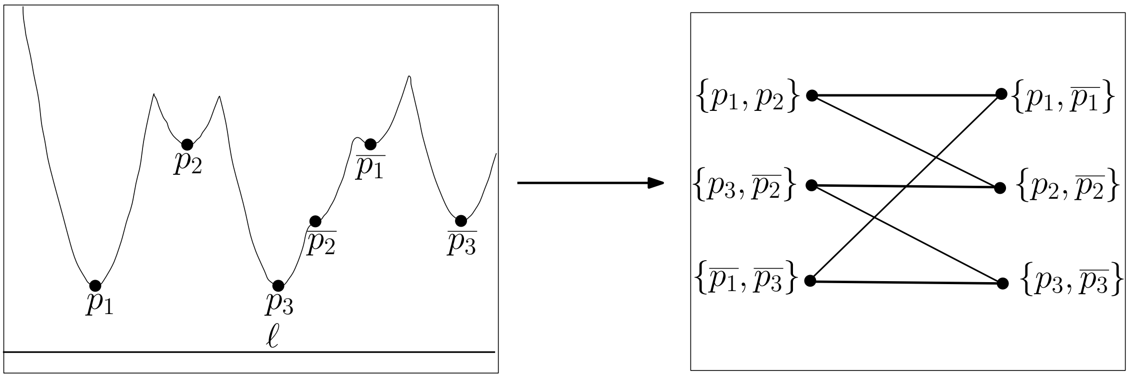

Figure 5.1 : Creating the bipartite graph from the lower envelope

Now we have the following lemma.

Lemma 5.1.1. $G = (U,V,E)$ *contains a perfect matching* M*.*

Proof. Each vertex in U and V has degree at least 1 and at most 2. Also if vertex B_i in U has degree one, then the vertex Q_j to which it is connected in V also has degree one (as it implies that both $B_i = Q_j = \{q_i, \overline{q_i}\}$). Similarly degree two vertices in U are connected to degree two vertices in V. Thus, we note that every connected component is either an even cycle or an edge. Every subset W of U has a set of neighbours $N_G(W)$ such that $|W| \leq |N_G(W)|$ (here the neighbours of W is the set of vertices in V to which vertices in W are connected). Hence by Hall's marriage theorem [40], G has a perfect matching M. $\square$

Let M be a perfect matching in G. Now for each edge (B_i, Q_j) selected in the matching M, if $|B_i \cap Q_j| = 1$ then include $B_i \cap Q_j$ in $P^{''}$, else if $|B_i \cap Q_j| = 2$ then we include one arbitrary point of $B_i \cap Q_j$ in $P^{''}$. Observe that from each pair of points in P_{pair}, only one point is selected. Thus $P^{''}$ is conflict-free. Now we have following lemma.

Lemma 5.1.2. $d^F(P^{''},\ell) \leq 3d^F(Q,\ell)$.

Proof. Since $d^F(Q,\ell) = d^F(P^{'},\ell)$, it is enough to show that $d^F(P^{''},\ell) \leq 3d^F(P^{'},\ell)$. Let π be the sorted order of points in $P^{'}$ along X-axis. We first prove the following claim.

Claim 5.1.1. *For any point $s \in P^{'}$, at least one among s, its predecessor in π and its successor in π, is in P''.*

Proof. We claim that (i) no three consecutive points from π can be in $P' \setminus P''$. For any three consecutive points q_1, q_2, q_3, either one of them does not belong to P_{pair} and thus belongs to P'' or one among $\{q_1, q_2\}$ and $\{q_2, q_3\}$ belongs to P_{pair}. From the construction of P'', we have one point from every pair in P_{pair}. Thus we include one among q_1, q_2, q_3, in P''. Now we claim that (ii) at least one among the first two points in π is in P''. Let s_1 and s_2 be the first two points in π. If $\{s_1, s_2\} \not\subseteq P_{pair}$, then $P'' \cap \{s_1, s_2\} \neq \emptyset$. Otherwise $B_1 = \{s_1, s_2\}$ and by the construction of P'', we have that $P'' \cap \{s_1, s_2\} \neq \emptyset$. Similarly we can prove that (iii) at least one among the last two points in π is in P''.

The claim follows from the statements $(i), (ii)$ and (iii). $\square$

Let $d = d^F(P^{'}, \ell)$. Now we prove that $d^F(P^{''}, \ell) \leq 3d$. Towards that it is enough to prove that for any point on ℓ, there is a point in $P^{''}$, which is at a distance at most $3d$. Let z be a point in ℓ. Since $d = d^F(P^{'}, \ell)$, there is a point s in $P^{'}$ such that $d(z,s) \leq d$. Now we show that there is a point $s' \in P''$ such that $d(z,s') \leq 3d$. If $s \in P''$, then we set $s' = s$. Otherwise, by Claim 5.1.1, either its successor or its predecessor in π belongs to P''. Let s' be a point in P'' which is either successor of s or predecessor of s. Since $d = d^F(P^{'}, \ell)$, there is a point t on ℓ such that $d(t,s) \leq d$ and $d(t,s') \leq d$. Now we have $d(z,s) \leq d$, $d(s,t) \leq d$ and $d(t,s') \leq d$. Hence by triangular inequality, we get $d(z,s') \leq 3d$. This completes the proof of the lemma. $\square$

Hence we have the following theorem.

Theorem 5.1.3. *There is a* 3*-approximation algorithm for* CONFLICT-FREE FRÉCHET DISTANCE.