6

Approximation for Conflicts as Geometric Graphs

In this chapter we continue our study of conflicts in approximation paradigm. Here we consider the conflict graph to be an intersection graph of some geometric objects.

We define some notations specific to this chapter before recalling the problems. Here, the universe is represented by a set of points denoted as $P = \{p_1, p_2, \dots, p_n\}$. We denote the set of geometric objects as $\mathscr{O} = \{O_1, O_2, \dots, O_m\}$. Most of geometric covering problems consider the intersection graph on the set of geometric objects. In the intersection graph, set of objects represents vertices and there is an edge between two objects if and only if the two objects intersect. The intersection graph of geometric objects $\mathscr{O}$ is denoted as $\mathscr{COV}(V, E_v)$ (hence $V = \mathscr{O}$). We assume that the underlying representation of the geometric objects and points is given with $\mathscr{COV}$. Further let $\mathscr{CG}(U, E_u)$ represents the conflict graph where $U = \mathscr{O}$. We call a set of geometric objects *conflict free* if they form an independent set in $\mathscr{CG}$. Also, we define a bijective function $f : U \rightarrow V$ such that $v_i \in V$ and $u_i \in U$ denotes the vertex corresponding to same object $O_i \in \mathscr{O}$ for all $1 \leq i \leq m$. For any subset of objects $F \subseteq V$, let $p(F)$ be the set of points covered by objects in F. With slight abuse of notation, for any subset of objects $F \subseteq U$ we also denote the set of points covered by $f(F)$ as $p(F)$. Formally, $p : 2^U \cup 2^V \rightarrow \mathbb{N}$ is the function such that

$$\text{p(F)} = \begin{cases} \{x | x \in P \text{ and } \exists F_i \in F \text{ such that } x \in f(F_i) \cap P\} & \text{if } F \in 2^U \\ \{x | x \in P \text{ and } \exists F_i \in F \text{ such that } x \in F_i \cap P\} & \text{if } F \in 2^V \end{cases}$$

Assume, $OPT \subseteq V$ represents the optimal solution to the covering problem.

In our first problem, we consider the geometric objects to be unit intervals. Also, the conflict graph is given as some other intersection graph of set of m unit intervals. Notice that the intersection graph in covering and conflict are different graphs and need not be isomorphic. The maximization version of the problem is defined as follows.

MAX UNIT INTERVAL CF-SC **Input:** A set of points P on X-axis, a set of unit intervals $\mathscr{O}$ on X-axis, a unit interval graph as conflict graph $\mathscr{CG}$. **Question:** Maximize the number of points covered using a set of conflict free unit intervals?

In our next problem, we are given a set of points P, that lies in $\mathbb{R}^2$ and the geometric objects are unit disks. The conflict graph is given as some other intersection graph on a set of m unit disks. The intersection graph in covering and conflict are different graphs and need not be isomorphic. We define the problem as follows,

MAX UNIT DISK - UNIT INTERVAL CF-SC **Input:** A set of points P in $\mathbb{R}^2$, a set of unit intervals $\mathscr{O}$, a unit disk graph as conflict graph CG. **Question:** Maximize the number of points covered using a set of conflict free unit intervals?

We assume that we are given a unit disk representation graph in our above problem.

Now we mention results that we prove in this chapter.

Results in this Chapter We propose a general framework where if the geometric graphs $\mathscr{COV}$ and $\mathscr{CG}$ satisfy some properties, then GRAPHICAL CONFLICT FREE SET COVER admits constant factor approximation where the constant is based on those properties. As an application to this, we give an 8-factor approximation algorithm for MAX UNIT INTERVAL CF-SC in section 6.2. Later in section 6.3, we prove the approximation hardness of the problem by showing that that the problem is APX-hard. Hence, we prove that MAX UNIT INTERVAL CF-SC does not admit PTAS under standard computer theoretic assumptions. We also present a 36-factor approximation algorithm for MAX UNIT DISK - UNIT INTERVAL CF-SC in section 6.2. We show that this problem is also APX-hard in 6.2. We prove APX-hardness for more general case considered in Chapter 4 where conflict graph is tree or 1-arboricity graph (thus holds for higher arboricity graphs too) in point interval covering. We also show problem is APX-hard when both $\mathscr{CG}$ and $\mathscr{COV}$ are unit coin graphs.

6.1 A GENERAL FRAMEWORK

Here, we say an object $O \in \mathscr{O}$ covers a point p if $p \in O$. For any $O \in \mathscr{O}$, we denote the points covered by O by $p(O)$. With slight abuse of notation, for any subset of geometric objects $\mathscr{O}' \subseteq \mathscr{O}$ we define $p(\mathscr{O}') = \cup_{O \in \mathscr{O}'} p(O)$. We assume that $v_i \in \mathscr{COV}(V, E_v)$ and $u_i \in \mathscr{CG}(U, E_u)$ denotes the same geometric object O_i for all $1 \leq i \leq m$. The objective of this chapter is to present an efficient approximation algorithm for Geometric Conflict Free Covering problem when $\mathscr{COV}$ and $\mathscr{CG}$ satisfies certain conditions.

We say a Geometric Conflict Free Covering follows the (α, β, γ)-property if the following restrictions hold.
The restrictions on $\mathscr{COV}$ are as follows:

(i) The vertices in V can be divided into α many color classes $\{\mathscr{V}_1, \mathscr{V}_2, \dots \mathscr{V}_\alpha\}$ where each geometric object in V belongs to exactly one color class.

(ii) Each color class $\mathscr{V}_i$ can be divided into disjoint subsets $\{V_{i1}, V_{i2}, \dots V_{ia}\}$ where $\mathscr{V}_i = \cup_{1 \leq j \leq a} V_{ij}$

(iii) The set of geometric objects corresponding to the vertices in V_{ij} be $\mathscr{O}_{ij}$. In any optimal solution the points $p(\mathscr{O}_{ij})$ can be covered by at most β many objects.

We have the following restriction on $\mathscr{CG}$:

(i) Vertices in U can be divided into l many color classes $\{\mathscr{U}_1, \mathscr{U}_2 \dots \mathscr{U}_\gamma\}$ where each geometric object in U belongs to exactly one color class.

(ii) Each color class $\mathscr{U}_i$ is a union of disjoint subsets $\{U_{i1}, U_{i2}, \dots U_{ic}\}$ where each U_{ij} forms a clique.

We denote such a Geometric Conflict Free Covering problem by (α, β, γ)-GCFCOV. For example in Figure 6.1(above) the covering objects (intervals) are divided into two classes, $\mathscr{V}_1$ and $\mathscr{V}_2$ (dashed). Each of the color classes is a collection of two cliques. in Figure 6.1(below) $\mathscr{CG}$ is divided into two classes, $\mathscr{U}_1$ and $\mathscr{U}_2$ (doted).

Lemma 6.1.1. $(1,1,1)$-GCFCOV *can be solved in polynomial time.*

Proof. From the definition of GCFCOV, vertices V of $\mathscr{COV}$ can be divided into disjoint subsets $\{V_{11}, V_{12}, \dots V_{1a}\}$ where $V = \cup_{1 \leq j \leq a} V_{1j}$. Also, $p(\mathscr{O}_{1j})$ can be covered by at most one object. Thus from each subset V_{1j} selecting one object into the optimal solution suffices. From the restriction on $\mathscr{CG}$, vertices U of $\mathscr{CG}$ can be partitioned into disjoint clique $\{U_{11}, U_{12}, \dots U_{1c}\}$. Therefore optimal solution (or any feasible solution) can contain at most one object from each U_{1i}.

In order to show that the problem is polynomial time solvable, consider the following weighted bipartite graph $G_b(V_b, E_b)$ where $V_b = \{V_{11}, V_{12}, \dots V_{1a}\} \cup \{U_{11}, U_{12}, \dots U_{1c}\}$. There is an edge between V_{1i} and U_{1j} if and only if there is at least one object O_k such that $v_k \in V_{1i}$ and $u_k \in U_{1j}$. The weight of the edge (V_{1i}, U_{1j}) is equals to $\max_{O_k} |p(O_k)|$ where the maximum is taken over all objects O_k such that $v_k \in V_{1i}$ and

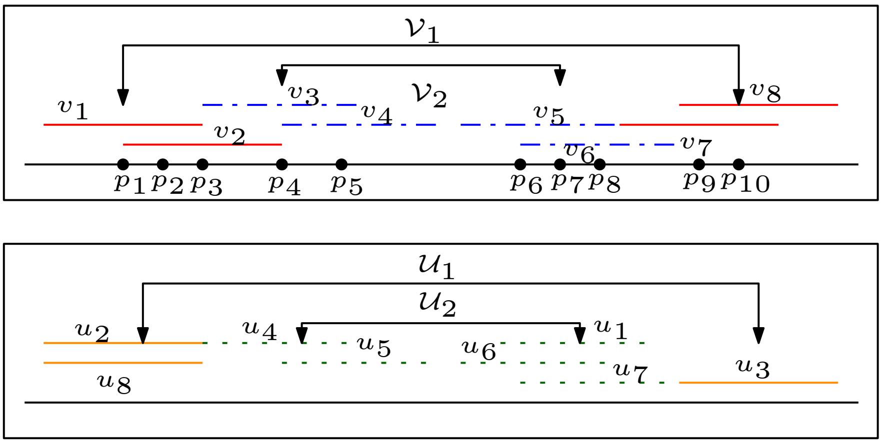


Figure 6.1 : Special Geometric covering

$u_k \in U_{1j}$. Thus each edge in G_b represents an object. Any matching in G_b is a collection of objects, O_M. Let M^* denotes the maximum weighted matching in G_b. We have the following claim.

Claim 6.1.1. *O_{M^*} is an optimal solution to* $(1,1,1)$-GCFCOV.

Proof. By definition any solution to $(1,1,1)$-GCFCOV is a matching in G_b. Observe that the points covered by the objects represented by the edges in the matching is disjoint because each subset V_{1i} is disjoint from any other. Hence the total points covered by all the objects in a matching is the summation of the edge weights in the matching, i.e. cost of matching. Thus the claim holds. □

As the matching in bipartite graph can be solved in polynomial time so is $(1,1,1)$-GCFCOV.

□

Theorem 6.1.1. *There is an $\alpha * \beta * \gamma$ factor approximation algorithm for* (α,β,γ)-GCFCOV.

Proof. Consider an instance $(P,\mathscr{O},\mathscr{CG})$ of (α,β,γ)-GCFCOV. Let OPT be any optimal solution. The vertex corresponding to any object $O \in \mathscr{O}$ belongs to one of the α color classes of $\mathscr{COV}$ and one of the β color classes of $\mathscr{CG}$. Thus each object has one of the $\alpha * \beta$ colors associated with it. For any $(i,j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta$, let OPT_{ij} denotes the objects in OPT with color (i,j). Similarly the objects in $\mathscr{O}$ can also be divided into color classes and we denote it by $\mathscr{O}_{ij}$. By the optimal solution of the color class $OPT(\mathscr{O}_{ij})$ denote the maximum points in P that can be covered by any conflict free subset of objects from $\mathscr{O}_{ij}$. Observe $OPT(\mathscr{O}_{ij}) \geq OPT_{ij}$. By pigeonhole principle $\max_{ij} OPT_{ij} \geq \frac{OPT}{\alpha*\beta}$. Thus by computing the optimal solution to each color class and choosing the maximum, we have a $\alpha * \beta$ factor approximate solution to (α,β,γ)-GCFCOV. From now onwards we assume that we are interested in finding an optimal solution for the color class (i,j).

Recall each color class $\mathscr{V}_i$ can be divided into disjoint subsets $\{V_{k1}, V_{k2}, \ldots V_{ka}\}$ and $OPT(\mathscr{O}_{ij}) \cap V_{kl} \leq \gamma$. For each subset V_{kl} let λ_{kl} denotes the object in $OPT(\mathscr{O}_{ij}) \cap V_{kl}$ which covers the maximum points among the objects in $OPT(\mathscr{O}_{ij}) \cap V_{kl}$. Let Λ be the collection of all such objects. Observe $|p(\Lambda) \geq \frac{OPT(\mathscr{O}_{ij})}{\gamma}|$. Hence if we can solve the problem with added restriction that from each subset V_{kl} only one object can be chosen then it is a γ approximation. This problem is same as solving $(1,1,1)$-GCFCOV. From Lemma 6.1.1 we know $(1,1,1)$-GCFCOV is polynomial time solvable. Thus the result holds.

□

We have following corollary to the above theorem.

Corollary 6.1.1. *We have following results to* GRAPHICAL CONFLICT FREE SET COVER *for different restrictions on $\mathscr{CG}$ and $\mathscr{COV}$ as mentioned below.*

1. *If $\mathscr{COV}$ is α colorable and $\mathscr{CG}$ is γ colorable, then we have $\alpha * \gamma$ factor approximation algorithm for* GRAPHICAL CONFLICT FREE SET COVER*.*

2. *If $\mathscr{CG}$ is γ colorable and covering problem $\mathscr{COV}$ (without $\mathscr{CG}$) is polynomial time solvable then we have γ factor approximation algorithm for* GRAPHICAL CONFLICT FREE SET COVER*.*

In the next section we provided several geometric graph classes which supports the (α, β, γ)-GCFCOV properties.

6.2 DIFFERENT GEOMETRIC GRAPH CLASSES THAT FITS THE RESTRICTIONS

Now let's recall MAX UNIT INTERVAL CF-SC.

> MAX UNIT INTERVAL CF-SC
> **Input:** A set of points P on X-axis, a set of unit intervals $\mathscr{O}$ on X-axis, a unit interval graph as conflict graph $\mathscr{CG}$.
> **Question:** Maximize the number of points covered using a set of conflict free unit intervals?

We make the following claim.

Theorem 6.2.1. MAX UNIT INTERVAL CF-SC *is equivalent to* $(2,2,2)$-GCFCOV *and thus admits* 8*-factor approximation algorithm.*

Proof. Without loss of generality assume that all the intervals in $\mathscr{COV}$ lies between 0 and q on X-axis and all the intervals in $\mathscr{CG}$ lies between 0 and r where $q, r \in \mathbb{N}$. Also, let every point $p \in P$ is covered by at least one interval in $\mathscr{COV}$.

We define the set of points $C_o = \{(x,0)|x \text{ is odd integer}\}$, $C_e = \{(x,0)|x \text{ and is even integer}\}$. Without loss of generality, we can assume that none of the unit intervals in $\mathscr{COV}$ and $\mathscr{CG}$ has end points in C_o or C_e. Now we divide the intervals in $\mathscr{CG}$ into two disjoint color classes where $\mathscr{U}_1$ contains intervals in $\{O \in U | O \cap C_o \neq \emptyset\}$ and $\mathscr{U}_2$ contains intervals in $\{O \in U | O \cap C_e \neq \emptyset\}$. Similarly, we divide the intervals in $\mathscr{COV}$ into two disjoint color classes $\mathscr{V}_1$ contains intervals in $\{O \in V | O \cap C_o \neq \emptyset\}$ and $\mathscr{V}_2$ contains intervals in $\{O \in V | O \cap C_e \neq \emptyset\}$.

We notice that the intervals in $\mathscr{V}_i$ forms disjoint cliques for $i \in \{1,2\}$. We denote the set of intervals in each such clique as V_{ij} for $i \in \{1,2\}, 0 \leq j \leq \lceil \frac{q}{2} \rceil$. Similarly, the intervals in color classes in $\mathscr{CG}$ can be further partitioned into disjoint sets U_{ij} for $i \in \{1,2\}, 0 \leq j \leq \lceil \frac{r}{2} \rceil$ where set of intervals in each U_{ij} forms a clique in $\mathscr{CG}$.

We make the following claim.

Claim 6.2.1. *It suffices for any optimal solution to* MAX UNIT INTERVAL CF-SC *to contain at most two intervals from any V_{ij} where $i \in \{1,2\}, 0 \leq j \leq \lceil \frac{q}{2} \rceil$.*

Proof. Suppose not. Assume we need three intervals $O_1 = [a_1, a_1+1], O_2 = [a_2, a_2+1], O_3 = [a_3, a_3+1] \in$

V_{ij}. Without loss of generality, assume $a_1 < a_2 < a_3$. Then $p(O_1) \cup p(O_2) \cup p(O_3) = p(O_1) \cup p(O_3)$ as O_1, O_2, O_3 are unit intervals forming a clique. Hence, we can remove O_2 from the solution. □

Now, we can see that Max Unit Interval CF-SC follows (α, β, γ) property where $\alpha = \beta = \gamma = 2$. Hence the theorem is proved. □

Algorithm 2 is 8-factor approximation algorithm for Max Unit Interval CF-SC.

Algorithm 2: Approximation algorithm for Max Unit Interval CF-SC

```
Divide intervals in 𝒞𝒢 into two disjoint color classes where 𝒰1 contains intervals in
   {O ∈ U | O ∩ Co ≠ ∅} and 𝒰2 contains intervals in {O ∈ U | O ∩ Ce ≠ ∅}.
Further partition the intervals in each color class 𝒰i into disjoint sets Uij for i ∈ {1,2}, 0 ≤ j ≤ ⌈r/2⌉
   where intervals in each Uij form a clique. Let 𝒰j* = {Uj1, Uj2, ... U⌈r/2⌉} where j ∈ {1,2}.
Similarly, we divide the intervals in 𝒞𝒪𝒱 into two disjoint color classes where 𝒱1 contains intervals in
   {O ∈ V | O ∩ Co ≠ ∅} and 𝒱2 contains intervals in {O ∈ V | O ∩ Ce ≠ ∅}.
Again partition the intervals in each color class 𝒱i into disjoint sets Vij for i ∈ {1,2}, 0 ≤ j ≤ ⌈q/2⌉ where
   intervals in each Vij form a clique. Let 𝒱i* = {Vi1, Vi2, ... V⌈q/2⌉} where i ∈ {1,2}.
APPROX ← ∅
p(APPROX) ← ∅.
Forall Vb = {𝒱1*, 𝒱2*} and Ub ∈ {𝒰1*, 𝒰2*}
{
      Construct the weighted bipartite graph Gb(Vb, Ub, Eb).
      There is an edge (Vij, Ulp) ∈ Gb if and only if there exists Ok ∈ 𝒪 such that vk ∈ Vij
      and uk ∈ Ulp.
      The weight of the edge (Vij, Ulp) is equal to max_{Ok} |p(Ok)| where the maximum is taken over
      all intervals Ok such that vk ∈ Vij and uk ∈ Ulp. Thus each edge in Gb represents an unit
      interval.
      Let M* denote the maximum weighted matching in Gb and O_{M*} be corresponding collection of
      unit intervals.
            If(p(APPROX) < p(O_{M*}))
            {
              APPROX ← O_{M*}
              p(APPROX) ← p(O_{M*})
            }
}
```

Now let us recall Max Unit Disk - Unit Interval CF-SC.

> Max Unit Disk - Unit Interval CF-SC
> **Input:** A set of points P in $\mathbb{R}^2$, a set of unit intervals $\mathscr{O}$, a unit disk graph as conflict graph CG.
> **Question:** Maximize the number of points covered using a set of conflict free unit intervals?

Here we assume for each $O_i \in \mathscr{O}$, that we have a corresponding vertex $v_i \in \mathscr{COV}(V, E_v)$. Also, let us denote the set of unit disks corresponding to vertices in $\mathscr{CG}(U, E_u)$ again by $\mathscr{O}$. For each $v_i \in V$, we have corresponding vertex $u_i \in U$. Hence, $O_i \in \mathscr{O}$ represents a unit interval v_i if we talk about $\mathscr{COV}$ and unit disk u_i if we talk about $\mathscr{CG}$. Assume all the disks in $\mathscr{CG}$ are lying in XY-plane inside the square with vertices $\{(0,0),(0,p),(p,0),(p,p)\}$ where $p \in \mathbb{N}$. Divide the square further into p^2 unit (1×1) squares. Now we assign numbers between 0 to 9 to each square. It is done by repeating the grid of 3×3 horizontally and vertically starting from $(p,0)$. The 3×3 grid to be repeated is shown in the Figure 6.2. Label the squares as per grid labelling.

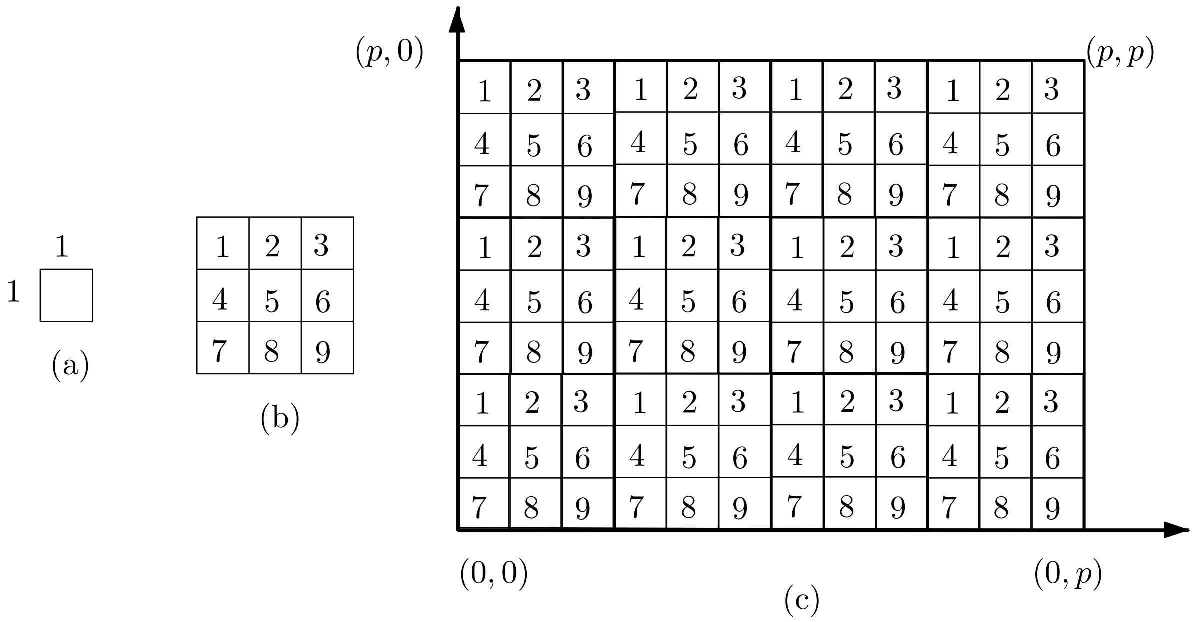


Figure 6.2 : Division of XY plane in $\mathscr{CG}$

Let the set of squares with label i where $1 \le i \le 9$ is denoted by S_{CG_i}. Also, let $S_{CG} = \bigcup_{i=1}^{9} S_{CG_i}$. Without loss of generality, we can assume that each unit disk in $\mathscr{CG}$ has its centre strictly lying inside the square $s \in S_{CG}$. We denote the coordinates of centre of disk O by $cen(O)$. Let $\mathscr{U}_i$ contains the disks in $\{O|O \in \mathscr{CG}$ *and* $cen(D) \in s$ *where* $s \in S_{CG_i}\}$ for all $1 \le i \le 9$. We can further partition each $\mathscr{U}_i$ into sets U_{ij} where each U_{ij} correspond to disks with centre in s_k for $1 \le i \le 9, 1 \le j \le \lceil \frac{p^2}{2} \rceil, s_k \in S_{CG_i}$.

We define the set of points $C_o = \{(x,0)|x$ is odd integer$\}$, $C_e = \{(x,0)|x$ and is even integer$\}$. Without loss of generality, we can assume that none of the unit intervals in $\mathscr{COV}$ has end points in C_o or C_e. Now we divide the intervals in $\mathscr{COV}$ into two disjoint color classes where $\mathscr{V}_1$ contains intervals in $\{O \in V | O \cap C_o \ne \emptyset\}$ and $\mathscr{V}_2$ contains intervals in $\{O \in V | O \cap C_e \ne \emptyset\}$. Again partition the intervals in each color class $\mathscr{V}_i$ into disjoint sets V_{ij} for $i \in \{1,2\}, 0 \le j \le \lceil \frac{q}{2} \rceil$ where intervals in each V_{ij} form a clique.

We have the following theorem.

Theorem 6.2.2. Max Unit Disk - Unit Interval CF-SC *is equivalent to* $(2,9,2)$-GCFCOV *and thus admits a* 36*-factor approximation algorithm.*

Proof. Using the claim in Theorem 6.2.1, we notice that it suffices for any optimal solution to Max Unit Disk - Unit Interval CF-SC to contain at most two intervals from any V_{ij} where $i \in \{1,2\}, 0 \le j \le \lceil \frac{q}{2} \rceil$. Thus with above partition of $\mathscr{CG}$ and $\mathscr{COV}$, Max Unit Disk - Unit Interval CF-SC follows (α,β,γ) property where $\alpha = \gamma = 2$ and $\beta = 9$. Hence by Theorem 6.1.1, Max Unit Disk - Unit Interval CF-SC admits 36-factor approximation algorithm. □

Algorithm 3 is 36-factor approximation algorithm for Max Unit Disk - Unit Interval CF-SC.

On similar lines, we now state few more results based on Corollary 6.1.1.

Algorithm 3: Approximation algorithm for MAX UNIT DISK - UNIT INTERVAL CF-SC

Divide disks in $\mathscr{CG}$ into nine disjoint color classes where $\mathscr{U}_i$ contains the disks in $\{O|O \in \mathscr{CG}$ *and* $cen(D) \in s$ *where* $s \in S_{CG_i}\}$ for all $1 \leq i \leq 9$.
Further partition each $\mathscr{U}_i$ into sets U_{ij} where each U_{ij} corresponds to disks with centre in s_k for $1 \leq i \leq 9, 1 \leq j \leq \lceil \frac{p^2}{2} \rceil, s_k \in S_{CG_i}$. Let $\mathscr{U}_j^* = \{U_{j1}, U_{j2}, \ldots U_{\lceil \frac{p^2}{9} \rceil}\}$ where $j \in \{1, 2 \ldots 9\}$.
Similarly, we divide the intervals in $\mathscr{COV}$ into two disjoint color classes where $\mathscr{V}_1$ contains intervals in $\{O \in V | O \cap C_o \neq \emptyset\}$ and $\mathscr{V}_2$ contains intervals in $\{O \in V | O \cap C_e \neq \emptyset\}$.
Again partition the intervals in each color class $\mathscr{V}_i$ into disjoint sets V_{ij} for $i \in \{1, 2\}, 0 \leq j \leq \lceil \frac{q}{2} \rceil$ where intervals in each V_{ij} form a clique. Let $\mathscr{V}_i^* = \{V_{i1}, V_{i2}, \ldots V_{\lceil \frac{q}{2} \rceil}\}$ where $i \in \{1, 2\}$.
$APPROX \leftarrow \emptyset$
$p(APPROX) \leftarrow \emptyset$.
For all $V_b = \{\mathscr{V}_1^*, \mathscr{V}_2^*\}$ and $U_b \in \{\mathscr{U}_1^*, \mathscr{U}_2^*, \ldots \mathscr{U}_9^*\}$ {
    Construct the weighted bipartite graph $G_b(V_b, U_b, E_b)$.
    There is an edge $(V_{ij}, U_{lp}) \in G_b$ if and only if there exists $O_k \in \mathscr{O}$ such that $v_k \in V_{ij}$
    and $u_k \in U_{lp}$.
    The weight of the edge (V_{ij}, U_{lp}) is equal to $\max_{O_k} |p(O_k)|$ where the maximum is taken over
    all intervals O_k such that $v_k \in V_{ij}$ and $u_k \in U_{lp}$. Thus each edge in G_b represents an interval.
    Let M^* denotes the maximum weighted matching in G_b and O_{M^*} to be corresponding collection
    of unit intervals.
        If($p(APPROX) < p(O_{M^*})$)
        {
            $APPROX \leftarrow O_{M^*}$
            $p(APPROX) \leftarrow p(O_{M^*})$
        }
}

Theorem 6.2.3. *We have following results to* GRAPHICAL CONFLICT FREE SET COVER *for different restrictions on* $\mathscr{CG}$ *and* $\mathscr{COV}$ *as mentioned below.*

1. *If* $\mathscr{CG}$ *is* d*-arboricity graph and* $\mathscr{COV}$ *is also* d*-arboricity graph, then we have* $(d+1)^2$ *factor approximation to* GRAPHICAL CONFLICT FREE SET COVER.

2. *If* $\mathscr{CG}$ *is* d*-arboricity graph and* $\mathscr{COV}$ *is interval graph, then we have* $(d+1)$ *factor approximation to* GRAPHICAL CONFLICT FREE SET COVER.

Above theorem is based on the fact that d-arboricity graph is $(d+1)$ colorable in polynomial time.

6.3 APPROXIMATION HARDNESS

Next, we consider another problem 2-PATH INTERVAL CF-SC and show that it is APX-hard using reduction from MAX 2-SAT 3. This further helps in proving APX-hardness for many graph classes. Let two length path denotes a tree on three vertices. We define 2-PATH INTERVAL CF-SC as follows.

2-PATH INTERVAL CF-SC
Input: A universe P of n points, a set of m unit intervals $\mathscr{I}$ on X-axis, a conflict graph $\mathscr{CG}$ which is a disjoint union of connected components where connected components are either isolated vertices, single edges or 2 length paths.
Question: Find the assignment to n variables so as maximum number of clauses can be satisfied.

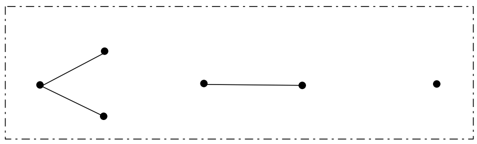
We call a graph that is disjoint union of isolated vertices, single edges and 2 length paths as a 2-*path graph*. Again intersection graph $\mathscr{COV}(V,E)$ and conflict graph $\mathscr{CG}(U,E')$ both on set of intervals $\mathscr{I}$ are different graphs and need not be isomorphic. The functions f and p are defined as used previously.

The problem MAX 2-SAT 3 is defined as follows.

MAX 2-SAT 3
Input: A 2 CNF formula Π with set of n variables $X = \{x_1, x_2, \dots, x_m\}$ and m clauses $C = \{c_1, c_2, \dots, c_n\}$ such that each variable occurs in at most 3 clauses.
Question: Find the assignment to n variables so as maximum number of clauses can be satisfied.

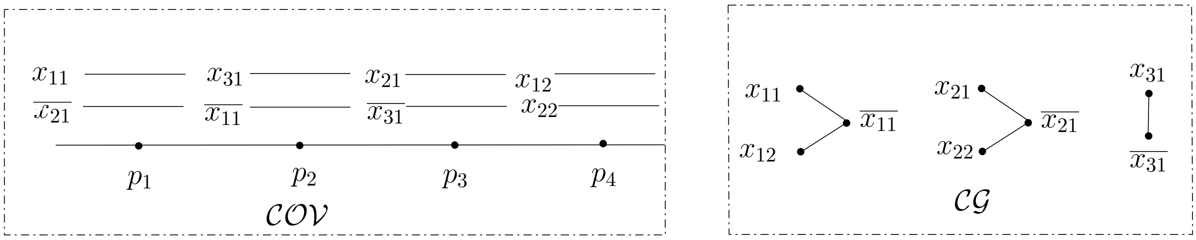
This problem has been proved to be APX-hard by Ausiello and et. al. in [8],[9].

We use the following reduction for 2-PATH INTERVAL CF-SC. For each clause $c_i \in C$, take a point $p_i = (i + .5i)$. These points constitute the set P. Consider the variable $x_j \in c_i$. Let it is y^{th} occurrence of the variable x_j in Π. Construct an unit interval v_{jw} covering point p_i only. Assume the variable $\overline{x_j} \in c_l$ and it is w^{th} occurrence of $\overline{x_j}$ in Π. Construct a unit interval $\overline{v_{jy}}$ covering point p_i only. Do it for all variables in Π. Such a set of intervals constitute $V \in \mathscr{COV}$. Now we need to construct $\mathscr{CG}$. Assume the variable $x_i \in X$ has j many occurrences $X_i = \{x_{i1}, x_{i2}, \dots, x_{ij}\}$ and $\overline{x_i} \in X$ has ℓ many occurrences $\overline{X_i} = \{\overline{x_{i1}}, \overline{x_{i2}}, \dots, \overline{x_{i\ell}}\}$. Then we construct a complete bipartite graph $I = \{X_i, \overline{X_i}, E\}$. Do it for all the variables in X. We take $\mathscr{CG} = \bigcup_{i=1}^{n} X_i$. Notice that as a variable can occur at most thrice in Π, thus $\mathscr{CG}$ consists of union of disjoint sub-graphs from Figure 6.3,(a). Hence CG is 2-path graph. The reduction is illustrated through an example in Figure 6.3(b).



(a)

$$\Pi = \underbrace{(x_1 \vee \overline{x_2})}_{c_1} \wedge \underbrace{(x_3 \vee \overline{x_1})}_{c_2} \wedge \underbrace{(x_2 \vee \overline{x_3})}_{c_3} \wedge \underbrace{(x_1 \vee x_2)}_{c_4}$$



(b)

Figure 6.3 : Reduction to MAX UNIT INTERVAL CF-SC from MAX 2-SAT 3

Claim 6.3.1. *Suppose optimal solution of an instance of* MAX 2-SAT 3 *satisfies m' clauses. Then optimal solution to corresponding instance of* MAX UNIT INTERVAL CF-SC *also covers m' points and vice versa.*

Proof. Consider the first case where OPT is the set of m' clauses satisfied in optimal solution to the instance of MAX 2-SAT 3. For each clause c_i that is satisfied in OPT, at least one of the literal should be true. Hence, there exists a variable $x \in c_i$ such that either x is true or $\overline{x}$ is true. Without loss of generality let the literal be x. Take all the unit intervals corresponding to x and their occurrences in Π into solution $MYSOL$. These unit intervals will cover the point p_i corresponding to c_i and other points corresponding to the clauses which x satisfies. Hence, $MYSOL$ covers at least m' points. Consider the clause c_j such that c_j is not satisfied in OPT. Hence all the literals are false in c_j. Without loss of generality, let $y \in c_j$ such that y is false. Hence, $\overline{y}$ is true. Thus, we have already picked unit intervals corresponding to y in $MYSOL$. These are in conflict with the intervals corresponding to y in $\mathscr{CG}$ and thus intervals corresponding to y can not be picked in $MYSOL$. Hence, $MYSOL$ covers at least m' points. Thus if OPT satisfies m' clauses of Π then $MYSOL$ covers at least m' points.

Now consider OPT to be the set of unit intervals in the optimal solution to the instance of Max Unit Interval CF-SC that covers m' points. Let interval $I \in OPT$ covers point p_i. Without loss of generality let x be the literal corresponding to I in Π. Put the value of x true in Π and include c_i in $MYSOL$. Also include all other clauses satisfied by x into $MYSOL$. Due to construction of $\mathscr{CG}$, the assignment to the variables is valid. So it for all the intervals in OPT. Thus, $|MYSOL| \geq m'$. Now let us consider the point p_j that is covered by two intervals I_1 and I_2. Assume $I_1, I_2 \notin OPT$. Then $\overline{I_1}$ and $\overline{I_2}$ are in OPT. If not then we can include one of I_1 or I_2 in OPT without losing validity of solution which in turns increases the number of points covered and thus contradicting optimality of OPT. So, we can not make either of literals corresponding to I_1 and I_2 true and thus can not satisfy clause c_j. Hence, the number of clauses satisfied is at most m'. Thus, if OPT covers m' points then $MYSOL$ has m' satisfied clauses.

This proves the claim. □ □

Using the above claim, we can prove APX-hardness of 2-Path Interval CF-SC using strict reduction. This in turn proves the following theorem.

Theorem 6.3.1. *The Graphical Conflict Free Set Cover is APX-hard for the following classes of conflict graph $\mathscr{CG}$ an intersection graph $\mathscr{COV}$.*

1. *$\mathscr{CG}$ is a tree and $\mathscr{COV}$ is an intersection graph on intervals lying on a line.*
2. *$\mathscr{CG}$ is a d arboricity graph for $d \geq 1 \in \mathbb{N}$ and $\mathscr{COV}$ is an intersection graph on intervals lying on a line.*
3. *$\mathscr{CG}$ is a unit interval graph and $\mathscr{COV}$ is an intersection graph on unit intervals lying on a line. Thus it holds for general case of interval graphs too.*
4. *$\mathscr{CG}$ is a unit disc graph and $\mathscr{COV}$ is an intersection graph on unit disks lying in $\mathbb{R}^2$. Thus it holds for general case of disks graphs too.*
5. *$\mathscr{CG}$ is a unit interval graph and $\mathscr{COV}$ is an intersection graph on unit coins lying in $\mathbb{R}^2$. Thus it holds for general case of coins graphs too.*