3

Metric Dynamics of a Finite Family

Let (X,d) be a compact metric space and let $\mathbb{F} = \{f_1, f_2, \dots, f_k\}$ be a finite family of continuous surjective self maps on X. In this chapter, we investigate various metric related dynamical properties of the non-autonomous system generated by a finite family $\mathbb{F}$. We relate the dynamics of the non-autonomous system $(X,\mathbb{F})$ with the dynamics of the autonomous system $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$. We investigate properties like equicontinuity, minimality and various forms of sensitivities for two systems. In the process, we show that, while equivalence of properties like equicontinuity, sensitivity, Li-Yorke sensitivity holds unconditionally for two systems, minimality is equivalent for two systems when space X is connected. We give an example to show the necessity of condition imposed. We also investigate properties like proximality, distality and Li-Yorke chaos for non-autonomous system generated.

3.1 EQUICONTINUITY AND MINIMALITY

Proposition 3.1.1 *Let $(X,\mathbb{F})$ be a non-autonomous dynamical system generated by finite family $\mathbb{F}$. The system $(X,\mathbb{F})$ is equicontinuous if and only if $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$ is equicontinuous.*

Proof. Let $(X,\mathbb{F})$ be equicontinuous and let $\varepsilon > 0$ be given. As $(X,\mathbb{F})$ is equicontinuous, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(\omega_n(x), \omega_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$. In particular, $d(\omega_{nk}(x), \omega_{nk}(y)) < \varepsilon$ for all $n \in \mathbb{N}$ and hence $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$ is equicontinuous.

Conversely, let $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$ be equicontinuous and let $\varepsilon > 0$ be given. Then, as the family $\{g_r = f_r \circ \dots \circ f_2 \circ f_1 : r = 1,2,\dots k\}$ is a finite family of continuous maps, there exists $\eta > 0$ ($\eta < \varepsilon$) such that $d(x,y) < \eta$ implies $d(g_r(x), g_r(y)) < \varepsilon$ (for $r = 1,2,\dots k$). As $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$ is equicontinuous, there exists $\delta > 0$ ($\delta < \eta$) such that $d(x,y) < \delta$ implies $d(\omega_{nk}(x), \omega_{nk}(y)) < \eta$ for all $n \in \mathbb{N}$. Consequently, $d(x,y) < \delta$ ensures $d(g_r(\omega_{nk}(x)), g_r(\omega_{nk}(y))) < \varepsilon$ for all $r \in \{1,2,\dots k\}$ and $n \in \mathbb{Z}^+$. As any point $\omega_m(x)$ can be written as $g_r(\omega_{nk}(x))$ for some $r \in \{1,2,\dots k\}$ and $n \in \mathbb{Z}^+$, we have $d(\omega_m(x), \omega_m(y)) < \varepsilon$ for all $m \in \mathbb{N}$ and hence $(X,\mathbb{F})$ is equicontinuous. □

Proposition 3.1.2 *Let $(X,\mathbb{F})$ be a non-autonomous dynamical system generated by finite family of maps. If the system $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$ is minimal then the system $(X,\mathbb{F})$ is minimal. Further if space X is connected then converse is also true.*

Proof. Let $(X,\mathbb{F})$ be minimal and let $x \in X$. As orbit of x is dense in $(X,\mathbb{F})$, for each $y \in X$ there exists a sequence (m_i) of natural numbers and $r \in \{1,2,\dots k\}$ such that $\omega_{km_i+r}(x)$ converges to y (follows from the fact that the generating family $\mathbb{F}$ is finite). For any, $b \in X$, we say a is related to b in $\mathbb{F}$-sense (denoted as $a\mathbb{F}b$) if there exists $m \in \{1,2,\dots k\}$ and sequences (s_i) and (t_i) of natural numbers such that $(\omega_{ks_i+m}(x), \omega_{kt_i+m}(x))$ converges to (a,b) (in the product topology). Note that the relation defines an equivalence relation on X and hence partitions X into k disjoint sets $C_1, C_2, \dots C_k$ of X. Further, as each C_k is closed (and hence clopen), connectedness of X implies $C_1 = C_2 = \dots = C_k = X$and hence orbit of x is dense in $(X, f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)$.

Conversely, as orbit of any point x under $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is contained in orbit of x in $(X, \mathbb{F})$, minimality of $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ implies minimality of $(X, \mathbb{F})$ and hence the proof of converse is complete. □

Remark 3.1.1 The above result establishes the equivalence of minimality for the two systems $(X, \mathbb{F})$ and $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ when the space X is connected. Although the proof of converse is trivial, the proof for the forward part partitions the space X into k (atmost) disjoint non-empty clopen sets C_r, where C_r is the set of limit points of the sequence $(\omega_{nk+r}(x))$ $(r = 1, 2, \ldots k)$. Consequently, if X is connected, the generated sets coincide which in turn implies the denseness of orbit of x (for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$) and hence minimality of the two systems is equivalent. However, the equivalence holds good only when the space X is connected and fails to hold good in the absence of the stated condition. We now give an example in support of our claim.

Example 3.1.1 *Let $S_r = \{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ $(r = 1, 2)$ and let $X = S_1 \cup S_2$. Let $\alpha \in \mathbb{R}$ be an irrational multiple of 2π. Define $f_1, f_2 : X \to X$ as*

$$f_1(x) = \begin{cases} (r+1)e^{i(\theta+\alpha)} & \text{for } r = 1 \\ (r-1)e^{i(\theta+2\alpha)} & \text{for } r = 2 \end{cases}$$

$$f_2(x) = \begin{cases} (r+1)e^{i(\theta+2\alpha)} & \text{for } r = 1 \\ (r-1)e^{i(\theta+\alpha)} & \text{for } r = 2 \end{cases}$$

It may be noted that both f_1 and f_2 map S_1 to S_2 (and vice-versa) with an additional rotation of angle α (or 2α). Further as orbit of any point x in S_i is a rotation on S_i by angle 2α (or 4α) at even iterates and visits the other component of the space X via a rotation by angle 2α (or 4α) at odd iterates, the system $(X, \mathbb{F})$ is minimal. However, as $f_2 \circ f_1$ keeps each S_i invariant, the system $(X, f_2 \circ f_1)$ is not minimal.

Remark 3.1.2 The discussions above establish that if the space X is connected, the system $(X, \mathbb{F})$ is minimal if and only if the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is minimal. Further, it may be noted that connectedness is indeed required for the result to hold good and the derived result does not hold good in absence of the condition imposed (Example 3.1.1). It may be noted that the proof of proposition 3.1.2 does not require the maps f_i in the generating family to be distinct. Thus for an autonomous system (X, f), a similar proof establishes the minimality of f^k(from minimality of f) when the space X is connected. Consequently, if the space X is connected, an autonomous system (X, f) is minimal if and only if (X, f^m) is minimal (for each $m \in \mathbb{N}$). Further, connectedness is once again needed and the derived conclusion does not hold good when the stated condition is dropped. Hence we obtain the following corollary.

Corollary 3.1.1 *If the space X is connected, then (X, f) is minimal if and only if (X, f^m) is minimal for each $m \in \mathbb{N}$. Further, there exists a disconnected space X and a continuous self map f on X such that (X, f) is minimal but (X, f^2) is not minimal.*

Proof. The proof follows from the discussions in Remark 3.1.2. The conclusion follows directly from Example 3.1.1 as (X, f_1) is minimal but (X, f_1^2) is not minimal. □

3.2 VARIOUS FORMS OF SENSITIVITIES

Proposition 3.2.1 *Let $(X, \mathbb{F})$ be non-autonomous dynamical system generated by finite family of maps. The system $(X, \mathbb{F})$ is sensitive if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is sensitive.*

Proof. Let $(X, \mathbb{F})$ be sensitive (with sensitivity constant δ). As the family $\{g_r = f_r \circ f_{r-1} \cdots \circ f_1 : r = 1, 2, \cdots, k-1\}$ is finite family of continuous (uniformly continuous) maps, there exists $\eta > 0$ such that $d(x,y) < \eta$ ensures $d(g_r(x), g_r(y)) < \delta$ for all $x, y \in X$. We claim that η is sensitivity constant for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. Note that if there exists open set U such that $diam(\omega_{nk}(U)) < \eta \; \forall \; n$ then, $diam(g_r(\omega_{nk}(U))) < \delta$ for all $r = 1, 2, \cdots, k$ and $n \in \mathbb{N}$. As $\{g_r(\omega_{nk}(U)) : r = 1, 2, \cdots k, n \in \mathbb{N}\}$ coincides with the trajectory of the open set U under $(X, \mathbb{F})$, $diam(\omega_n(U)) < \delta \; \forall n \in \mathbb{N}$ which contradicts sensitivity of the system $(X, \mathbb{F})$ and hence any open set U expands (to size more than η) for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. Thus $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is sensitive (with sensitivity constant η) and the proof of forward part is complete.

Conversely, as orbit of any point x under $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is contained in orbit of x in $(X, \mathbb{F})$, sensitivity of $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ implies sensitivity of $(X, \mathbb{F})$ and hence the proof of converse is complete. □

Remark 3.2.1 The above proof establishes the equivalence of sensitivity for the two systems $(X, \mathbb{F})$ and $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. While proof in one of the directions is trivial, the other direction uses the fact that any continuous function on a compact metric space is uniformly continuous. However, the proof does not preserve the sensitivity constant and hence the two systems may be sensitive with different constant of sensitivity. It may be noted that cofinite sensitivity of $(X, \mathbb{F})$ ensures cofinite sensitivity of $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. Also, a proof similar to proof of proposition 3.2.1 (considering the family $\{h_r = f_k \circ f_{k-1} \circ \cdots \circ f_{k-r} : r = 0, 1, 2, \cdots, k-1\}$ and proving that common constant of uniform continuity is sensitivity constant for $(X, \mathbb{F})$) establishes that cofinite sensitivity of $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ ensures cofinite sensitivity of $(X, \mathbb{F})$ and hence cofinite sensitivity is equivalent for the two systems. Thus we get the following corollary.

Corollary 3.2.1 *The system $(X, \mathbb{F})$ is cofinite sensitive if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is cofinite sensitive.*

Proof. The proof follows directly from discussions in Remark 3.2.1. □

3.3 LI-YORKE SENSITIVITY AND LI-YORKE CHAOS

Proposition 3.3.1 *Let $(X, \mathbb{F})$ be a non-autonomous dynamical system govern by finite family of maps. (x, y) is proximal in $(X, \mathbb{F})$ if and only if (x, y) is proximal in $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$.*

Proof. Let (x_1, x_2) be proximal for $(X, \mathbb{F})$ and let (r_n) be the sequence of natural numbers such that $\lim\limits_{n\to\infty} d(\omega_{r_n}(x_1), \omega_{r_n}(x_2)) = 0$. As X is compact, without loss of generality (by passing on subsequence which we again denote by (r_n), we obtain an element $z \in X$ such that $(\omega_{r_n}(x_1), \omega_{r_n}(x_2))$ converges to (z, z). As the family $\mathbb{F}$ is finite, there exists a subsequence (r_{n_l}) of (r_n), $s \in \{1, 2, \ldots k\}$ and a sequence (km_{n_l}) (of multiples of k) such that $\omega_{r_{n_l}}(x_i) = f_s \circ f_{s-1} \circ \ldots \circ f_1(\omega_{km_{n_l}})(x_i)$ (for $i = 1, 2$). As $\omega_{r_{n_l}}(x_i)$ converges (to z) and $f_k \circ f_{k-1} \cdots \circ f_{s+1}$ is continuous, $f_k \circ f_{k-1} \cdots \circ f_{s+1}(\omega_{r_{n_l}}(x_i))$ converges to $f_k \circ f_{k-1} \cdots \circ f_{s+1}(z)$ (for $i = 1, 2$) or $\omega_{k(m_{n_l}+1)}(x_i)$ converges to $f_k \circ f_{k-1} \cdots \circ f_{s+1}(z)$(for $i = 1, 2$). Consequently, $\lim\limits_{l\to\infty} d(\omega_{k(m_{n_l}+1)}(x_1), \omega_{k(m_{n_l}+1)}(x_2)) = 0$ and hence (x_1, x_2) is proximal for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$.

Conversely, as orbit of any point x under $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is a subset of orbit of x under $(X, \mathbb{F})$, proximality of the pair (x_1, x_2) for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ ensures proximality of (x_1, x_2) for $(X, \mathbb{F})$ and hence the proof of converse is complete. □

Remark 3.3.1 The above result establishes the equivalence of proximal pairs for the two systems $(X, \mathbb{F})$ and $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. While proof of the converse is straightforward, the forward part uses the fact that if k is a fixed natural number and (r_n) is a sequence of natural numbers then there exists a subsequence (r_{n_l}) of (r_n) such that $((r_{n_l})$ modulo $k)$ is constant and hence the set of proximal pairs for the two systems coincide. Further, as equivalence of proximal pairs for two systems ensures equivalence of distal pairs, the system $(X,\mathbb{F})$ is distal if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is distal. Hence we get the following corollary.

Corollary 3.3.1 *Let $(X,\mathbb{F})$ be non-autonomous dynamical system generated by finite family of maps. $(X,\mathbb{F})$ is distal $\Leftrightarrow$ $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is distal.*

Proof. The proof follows from discussions in Remark 3.3.1. □

Proposition 3.3.2 *$(X,\mathbb{F})$ is Li-Yorke sensitive if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is Li-Yorke sensitive.*

Proof. Let $(X,\mathbb{F})$ be Li-Yorke sensitive (with sensitivity constant δ) and let $x \in X$. For any neighbourhood U of x, there exists $y \in U$ such that $\liminf\limits_{n\to\infty} d(\omega_n(x), \omega_n(y)) = 0$ and $\limsup\limits_{n\to\infty} d(\omega_n(x), \omega_n(y)) > \delta$. As δ is constant of sensitivity, a proof similar to proposition 3.2.1 ensures existence of $\eta > 0$ such that $\limsup\limits_{n\to\infty} d(\omega_{nk}(x), \omega_{nk}(y)) > \eta$. Further, as a pair is proximal for $(X,\mathbb{F})$ if and only if it is proximal for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$, (x,y) is δ Li-Yorke pair for $(X,\mathbb{F})$ if and only if (x,y) is η Li-Yorke pair for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. Hence, $(X,\mathbb{F})$ is Li-Yorke sensitive implies $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is Li-Yorke sensitive and the proof of forward part is complete.

Conversely, as orbit of any point x under $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is contained in orbit of x in $(X,\mathbb{F})$, Li-Yorke sensitivity of $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ implies Li-Yorke sensitivity of $(X,\mathbb{F})$ and hence the proof of converse is complete. □

Corollary 3.3.2 *Let $(X,\mathbb{F})$ be a non-autonomous dynamical system generated by finite family $\mathbb{F}$ then, $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is Li-Yorke chaotic $\Leftrightarrow$ $(X,\mathbb{F})$ is Li-Yorke chaotic.*

Proof. As (x,y) is a Li-Yorke pair of the system $(X,\mathbb{F})$ if and only if (x,y) is a Li-Yorke pair of the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$. Hence, the system $(X,\mathbb{F})$ is Li-Yorke chaotic if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_2 \circ f_1)$ is Li-Yorke chaotic. □

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