4

Dynamics of a Uniformly Convergent Sequence

In this chapter, we investigate the dynamical behaviour of a non-autonomous system generated by a uniformly convergent sequence of continuous surjective self maps on a compact metric space X. In particular, if $\mathbb{F} = (f_n)$ is a sequence of continuous self maps converging uniformly to f, we relate the dynamical behaviour of the system $(X, \mathbb{F})$ with the dynamics of the limiting system (X, f). In the process, we investigate properties like equicontinuity, various notions of mixing and sensitivities, minimality and proximality for the two systems. We show that if the family $\mathbb{F}$ is feeble open, topological mixing is equivalent for two systems. In addition, if $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively, properties like minimality, various notions of mixing and different forms of sensitivities are also equivalent for two systems. However, feeble openness is indeed needed to establish the derived results and the results obtained do not hold good when feeble openness is dropped. We give an example in support of our claim. Further, we prove that if the family $\mathbb{F}$ is bijective, equicontinuity is equivalent for two systems and hence (under collective convergence) many of the dynamical properties of non-autonomous dynamical systems can be approximated by an autonomous dynamical system. We give conditions under which collective convergence can naturally be obtained for a non-autonomous system. In particular, we prove that if family $\mathbb{F}$ commutes with the limiting function f and $\sum\limits_{n=1}^{\infty} D(f_n, f) < \infty$ then $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$. We prove that if the limiting function f is an isometry and $\sum\limits_{n=1}^{\infty} D(f_n, f) < \infty$ then $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$. We finally investigate proximal pairs and proximal cells for non-autonomous dynamical systems. We prove that if the generating family $\mathbb{F}$ commutative and $\sum\limits_{n=1}^{\infty} D(f_n, f) < \infty$ then proximal pairs (cells) are dense in $(X, \mathbb{F})$ if and only if proximal pairs (cells) are dense in (X, f).

4.1 EQUICONTINUITY AND MINIMALITY

Proposition 4.1.1 *Let $(X, \mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) converging uniformly to f. If $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X, f) is equicontinuous $\Rightarrow$ $(X, \mathbb{F})$ is equicontinuous. Further, if $f_i's$ are bijective then, $(X, \mathbb{F})$ is equicontinuous $\Rightarrow$ (X, f) is equicontinuous.*

Proof. Let (X, f) be equicontinuous and let $\varepsilon > 0$ be given. By equicontinuity, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. As $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega_{r+k}^r, f^k) < \frac{\varepsilon}{3}$ $\forall r \geq r_0$ and $k \in \mathbb{N}$ and hence $d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) < \frac{\varepsilon}{3}$ $\forall k \in \mathbb{N}$, $x \in X$. As ω_{r_0} is continuous, choose $\eta_{r_0} > 0$ such that $d(x, y) < \eta_{r_0}$ implies $d(\omega_{r_0}(x), \omega_{r_0}(y)) < \delta$ and hence $d(f^k(\omega_{r_0}(x)), f^k(\omega_{r_0}(y))) < \frac{\varepsilon}{3}$ for all $k \in \mathbb{N}$. By triangle inequality, $d(x, y) < \eta_{r_0}$ implies $d(\omega_{r_0+k}(x), \omega_{r_0+k}(y)) < \varepsilon$ for all $k \in \mathbb{N}$. Further, as $\{\omega_1, \omega_2, \ldots, \omega_{r_0}\}$ is a finite set, there exists $\eta'_{r_0} > 0$ such that $d(x, y) < \eta'_{r_0}$ forces $d(\omega_i(x), \omega_i(y)) < \varepsilon$ for $i = 1, 2, \ldots, r_0$ and hence choosing $\eta = \min\{\eta_{r_0}, \eta'_{r_0}\}$ ensures equicontinuity of $(X, \mathbb{F})$.

Conversely, let f_i's be bijective and let $\varepsilon > 0$ be given. By equicontinuity of $(X,\mathbb{F})$, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(\omega_n(x),\omega_n(y)) < \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega^r_{r+k}, f^k) < \frac{\varepsilon}{3}$ $\forall r \geq r_0$ and $k \in \mathbb{N}$ and hence $d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) < \frac{\varepsilon}{3}$ $\forall k \in \mathbb{N}, x \in X$. As each f_i is bijective, ω_{r_0} is a homeomorphism and thus there exists $\eta_{r_0} > 0$ such that $d(x,y) < \eta_{r_0}$ implies $d(\omega^{-1}_{r_0}(x), \omega^{-1}_{r_0}(y)) < \delta$. Let $x,y \in X$ such that $d(x,y) < \eta_{r_0}$. Let $x_{r_0} = \omega^{-1}_{r_0}(x)$ and $y_{r_0} = \omega^{-1}_{r_0}(y)$. For any $k \in \mathbb{N}$, we have $d(f^k(x), f^k(y)) \leq d(f^k(\omega_{r_0}(x_{r_0})), \omega_{r_0+k}(x_{r_0})) + d(\omega_{r_0+k}(x_{r_0}), \omega_{r_0+k}(y_{r_0})) + d(\omega_{r_0+k}(y_{r_0}), f^k(\omega_{r_0}(y_{r_0}))) < \varepsilon$ and hence (X,f) is equicontinuous. □

Proposition 4.1.2 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) converging uniformly to f. If $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X,f) is minimal $\Leftrightarrow$ $(X,\mathbb{F})$ is minimal.*

Proof. Let (X,f) be minimal and let $x \in X$. Let $y \in X$ and let $\varepsilon > 0$ be fixed. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega^r_{r+k}, f^k) < \frac{\varepsilon}{2}$ $\forall r \geq r_0$ and $k \in \mathbb{N}$ and hence $d(\omega_{r_0+k}(z), f^k(\omega_{r_0}(z))) < \frac{\varepsilon}{2}$ $\forall k \in \mathbb{N},\ z \in X$. As (X,f) is minimal, orbit of $\omega_{r_0}(x)$(under f) is dense in X and there exists $k \in \mathbb{N}$ such that $f^k(\omega_{r_0}(x)) \in S(y, \frac{\varepsilon}{2})$. Also collective convergence ensures $d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) < \frac{\varepsilon}{2}$ and hence by triangle inequality, $d(\omega_{r_0+k}(x), y) \leq d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) + d(f^k(\omega_{r_0}(x)), y) < \varepsilon$ and hence $\omega_{r_0+k}(x) \in S(y,\varepsilon)$. As the proof holds good for any choice of $\varepsilon > 0$ and $y \in X$, orbit of x (under $\mathbb{F}$) is dense in X. Further, as the proof holds good for any $x \in X$, $(X,\mathbb{F})$ is minimal.

Conversely, let $(X,\mathbb{F})$ be minimal and let $x \in X$. Let $y \in X$ and let $\varepsilon > 0$ be fixed. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega^r_{r+k}, f^k) < \frac{\varepsilon}{2}$ $\forall r \geq r_0$ and $k \in \mathbb{N}$ which implies $d(\omega_{r_0+k}(z), f^k(\omega_{r_0}(z))) < \frac{\varepsilon}{2}$ $\forall k \in \mathbb{N},\ z \in X$. As $(X,\mathbb{F})$ is minimal, orbit of any $z \in \omega^{-1}_{r_0}(x)$(under $\mathbb{F}$) is dense in X and hence intersects $S(y, \frac{\varepsilon}{2})$. Further, as the orbit intersects $S(y,\eta)$ for each $\eta > 0$, the set $\{n : \omega_n(z) \in S(y, \frac{\varepsilon}{2})\}$ is infinite and there exists $k \in \mathbb{N}$ such that $\omega_{r_0+k}(z) \in S(y, \frac{\varepsilon}{2})$. Further, collective convergence ensures $d(\omega_{r_0+k}(z), f^k(\omega_{r_0}(z))) < \frac{\varepsilon}{2}$ and hence by triangle inequality, $d(f^k(\omega_{r_0}(z)), y) < \varepsilon$ or $f^k(x) \in S(y,\varepsilon)$. As the proof holds good for any choice of $\varepsilon > 0$ and $y \in X$, orbit of x (under f) is dense in X. As the proof holds good for any $x \in X$, (X,f) is minimal. □

Remark 4.1.1 The above results establish the relation between equicontinuity and minimality of the two systems (X,f) and $(X,\mathbb{F})$. We prove that Collective convergence of $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ is sufficient to establish the equivalence of minimality for two systems. While collective convergence is enough to establish the equicontinuity of $(X,\mathbb{F})$ from equicontinuity of (X,f), the proof uses additional assumption of bijectivity of the generating functions f_i to establish the converse.

4.2 TOPOLOGICAL DYNAMICS GENERATED BY A UNIFORMLY CONVERGENT SEQUENCE

Proposition 4.2.1 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) of feeble open maps converging uniformly to f. If $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X,f) is transitive $\Leftrightarrow$ $(X,\mathbb{F})$ is transitive.*

Proof. Let $\varepsilon > 0$ be given and let $U = S(x,\varepsilon)$ and $V = S(y,\varepsilon)$ be two non-empty open sets in X. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega^r_{r+k}, f^k) < \frac{\varepsilon}{2}$ $\forall r \geq r_0$ and $k \in \mathbb{N}$. Further, as the family $\mathbb{F}$ is feeble open, $int(\omega_{r_0}(U))$ is non-empty open and thus by transitivity of (X,f) (applied to open sets $U' = int(\omega_{r_0}(U))$ and $V' = S(y, \frac{\varepsilon}{2})$), there exists $m \in \mathbb{N}$ such that $f^m(U') \cap V' \neq \phi$. Consequently, there exists $u' \in U'$ such that $f^m(u') \in V'$. Further, as $U' = int(\omega_{r_0}(U))$, there exists $u \in U$ such that $f^m(\omega_{r_0}(u)) \in V'$. Also, collective convergence ensures

$d(\omega_{m+r_0}(u), f^m(\omega_{r_0}(u))) < \frac{\varepsilon}{2}$ and hence by triangle inequality $d(y, \omega_{m+r_0}(u)) < \varepsilon$ or $\omega_{m+r_0}(U) \cap V \neq \phi$. Finally, as the proof holds for any pair of non-empty open sets $S(x,\varepsilon), S(y,\varepsilon)$ in X, the proof holds for any pair of non-empty open sets in X and $(X,\mathbb{F})$ is transitive.

Conversely, let $\varepsilon > 0$ be given and let $S(x,\varepsilon)$ and $S(y,\varepsilon)$ be two non-empty open sets in X. As $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega_{r+k}^r, f^k) < \frac{\varepsilon}{2} \ \forall r \geq r_0$ and $k \in \mathbb{N}$. As transitivity of a system forces any open set U to visit $S(y, \frac{1}{m})$ for each m, the set of times when any non-empty open set U visits $S(y,\varepsilon)$ is infinite. Consequently for the pair $(U,V) = (\omega_{r_0}^{-1}(S(x,\varepsilon)), S(y,\frac{\varepsilon}{2}))$, there exists $k \in \mathbb{N}$ and $u \in U$ such that $d(\omega_{r_0+k}(u), y) < \frac{\varepsilon}{2}$. Further, collective convergence ensures $d(\omega_{r_0+k}(u), f^k(\omega_{r_0}(u))) < \frac{\varepsilon}{2}$ and hence by triangle inequality $d(y, f^k(\omega_{r_0}(u))) < \varepsilon$. As $\omega_{r_0}(u) \in S(x,\varepsilon)$, we have $f^k(S(x,\varepsilon)) \cap S(y,\varepsilon) \neq \phi$. As the proof holds for any choice of non-empty open sets $S(x,\varepsilon)$ and $S(y,\varepsilon)$, the system (X,f) is transitive. □

Remark 4.2.1 If the family $\mathbb{F}$ is feeble open and (ω_{n+k}^n) convergences collectively, then, the above result establishes the equivalence of transitivity for the two systems. However, the converse part does not use the feeble openness of the maps f_n and hence transitivity of the non-autonomous system is carried forward to (X,f) even when the family $\mathbb{F}$ is not feeble open. Further, if $N_f(U,V)(N_{\mathbb{F}}(U,V))$ denotes the set of times when an open set U visits V under f $(\mathbb{F})$, the proof establishes that for each pair of open sets U,V, there exists a pair of open sets U',V' of open sets such that the set of times of interactions of U and V under $\mathbb{F}$ contains the translates of set of times of interactions of U' and V' under f (and vice-versa). As the argument depends on the diameter of open sets and not on the sets themselves, a similar argument establishes equivalence of weak mixing for the two systems. Hence we get the following corollary.

Corollary 4.2.1 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) of feeble open maps converging uniformly to f. If $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X,f) is weakly mixing $\Leftrightarrow$ $(X,\mathbb{F})$ is weakly mixing.*

Proof. The proof follows from the discussions in Remark 4.2.1. □

Proposition 4.2.2 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ of feeble open maps. If (f_n) converges uniformly to f then, (X,f) is topologically mixing $\Leftrightarrow$ $(X,\mathbb{F})$ is topologically mixing.*

Proof. Let U be a non-empty open set in X. As f is topologically mixing there exists $\varepsilon > 0$ such that for any open set V, $D_H(V,X) < \varepsilon$ ensures $D_H(f(V),X) < \varepsilon$ (proof follows from the fact that if V is big enough then it cannot shrink significantly as f is topologically mixing). Further, as f_n converges uniformly to f, there exists $r_0 \in \mathbb{N}$ such that $D_H(f_n(V),X) < \varepsilon$ for all $n \geq r_0$. Note that topological mixing (of f) implies there exists $k_0 \in \mathbb{N}$ $(k_0 \geq r_0)$ such that $D_H(f^n(U),X) < \frac{\varepsilon}{2} \ \forall n \geq k_0$. Also, as $\omega_{n+k_0}^n$ converges to f^{k_0}, there exists $n_0 \in \mathbb{N}$ such that $D(\omega_{n+k_0}^n, f^{k_0}) < \frac{\varepsilon}{2} \ \forall n \geq n_0$ and hence $D_H(\omega_{n+k_0}^n(U),X) < \varepsilon \ \forall n \geq n_0$. Consequently, $D_H(\omega_{n+k}^n(U),X) < \varepsilon \ \forall n \geq n_0, k \geq k_0$. As the argument holds for arbitrarily small $\varepsilon > 0$, the proof ensures collective convergence (on open sets) and hence $(X,\mathbb{F})$ is topologically mixing.

Conversely, if non-autonomous system is topologically mixing, similar arguments establishes collective convergence of $\omega_{n+k}^n(U)$ (to X) and hence (X,f) is topologically mixing. □

Remark 4.2.2 The above proof establishes that if the non-autonomous system $(X,\mathbb{F})$ is generated by a uniformly convergent sequence of feeble open maps (f_n) then topological mixing is equivalent for the non-autonomous system $(X,\mathbb{F})$ and the limiting system (X,f). The proof uses the fact that for a topologically mixing system, when an open set becomes "large" then it stays "large" for all times. The observation helps in establishing collective convergence for the family $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ which in turn establishes equivalence of topological mixing for the two systems.

Proposition 4.2.3 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$. If (f_n) converges uniformly to f, then, x is periodic for $(X,\mathbb{F}) \Rightarrow x$ is periodic for (X,f).*

Proof. Let x_0 be periodic for $(X,\mathbb{F})$ with period k and let $\varepsilon > 0$ be given. As ω^n_{n+k} converges uniformly to f^k, there exists $n_0 \in \mathbb{N}$ such that $D(\omega^n_{n+k}, f^k) < \varepsilon \;\; \forall n \geq n_0$. Therefore, $d(\omega^{n_0k}_{n_0k+k}(x), f^k(x)) < \varepsilon$ for any $x \in X$ and hence $d(\omega^{n_0k}_{n_0k+k}(\omega_{n_0k}(x_0)), f^k(\omega_{n_0k}(x_0))) < \varepsilon$. As $\omega_{rk}(x_0) = x_0$ for all r, the above argument ensures $d(f^k(x_0), x_0) < \varepsilon$. As the result holds good for any $\varepsilon > 0$ we have $f^k(x_0) = x_0$ and hence x_0 is periodic for f. □

Example 4.2.1 *Let S^1 denote the unit circle and let $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ where $f_n : S^1 \to S^1$ is defined as $f_n(\theta) = \theta + \frac{1}{n^2}$. Then f_n's are rotations on unit circle converging uniformly to identity I and hence every point is periodic for (S^1, I). However, as $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2\pi$, the non-autonomous system $(S^1,\mathbb{F})$ does not have any periodic point and hence the converse of the above result is not true.*

Remark 4.2.3 Proposition 4.2.3 shows that any point periodic for the system $(X,\mathbb{F})$ is also periodic point for the limiting system (X,f). However, the above example shows that the converse do not hold good and proves that $(X,\mathbb{F})$ can be void of periodic points even when each point of X is periodic for (X,f). Further, it must be noticed that the proposition 4.2.3 preserves only the periodic behaviour not the period itself from the system $(X,\mathbb{F})$ to the system (X,f).

4.3 METRIC DYNAMICS OF UNIFORMLY CONVERGENT SEQUENCE

Proposition 4.3.1 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) of feeble open maps converging uniformly to f. If $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X,f) is sensitive $\Leftrightarrow$ $(X,\mathbb{F})$ is sensitive.*

Proof. Let (X,f) be sensitive with constant of sensitivity δ and let $U = S(u,\varepsilon)$ any non-empty open set in X. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\delta}{4}$. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $D(\omega^r_{r+k}, f^k) < \frac{1}{m} \;\; \forall r \geq r_0$ and $k \in \mathbb{N}$ and hence $d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) < \frac{1}{m} \;\; \forall k \in \mathbb{N},\;\; x \in X$. As f_n's are feeble open, $int(\omega_{r_0}(U)) \neq \phi$ and hence by sensitivity (of f) there exists $v_1, v_2 \in int(\omega_{r_0}(U))$ and $k \in \mathbb{N}$ such that $d(f^k(v_1), f^k(v_2)) > \delta$. As $v_1, v_2 \in \omega_{r_0}(U)$, there exists $v'_1, v'_2 \in U$ such that $v_1 = \omega_{r_0}(v'_1)$ and $v_2 = \omega_{r_0}(v'_2)$. Thus $d(f^k(\omega_{r_0}(v'_1)), f^k(\omega_{r_0}(v'_2))) > \delta$ and hence by triangle inequality $d(\omega_{r_0+k}(v'_1), \omega_{r_0+k}(v'_2)) > \delta - \frac{2}{m} > \frac{\delta}{2}$ and hence $(X,\mathbb{F})$ is sensitive.

Conversely, let $(X,\mathbb{F})$ be sensitive with sensitivity constant δ and let U be non-empty open in X. Let $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\delta}{4}$. As $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$, there exists $r_0 \in \mathbb{N}$ such that $d(\omega_{r_0+k}(x), f^k(\omega_{r_0}(x))) < \frac{1}{m} \;\; \forall k \in \mathbb{N},\;\; x \in X$. As sensitivity of $(X,\mathbb{F})$ ensures expansivity of each $\frac{1}{n}$-ball in X, the set $\{k : diam(\omega_k(V)) > \delta\}$ is infinite for any non-empty open set V and thus there exists $v_1, v_2 \in \omega^{-1}_{r_0}(U)$ and $k \in \mathbb{N}$ such that $d(\omega_{r_0+k}(v_1), \omega_{r_0+k}(v_2)) > \delta$. As $v_1, v_2 \in \omega^{-1}_{r_0}(U)$, there exists $v'_1, v'_2 \in U$ such that $v'_1 = \omega_{r_0}(v_1)$ and $v'_2 = \omega_{r_0}(v_2)$ and hence $d(\omega^{r_0}_{r_0+k}(v'_1), \omega^{r_0}_{r_0+k}(v'_2)) > \delta$. Thus by triangle inequality $d(f^k(v'_1), f^k(v'_2)) > \delta - \frac{2}{m} > \frac{\delta}{2}$ and hence (X,f) is sensitive. □

Remark 4.3.1 The above proof establishes the equivalence of sensitivity for the two systems $(X,\mathbb{F})$ and (X,f) under collective convergence. As the choice of $m(\frac{1}{m} < \frac{\delta}{4})$ is arbitrary and can be made finer, sensitivity constant is preserved and hence the two systems are sensitive with the same constant of sensitivity. Further, the above proof establishes that for any pair of distinct points $x, y \in X$ there exists points x', y' such that the set of times of separation of x and y in the non-autonomous system contains the translates of set of times of separation of x' and y' in the autonomous system (and vice-versa). Consequently, for any non-empty open set U, there exists a non-empty open set U' such that the set of times of expansivity of U in the non-autonomous system contains the translates of set of times of expansivity of U' in the autonomous system (and vice-versa) and hence cofinite sensitivity is equivalent for the two systems. Hence we get the following corollary.

Corollary 4.3.1 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a sequence (f_n) of feeble open maps converging uniformly to f. If $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D) then, (X,f) is cofinitely sensitive $\Leftrightarrow$ $(X,\mathbb{F})$ is cofinitely sensitive.*

Proof. The proof follows from discussions in Remark 4.3.1. □

Remark 4.3.2 The above proofs show that collective convergence along with feeble openness of the family $\mathbb{F}$ is sufficient to establish the equivalence of any notion of mixing and sensitivities for two systems. However, feeble openness of the family $\mathbb{F}$ is needed to establish the above results and the above results need not be hold good if feeble openness of the family $\mathbb{F}$ is relaxed. We now give an example in support of our claim.

Example 4.3.1 *Let I be the unit interval and let $f : I \to I$ be defined as*

$$f(x) = \begin{cases} 2x & \text{for } x \in [0,\frac{1}{2}] \\ 2-2x & \text{for } x \in [\frac{1}{2},1] \end{cases} \quad \text{and let}$$

$$g(x) = \begin{cases} 1 & \text{for } x \in [0,\frac{1}{2}] \\ 2-2x & \text{for } x \in [\frac{1}{2},1] \end{cases}$$

Let $(X,\mathbb{F})$ be the non-autonomous system generated by $\mathbb{F} = \{g, f, f, \ldots\}$. It may be noted that as $\omega_{n+k}^n = f^k($ for $n \geq 2)$, collective convergence of $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ is ensured. However, as (X,f) exhibits all forms of mixing and sensitivities but $(X,\mathbb{F})$ does not exhibit any form of mixing or sensitivity, feeble openness of the family $\mathbb{F}$ is indeed required for preserving any notion of mixing or sensitivity.

Proposition 4.3.2 *Let X be compact and let $(X,\mathbb{F})$ be a sensitive non-autonomous system generated by a family $\mathbb{F}$. Then for any $x \in X$, $Prox_{\mathbb{F}}(x)$ is dense in $X \Leftrightarrow LY_{\mathbb{F}}(x)$ is dense in X.*

Proof. For any $x \in X$, it may be noted that if $(X,\mathbb{F})$ is sensitive with sensitivity constant δ, then for any $y \in X$ and any U_y of y, there exists $y' \in U_y$ and $n \in \mathbb{N}$ such that $d(\omega_n(y), \omega_n(y')) > \delta$. By triangle inequality, $d(\omega_n(x), \omega_n(y)) > \frac{\delta}{2}$ or $d(\omega_n(x), \omega_n(y')) > \frac{\delta}{2}$ and hence the set of points $\frac{\delta}{2}$-sensitive to x are dense in X.

Let $\varepsilon > 0$ be fixed. For any $x \in X$ and any non-empty open subset U of X, let V be non-empty open such that $V \subset \overline{V} \subset U$. As $Prox_{\mathbb{F}}(x)$ is dense in X, there exists $y \in V$ such that the pair (x,y) is proximal and hence there exists $n_1 \in \mathbb{N}$ such that $d(\omega_{n_1}(x), \omega_{n_1}(y)) < \varepsilon$. By continuity, there exists a neighbourhood $U_1(\subset V)$ of y such that $d(\omega_{n_1}(x), \omega_{n_1}(u_1)) < \varepsilon$ for all $u_1 \in U_1$. As the set of points $\frac{\delta}{2}$-sensitive to x are dense in X, there exists $y_1 \in U_1$ and $m_1 \in \mathbb{N}$ such that $d(\omega_{m_1}(x), \omega_{m_1}(y_1)) > \frac{\delta}{2}$ and once again by continuity there exists a neighbourhood $V_1(\subset U_1 \subset V)$ of y_1 such that $d(\omega_{n_1}(x), \omega_{n_1}(v_1)) < \varepsilon$ and $d(\omega_{m_1}(x), \omega_{m_1}(v_1)) > \frac{\delta}{2}$ for all $v_1 \in V_1$. Hence for $\varepsilon > 0$ and any pair x, U (where $x \in X$ and U is non-empty open subset of X) there exists $n, m \in \mathbb{N}$ and a non-empty open subset U_ε of X (satisfying $U_\varepsilon \subset \overline{U_\varepsilon} \subset U$) such that $d(\omega_m(x), \omega_m(u)) > \frac{\delta}{2}$ and $d(\omega_n(x), \omega_n(u)) < \varepsilon$ for all $u \in U_\varepsilon$.

Let $x \in X$ and U be any non-empty open subset of X. By argument above there exists non-empty open set U_1, $U_1 \subset \overline{U_1} \subset U$ and $n_1, m_1 \in \mathbb{N}$ such that $d(\omega_{m_1}(x), \omega_{m_1}(y)) > \frac{\delta}{2}$ and

$d(\omega_{n_1}(x), \omega_{n_1}(y)) < \frac{1}{2}$ for all $y \in U_1$. Repeating the process for the pair (x, U_1), there exists U_2 satisfying $U_2 \subset \overline{U_2} \subset U_1$ and $n_2, m_2 \in \mathbb{N}$ such that $d(\omega_{m_2}(x), \omega_{m_2}(y)) > \frac{\delta}{2}$ and $d(\omega_{n_2}(x), \omega_{n_2}(y)) < \frac{1}{4}$ for all $y \in U_2$. Inductively, we obtain a decreasing sequence U_k of non-empty open subsets of X such that $U_k \subset \overline{U_k} \subset U_{k-1}$ and sequences $(n_k), (m_k) \in \mathbb{N}$ such that $d(\omega_{m_k}(x), \omega_{m_k}(y)) > \frac{\delta}{2}$ and $d(\omega_{n_k}(x), \omega_{n_k}(y)) < \frac{1}{2^k}$ for all $y \in U_k$. As X is compact $\bigcap_{k=1}^{\infty} \overline{U_k} \neq \phi$. Then for any $u \in \bigcap_{k=1}^{\infty} \overline{U_k}$, we have $d(\omega_{m_k}(x), \omega_{m_k}(u)) > \frac{\delta}{2}$ and $d(\omega_{n_k}(x), \omega_{n_k}(u) < \frac{1}{2^k}$ for all k and hence (x, u) is a Li-Yorke pair. As the argument holds good for any non-empty open set U, $LY_{\mathbb{F}}(x)$ is dense in X.

As every Li-Yorke pair is proximal, proof of the converse part is trivial. □

Remark 4.3.3 The above result shows that for sensitive systems, proximal cells are dense in X for $(X, \mathbb{F})$ if and only if Li-Yorke cells are dense for $(X, \mathbb{F})$. It may be noted that the result does not use the compactness of the space X completely and holds good for locally compact spaces also. Further, the proof does not use denseness of the proximal cell completely and establishes that for a sensitive system, if the proximal cell of a point x is dense in a neighbourhood of x then Li-Yorke cell is dense in the neighbourhood of x. Consequently, a similar argument establishes that for a sensitive system, if the proximal cell of a point x is dense in a neighbourhood of x then x is a point of Li-Yorke sensitivity. The result is a natural extension of the result established in [Akin and Kolyada, 2003] for the autonomous systems.

Corollary 4.3.2 *Let (X, f) be a compact sensitive system. Then for any $x \in X$, $Prox_f(x)$ is dense in $X \Leftrightarrow LY_f(x)$ is dense in X.*

We now give some conditions under which uniform convergence of (f_n) ensures collective convergence of $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$.

Proposition 4.3.3 *Let $(X, \mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ and let f be any continuous self map on X. If f is an isometry and $\sum_{n=1}^{\infty} D(f_n, f) < \infty$ then $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges to $\{f^k : k \in \mathbb{N}\}$ collectively (with respect to the metric D).*

Proof. It may be noted that It may be noted that $D(\omega_{n+r+1}^n, f^{r+1}) = D(f_{n+r+1}(\omega_{n+r}^n), f(f^r)) \leq D(f_{n+r+1}(\omega_{n+r}^n), f(\omega_{n+r}^n)) + D(f(\omega_{n+r}^n), f(f^r)) \leq D(f_{n+r+1}, f) + D(\omega_{n+r}^n, f^r)$. Thus $D(\omega_{n+k}^n, f^k) \leq \sum_{i=1}^{k} D(f_{n+i}, f)$ ensures $D(\omega_{n+r+1}^n, f^{r+1}) \leq \sum_{i=1}^{r+1} D(f_{n+i}, f)$ and hence by induction, $D(\omega_{n+k}^n, f^k) \leq \sum_{i=1}^{k} D(f_{n+i}, f)$ for any $k \in \mathbb{N}$. Further, if $\sum_{n=1}^{\infty} D(f_n, f) < \infty$, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} D(f_{n_0+i}, f) < \varepsilon$. Consequently, $D(\omega_{n+k}^n, f^k) < \varepsilon$ $\forall k \in \mathbb{N}, n \geq n_0$ and hence the family $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges collectively to $\{f^k : k \in \mathbb{N}\}$ (with respect to the metric D). □

Proposition 4.3.4 *Let $(X, \mathbb{F})$ be a non-autonomous system generated by $\mathbb{F}$ and let f be any continuous self map on X. If the family $\mathbb{F}$ commutes with f and $\sum_{n=1}^{\infty} D(f_n, f) < \infty$ then $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ converges to $\{f^k : k \in \mathbb{N}\}$ collectively (with respect to the metric D).*

Proof. Let $x \in X$ and n be a natural number. As f_k's commute with f, $d(f_2 \circ f_1(x), f^2(x)) \leq d(f_2 \circ f_1(x), f \circ f_1(x)) + d(f \circ f_1(x), f \circ f(x)) = d(f_2 \circ f_1(x), f \circ f_1(x)) + d(f_1 \circ f(x), f \circ f(x)) \leq D(f_2, f) + D(f_1, f)$. Proceeding inductively, if $d(\omega_n(x), f^n(x)) \leq \sum_{i=1}^{n} D(f_i, f)$, then, $d(\omega_{n+1}(x), f^{n+1}(x)) \leq d(f_{n+1}(\omega_n(x)), f(\omega_n(x))) + d(\omega_n(f(x)), f^n(f(x))) \leq D(f_{n+1}, f) + \sum_{i=1}^{n} D(f_i, f)$ (by induction) and hence for any $k \in \mathbb{N}$, $d(\omega_k(x), f^k(x)) \leq \sum_{i=1}^{k} D(f_i, f)$. Further, as $\sum_{n=1}^{\infty} D(f_n, f) < \infty$, observing the conclusion for $y = \omega_n(x)$ and family $\mathbb{F} = \{f_{n+i} : i \in \mathbb{N}\}$ yields $d(\omega_{n+k}(x), f^k(\omega_n(x))) \leq \sum_{i=1}^{k} D(f_{n+i}, f)$ or $D(\omega^n_{n+k}, f^k) \leq \sum_{i=1}^{k} D(f_{n+i}, f)$ and hence $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$ converges collectively (with respect to the metric D). □

Remark 4.3.4 The above propositions provide sufficient conditions under which uniform convergence of (f_n) ensures collective convergence of $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$. While proposition 4.3.3 establishes collective convergence of $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$, if the sequence (f_n) converges at sufficiently fast rate when the limit map is an isometry, proposition 4.3.4 establishes the collective convergence of $\{(\omega^n_{n+k}) : k \in \mathbb{N}\}$, if sequence (f_n) converges at sufficiently fast rate and limit map commutes with the sequence (f_n).

4.4 PROXIMAL PAIRS AND PROXIMAL CELLS

Proposition 4.4.1 *Let $(X, \mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ commuting with f. If $\sum_{n=1}^{\infty} D(f_n, f) < \infty$ then, (x, y) is proximal for $(X, f) \Rightarrow (x, y)$ is proximal for $(X, \mathbb{F})$.*

Proof. Let $\varepsilon > 0$ be given and let (x, y) be proximal for (X, f). As (x, y) is proximal, there exists a sequence (n_k) in $\mathbb{N}$ such that $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$. As $\sum_{n=1}^{\infty} D(f_n, f) < \infty$, there exists $r_0 \in \mathbb{N}$ such that $\sum_{n=r_0}^{\infty} D(f_n, f) < \frac{\varepsilon}{3}$. As (x, y) is proximal for (X, f), there exists $n_k \in \mathbb{N}$ such that $d(f^{n_k}(x), f^{n_k}(y)) < \eta$ and hence $d(\omega_{r_0}(f^{n_k}(x)), \omega_{r_0}(f^{n_k}(y))) < \frac{\varepsilon}{3}$ (by continuity of ω_{r_0}). Further, as (f_n) commutes with f, $d(\omega_{r_0+n_k}(u), f^{n_k}(\omega_{r_0}(u))) < \sum_{i=1}^{n_k} D(f_{r_0+i}, f) < \frac{\varepsilon}{3}$ for any element u in X and hence by triangle inequality, $d(\omega_{r_0+n_k}(x), \omega_{r_0+n_k}(y)) < \varepsilon$. As the proof works for any $\varepsilon > 0$, (x, y) is proximal for $(X, \mathbb{F})$. □

Proposition 4.4.2 *Let X be compact and let $(X, \mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ commuting with f. If $\sum_{n=1}^{\infty} D(f_n, f) < \infty$ then, proximal cell of each x is dense for $(X, f) \Leftrightarrow$ proximal cell of each x is dense for $(X, \mathbb{F})$.*

Proof. As every pair proximal for (X, f) is proximal for $(X, \mathbb{F})$, if Prox(x) is dense for (X, f) then Prox(x) is also dense for $(X, \mathbb{F})$ and the proof of forward part is complete.

Conversely let $Prox_{\mathbb{F}}(x)$ be dense for each $x \in X$. Fix $x \in X$ and let U be a non-empty open subset of X. Let U_1 be non-empty open such that $\overline{U_1} \subset U$. As $\sum_{n=1}^{\infty} D(f_n, f) < \infty$, for each $m \in \mathbb{Z}^+$ there exists $r_m \in \mathbb{N}$ such that $\sum_{n=r_m}^{\infty} D(f_n, f) < \frac{1}{3.2^m}$. Fix $m = 1$ and let $V_m = \omega_{r_m}^{-1}(U_m)$. Then V_m is open in X. Pick any $x_m \in \omega_{r_m}^{-1}(x)$. As $Prox_{\mathbb{F}}(x_m)$ is dense for $(X, \mathbb{F})$, there exists a $z_m = \omega_{r_m}^{-1}(y_m) \in V_m$ $(y_m \in U_m)$ such that (x_m, z_m) is proximal for $(X, \mathbb{F})$. Thus, there exists a sequence $(n_{k,m})$ in $\mathbb{N}$ such that $\lim_{n_{k,m}\to\infty} d(\omega_{n_{k,m}+r_m}(x_m), \omega_{n_{k,m}+r_m}(z_m)) = 0$. Choose $s_m \in \mathbb{N}$ such that $d(\omega_{s_m+r_m}(x_m), \omega_{s_m+r_m}(z_m)) < \frac{1}{3.2^m}$. Also, by proposition 4.3.4, $d(\omega_{s_m+r_m}(w), f^{s_m}(\omega_{r_m}(w))) < \sum_{i=1}^{s_m} D(f_{r_m+i}, f) < \frac{1}{3.2^m}$ for any $w \in X$ and hence by triangle inequality, we have $d(f^{s_m}(\omega_{r_m}(x_m)), f^{s_m}(\omega_{r_m}(z_m))) < \frac{1}{2^m}$ or $d(f^{s_m}(x), f^{s_m}(y_m)) < \frac{1}{2^m}$. By continuity, there exists neighbourhood U_{m+1} of y_m such that $\overline{U_{m+1}} \subset U_m$ and $d(f^{s_m}(x), f^{s_m}(y)) < \frac{1}{2^m}$ for all $y \in U_{m+1}$.

Repeating the process for pair (x,U_m) (for each m), we obtain a open set U_{m+1}, $\overline{U_{m+1}} \subset U_m \subset U$ and $s_m \in \mathbb{N}$ such that $d(f^{s_m}(x), f^{s_m}(y)) < \frac{1}{2^m}$ for all $y \in U_{m+1}$. As $\overline{U_{m+1}}$ is a nested decreasing sequence of closed sets in a compact metric space and hence $\bigcap_{m=1}^{\infty} \overline{U_m} \neq \phi$. Then, for any $u \in \bigcap_{m=1}^{\infty} \overline{U_m} \subset U$, as $u \in U_m$ for all m, we have $d(f^{s_m}(x), f^{s_m}(u)) < \frac{1}{2^m}$ for all m and hence the pair (x,u) is proximal for (X,f). As the proof holds good for any non-empty open set U in X, $Prox_f(x)$ is dense in X. □

Remark 4.4.1 The above result proves that proximal cell of each x is dense for (X,f) if and only if the proximal cell of each x is dense for $(X,\mathbb{F})$ and hence the dynamical behaviour of two systems is equivalent in this regard. It may be noted that Example 4.3.1 establishes that proximal pairs need not be carried over (from (X,f) to $(X,\mathbb{F})$) and hence the above conclusions may not hold good under the weaker assumption of collective convergence. However, the proof does not exploit the denseness condition and establishes that denseness of proximal cells in some neighbourhood is equivalent for the two systems. Further, by proposition 4.3.2, if a system is sensitive then denseness of proximal cells is equivalent to the denseness of Li-Yorke cells. Equivalently, if proximal cells are dense for a system then the system is sensitive if and only if it is Li-Yorke sensitive. As sensitivity is equivalent for feeble open systems (under collective convergence), we obtain the following corollary.

Corollary 4.4.1 *Let $(X,\mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ of feeble open maps commuting with f satisfying $\sum_{n=1}^{\infty} D(f_n,f) < \infty$. If $Prox(x)$ is dense in X for each $x \in X$, then (X,f) is Li-Yorke sensitive $\Leftrightarrow$ $(X,\mathbb{F})$ is Li-Yorke sensitive.*

Proof. The result is a direct consequence of proposition 4.3.1 and Remark 4.4.1. □

Proposition 4.4.3 *Let X be compact and let $(X,\mathbb{F})$ be a non-autonomous system generated by a family $\mathbb{F}$ commuting with f. If $\sum_{n=1}^{\infty} D(f_n,f) < \infty$ then, set of proximal pairs is dense for (X,f) $\Leftrightarrow$ set of proximal pairs is dense for $(X,\mathbb{F})$.*

Proof. As every pair proximal for (X,f) is proximal for $(X,\mathbb{F})$, if (X,f) has dense set of proximal pairs then $(X,\mathbb{F})$ has dense set of proximal pairs.

Conversely, let $(X,\mathbb{F})$ exhibit dense set of proximal pairs and let $U_1 \times U_2$ be any non-empty open set in $X \times X$. Let U'_i be non-empty open such that $\overline{U'_i} \subset U_i$. As $\sum_{n=1}^{\infty} D(f_n,f) < \infty$, for each $m \in \mathbb{Z}^+$ there exists $r_m \in \mathbb{N}$ such that $\sum_{n=r_m}^{\infty} D(f_n,f) < \frac{1}{3.2^m}$. Fix $m=0$ and let $U_{1,m} = U'_1$ and $U_{2,m} = U'_2$. For $i = 1,2$, let $V_{i,m} = \omega_{r_m}^{-1}(U_{i,m})$. Then $V_{1,m} \times V_{2,m}$ is open in $X \times X$. Consequently there exists a $(x_{1,m}, x_{2,m}) \in V_{1,m} \times V_{2,m}$ such that $(x_{1,m}, x_{2,m})$ is proximal for $(X,\mathbb{F})$ and hence there exists a sequence $(n_{k,m})$ in $\mathbb{N}$ such that $\lim_{n_{k,m}\to\infty} d(\omega_{n_{k,m}+r_m}(x_{1,m}), \omega_{n_{k,m}+r_m}(x_{2,m})) = 0$. Choose $s_m \in \mathbb{N}$ such that $d(\omega_{s_m+r_m}(x_{1,m}), \omega_{s_m+r_m}(x_{2,m})) < \frac{1}{3.2^m}$. Also, by proposition 4.3.4, $d(\omega_{s_m+r_m}(x_{i,m}), f^{s_m}(\omega_{r_m}(x_{i,m}))) < \sum_{i=1}^{s_m} D(f_{r_m+i}, f) < \frac{1}{3.2^m}$ and hence by triangle inequality, we have $d(f^{s_m}(\omega_{r_m}(x_{1,m})), f^{s_m}(\omega_{r_m}(x_{2,m}))) < \frac{1}{2^m}$. Note that $\omega_{r_m}(x_{i,m}) \in U_{i,m}$ and assuming $u_{i,m} = \omega_{r_m}(x_{i,m})$ yields $d(f^{s_m}(u_{1,m}), f^{s_m}(u_{2,m})) < \frac{1}{2^m}$. Thus, there exists neighbourhoods $U_{i,m+1}$ of $u_{i,m}$ such that $\overline{U_{i,m+1}} \subset U_{i,m}$ and $d(f^{s_m}(x), f^{s_m}(y)) < \frac{1}{2^m}$ for all $x \in U_{1,m+1}, y \in U_{2,m+1}$.

Repeating the process for each m, we obtain a nested sequence of open sets $U_{i,m+1}$, $\overline{U_{i,m+1}} \subset U_{i,m} \subset U_i$ such that $d(f^{s_m}(x), f^{s_m}(y)) < \frac{1}{2^m}$ for all $x \in U_{1,m+1}, y \in U_{2,m+1}$. As $\overline{U_{i,m+1}}$ is a nested decreasing sequence of closed sets in a compact metric space and hence $\bigcap_{m=0}^{\infty} \overline{U_{i,m}} \neq \phi$ $(i = 1,2)$. Then, for any

$u_1 \in \bigcap_{m=0}^{\infty} \overline{U_{1,m}} \subset U_1$, $u_2 \in \bigcap_{m=0}^{\infty} \overline{U_{2,m}} \subset U_2$, as $u_i \in U_{i,m}$ for all m, we have $d(f^{s_m}(u_1), f^{s_m}(u_2)) < \frac{1}{2^m}$ for all m and hence (u_1, u_2) is proximal for (X, f). As the proof holds good for any pair of non-empty open sets U_1, U_2 in X, the set of pairs proximal for (X, f) is dense in $X \times X$. $\square$

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