5

Generating Functions and Rearrangements of Non-autonomous Systems

In this chapter, we relate the dynamics of a non-autonomous dynamical system with the dynamics of its generating functions. We prove that in general there is no direct correspondence between the dynamical behaviour of a non-autonomous dynamical system and the dynamical behaviour of its generating functions. In particular, we give examples to prove that the dynamics of the generating functions need not carry forward to the non-autonomous system generated (and vice-versa). We also study the dynamics of truncations of a given non-autonomous system. We prove that if the family $\mathbb{F}$ is feeble open, properties like transitivity, mixing and various forms of sensitivities are equivalent for two systems. We also investigate properties like equicontinuity, minimality and proximality for the two systems. We use the results obtained to investigate the dynamics of various possible rearrangements of a given non-autonomous dynamical system. In particular, we prove that under derived conditions, the dynamics of a system is preserved under finite rearrangements. We give an example to show that the dynamics of a system need not be preserved under infinite rearrangement.

5.1 NON-AUTONOMOUS SYSTEMS AND ITS GENERATING FUNCTIONS

We now discuss the relation between the dynamics of non-autonomous system and dynamics of its generating functions.

Example 5.1.1 *Let S^1 be the unit circle and let $\theta \in (0,1)$ be an rational. Let $f_n : S^1 \to S^1$ be defined as $f_n(\phi) = \phi + 2\pi\frac{\theta}{3^n}$. As θ is rational, each map f_k has dense set of periodic points. However, as $\sum\limits_{n=1}^{\infty} \frac{\theta}{3^n} < 1$, for any $\beta \in S^1$, $\omega_n(\beta) \neq \beta \ \forall n$. Hence the non-autonomous system generated by $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ fails to have any periodic point.*

Example 5.1.2 *Let S^1 be the unit circle and let $\theta \in (0,1)$ be an irrational. Let $f_1, f_2 : S^1 \to S^1$ be defined as $f_1(\phi) = \phi + 2\pi\theta$ and $f_2(\phi) = \phi - 2\pi\theta$ respectively and let $(X,\mathbb{F})$ be the corresponding non-autonomous dynamical system. As each f_i is an irrational rotation, no point is periodic for any f_i. However as $f_1 \circ f_2 = Id$, the system $(S^1,\mathbb{F})$ has dense set of periodic points.*

Remark 5.1.1 The above examples shows that, dense periodicity for a non-autonomous dynamical system cannot be characterized in terms of the dense periodicity of the generating functions. While example 5.1.2 shows system may exhibit dense periodicity without any of the generating functions exhibiting dense periodicity, example 5.1.1 proves that the system may fail to have a dense set of periodic points even when all its generating functions exhibit the same.

Example 5.1.3 *Let I be the unit interval and let (q_n) be an enumeration of rationals in I. Let $f_n : I \to I$ be defined as $f_n(x) = q_n$ for all $x \in I$. Then each f_n is a constant map but the system $(X,\mathbb{F})$ generated by $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ is minimal.*

Remark 5.1.2 In example, 5.1.2 the system $(X,\mathbb{F})$ fails to be minimal as $f_2 \circ f_1 = id$. However, f_1 and f_2 are minimal as θ is irrational. Thus, the example shows that even if each of the maps f_i are minimal, the non-autonomous system generated by $\mathbb{F}$ need not be minimal. On the other hand, example 5.1.3 shows that the non-autonomous system can exhibit minimality without any of the maps f_i being minimal. Hence, minimality of a non-autonomous system, in general, cannot be characterized in terms of minimality of its generating functions.

Example 5.1.4 *Let I be the unit interval and let $f_1, f_2 : I \to I$ be defined as*

$$f_1(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{2} - x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in [0, \frac{1}{2}] \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Let $\mathbb{F} = \{f_1, f_2\}$ and let $(X, \mathbb{F})$ be the corresponding non-autonomous system. As $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$ are invariant for f_1 and f_2 respectively, none of the f_i are transitive. However, the map

$$f_2 \circ f_1(x) = \begin{cases} \frac{1}{2} - 2x & \text{if } x \in [0, \frac{1}{4}] \\ 4x - 1 & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

is transitive and hence the non-autonomous system $(X, \mathbb{F})$ is transitive.

Example 5.1.5 *Let $\sum = \{0, 1\}^{\mathbb{N}}$ be the collection of two-sided sequences of 0 and 1 endowed with the product topology. Let $\sigma_1, \sigma_2 : \sum \to \sum$ be defined as $\sigma_1(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots) = (\ldots x_{-2}x_{-1}x_0.x_1x_2\ldots)$ and $\sigma_2(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots) = (\ldots x_{-2}.x_{-1}x_0x_1x_2\ldots)$. Then σ_1, σ_2 are the shift operators and are continuous with respect to the product topology. Let $\mathbb{F} = \{\sigma_1, \sigma_2\}$ and let $(X, \mathbb{F})$ be the corresponding non-autonomous system. It can be seen that each σ_i is transitive. However as $\sigma_1 \circ \sigma_2 = id$, the system generated is not transitive.*

Remark 5.1.3 The above two examples show that transitivity of a general non-autonomous system is not equivalent to transitivity of its generating functions. While the example 5.1.4 shows that the non-autonomous system can exhibit transitivity without any of the generating functions being transitive, example 5.1.5 shows that the non-autonomous system fails to be transitive even each of its generating functions exhibits the same.

Example 5.1.6 *Let I be the unit interval and let f_1, f_2 be defined as*

$$f_1(x) = \begin{cases} 2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ -x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Remark 5.1.4 In example 5.1.6 none of the maps f_i are weakly mixing (topologically mixing). But the system $(X, \mathbb{F})$ generated by $\mathbb{F} = \{f_1, f_2\}$ is weakly mixing (topologically mixing), as for any open set U, there exists a natural number n such that $\omega_n(U) = [0, 1]$. Hence the non-autonomous system $(X, \mathbb{F})$ is weakly mixing (topologically mixing). The non-autonomous dynamical system generated in example 5.1.6 exhibits weakly mixing (topological mixing) without any of its components f_i exhibiting the same. Also example 5.1.5 shows that the non-autonomous system generated need not exhibit weakly mixing (topological mixing) even if each of the generating functions exhibit weakly mixing (topological mixing). Hence weakly mixing (topologically mixing) of a non-autonomous system cannot be characterized in terms of weakly mixing (topologically mixing) of its generating components.

Example 5.1.7 *Let $I \times S^1$ be the unit cylinder. Let $f_1, f_2 : I \times S^1 \to I \times S^1$ be defined as $f_1((r,\theta)) = (r, \theta + r)$ and $f_2((r,\theta)) = (r, \theta - r)$ respectively. Let $\mathbb{F} = \{f_1, f_2\}$ and let $(X, \mathbb{F})$ be the corresponding non-autonomous system. As points at different heights of the cylinder are rotating with different speeds, each of the maps f_i are cofinitely sensitive [Sharma and Nagar, 2010]. However as $f_2 \circ f_1 = Id$, the system $(I \times S^1, \mathbb{F})$ is not sensitive.*

Remark 5.1.5 Example 5.1.7 shows that the non-autonomous system generated need not be sensitive even when each of the maps f_i are sensitive. Also, example 5.1.4 proves that the non-autonomous system can exhibit sensitivity without any of the maps f_i being sensitive. Hence, sensitivity of the non-autonomous system in general cannot be characterized in terms of sensitivity of its generating functions.

Example 5.1.8 *Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined as $f_1(x) = |x|$ and $f_2(x) = 2x - 1$. Let $\mathbb{F} = \{f_1, f_2\}$ and let $(X, \mathbb{F})$ be the corresponding non-autonomous system. Then f_1 and f_2 fail to be Li-Yorke chaotic. However, as $f_2(f_1(-\frac{7}{9})) = \frac{5}{9}, f_2(f_1(\frac{5}{9})) = \frac{1}{9}, f_2(f_1(\frac{1}{9})) = -\frac{7}{9}$, the map $f_2 \circ f_1(x) : R \to R$ possesses a period 3 point and hence is Li-Yorke Chaotic. Consequently, $(X, \mathbb{F})$ is Li-Yorke chaotic.*

Remark 5.1.6 The above example shows that the non-autonomous system may be Li-Yorke chaotic without any of the generating members being Li-Yorke chaotic. Also, example 5.1.5 shows that the non-autonomous system may not be Li-Yorke chaotic even when all the generating functions are Li-Yorke chaotic. Hence Li-Yorke chaoticity of a non-autonomous system cannot be characterized in terms of Li-Yorke chaoticity of its generating functions.

5.2 DYNAMICS OF TRUNCATED SYSTEM

For any $k \in \mathbb{N}$, we define the family $\mathbb{F}_k = \{f_n : n \geq k+1\}$.

Proposition 5.2.1 *$(X, \mathbb{F})$ is minimal $\Leftrightarrow$ $(X, \mathbb{F}_k)$ is minimal.*

Proof. Let $(X, \mathbb{F})$ be minimal and let $x \in X$. As each f_k is surjective, $\omega_k^{-1}(x)$ is non-empty. Further, as $(X, \mathbb{F})$ is minimal, orbit of any $y \in \omega_k^{-1}(x)$ (under $\mathbb{F}$) is dense in X. As orbit of x (under $\mathbb{F}_k$) and orbit of y (under $\mathbb{F}$) differ by finitely many points (atmost k), denseness of orbit of y (under $\mathbb{F}$) implies denseness of orbit of x (under $\mathbb{F}_k$) and hence $(X, \mathbb{F}_k)$ is minimal.

Conversely, let $x \in X$ and $y = \omega_k(x)$. As $(X, \mathbb{F}_k)$ is minimal, orbit of y (under $\mathbb{F}_k$) is dense in X. Further as orbit of y (under $\mathbb{F}_k$) and orbit of x (under $\mathbb{F}$) differ by finitely many points (atmost k), denseness of orbit of y (under $\mathbb{F}_k$) ensures denseness of orbit of x (under $\mathbb{F}$) and hence $(X, \mathbb{F})$ is minimal. □

Proposition 5.2.2 *$(X, \mathbb{F}_k)$ is equicontinuous $\Leftrightarrow$ $(X, \mathbb{F})$ is equicontinuous.*

Proof. Let $(X, \mathbb{F}_k)$ be equicontinuous and let $\varepsilon > 0$ be given. As $(X, \mathbb{F}_k)$ is equicontinuous, there exists $\rho > 0$ ($\rho < \varepsilon$) such that $d(x,y) < \rho$ implies $d(\omega_n^k(x), \omega_n^k(y)) < \varepsilon \ \forall \ n \geq k+1$. Also as the set $\{f_1, f_2 \circ f_1, \ldots, f_k \circ f_{k-1} \circ \ldots \circ f_1\}$ is finite, there exists $\eta > 0$ such that $d(x,y) < \eta$ ensures $d(f_r \circ f_{r-1} \circ \ldots \circ f_1(x), f_r \circ f_{r-1} \circ \ldots \circ f_1(y)) < \rho$ for $r \in \{1, 2, \ldots, k\}$ or $d(\omega_r(x), \omega_r(y)) < \rho$ for $r \in \{1, 2, \ldots, k\}$. In particular, $d(x,y) < \eta$ gives $d(\omega_k(x), \omega_k(y)) < \rho$ which further implies $d(\omega_n^k(\omega_k(x)), \omega_n^k(\omega_k(y))) < \varepsilon \ \forall \ n \geq k+1$ (by equicontinuity of $(X, \mathbb{F}_k)$) or $d(\omega_n(x), \omega_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$ and hence $(X, \mathbb{F})$ is equicontinuous.

Conversely, let $(X, \mathbb{F})$ be equicontinuous and let $\varepsilon > 0$ be given. As $(X, \mathbb{F})$ is equicontinuous, there exists $\delta > 0$ such that $d(x,y) < \delta$ ensures $d(\omega_n(x), \omega_n(y)) < \varepsilon \ \forall n \in \mathbb{N}$. Let $x \in X$ and let $\mathcal{N}_x = \{S(x, \frac{1}{n}) \ : n \in \mathbb{N}\}$ be the neighbourhood base at x. As $\bigcap\limits_{U \in \mathcal{N}_x} \omega_k^{-1}(U) = \omega_k^{-1}(\{x\})$, there exist $U \in \mathcal{N}_x$

such that $\omega_k^{-1}(U) \subset \bigcup_{y\in\omega_k^{-1}(\{x\})} S(y,\delta)$ or there exist $m\in\mathbb{N}$ such that $\omega_k^{-1}(S(x,\frac{1}{m})) \subset \bigcup_{y\in\omega_k^{-1}(\{x\})} S(y,\delta)$. Consequently, if $d(x,z) < \frac{1}{m}$ for any $u \in \omega_k^{-1}(z)$, $d(u,y) < \delta$ for some $y \in \omega_k^{-1}(\{x\})$ and hence $d(\omega_n(u),\omega_n(y)) < \varepsilon$ for all $n\in\mathbb{N}$. As $\omega_k(u) = z$, $\omega_k(y) = x$ and $\omega_k^n \circ \omega_k = \omega_n$, we obtain $d(\omega_n^k(x),\omega_n^k(z)) < \varepsilon$ for all $n \geq k+1$ and hence the truncated system is equicontinuous at x. As the proof holds for any $x\in X$, $(X,\mathbb{F}_k)$ is equicontinuous and hence equicontinuity is equivalent for the two systems. □

Remark 5.2.1 The above proofs establish the equivalence of minimality and equicontinuity for the two systems. While equivalence of minimality for the two systems follows from the fact that denseness of a set is not altered by addition or deletion of finitely many points, equivalence of equicontinuity is established by working on each of the fibres of the inverse function (fibres of f^{-1} are functions $g = f^{-1}|_M$ where M is maximal subset of X such that $f|_M$ is injective). The result is intuitive in nature and establishes the fact that addition of finitely many maps cannot generate sensitivity in a non-sensitive system.

Proposition 5.2.3 *If $\mathbb{F}$ is commutative then, (x,y) is proximal for $(X,\mathbb{F}_k) \Rightarrow (x,y)$ is proximal for $(X,\mathbb{F})$. Further if the family $\mathbb{F}$ is bijective then (x,y) is proximal for $(X,\mathbb{F}) \Rightarrow (x,y)$ is proximal for $(X,\mathbb{F}_k)$.*

Proof. If (x,y) is proximal for $(X,\mathbb{F}_k)$ then there exists a sequence (n_r) of positive integers such that $\lim\limits_{r\to\infty} d(\omega_{n_r}^k(x),\omega_{n_r}^k(y)) = 0$. As X is compact, there exists $z\in X$ and a subsequence (n_{r_l}) of (n_r) such that $\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(x) = \lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(y) = z$. Thus we get, $f_k\circ f_{k-1}\circ\ldots\circ f_1(\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(x)) = f_k\circ f_{k-1}\circ\ldots\circ f_1(\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(y)) = f_k\circ f_{k-1}\circ\ldots\circ f_1(z)$ or $\lim\limits_{l\to\infty} f_k\circ f_{k-1}\circ\ldots\circ f_1(\omega_{n_{r_l}}^k(x)) = \lim\limits_{l\to\infty} f_k\circ f_{k-1}\circ\ldots\circ f_1(\omega_{n_{r_l}}^k(y)) = f_k\circ f_{k-1}\circ\ldots\circ f_1(z)$ (as $f_k\circ f_{k-1}\circ\ldots\circ f_1$ is continuous). Consequently, $\lim\limits_{l\to\infty}\omega_{n_{r_l}}(x) = \lim\limits_{l\to\infty}\omega_{n_{r_l}}(y) = f_k\circ f_{k-1}\circ\ldots\circ f_1(z)$ (as $\mathbb{F}$ is commutative) and hence (x,y) is proximal for $(X,\mathbb{F})$.

Conversely, let (x,y) be proximal for $(X,\mathbb{F})$. Thus, there exists sequence (n_r) of natural numbers such that $\lim\limits_{r\to\infty} d(\omega_{n_r}(x),\omega_{n_r}(y)) = 0$. Consequently, there exists a subsequence (n_{r_l}) of (n_r) and $z\in X$ such that $\lim\limits_{l\to\infty}\omega_{n_{r_l}}(x) = \lim\limits_{l\to\infty}\omega_{n_{r_l}}(y) = z$. As $\omega_{n_{r_l}} = \omega_{n_{r_l}}^k\circ(f_k\circ f_{k-1}\circ\ldots\circ f_1)$ and the family $\mathbb{F}$ is commutative, we obtain $\lim\limits_{l\to\infty} f_k\circ f_{k-1}\circ\ldots\circ f_1(\omega_{n_{r_l}}^k(x)) = \lim\limits_{l\to\infty} f_k\circ f_{k-1}\circ\ldots\circ f_1(\omega_{n_{r_l}}^k(y)) = z$ or $f_k\circ f_{k-1}\circ\ldots\circ f_1(\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(x)) = f_k\circ f_{k-1}\circ\ldots\circ f_1(\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(y)) = z$ (as $f_k\circ f_{k-1}\circ\ldots\circ f_1$ is continuous). As each f_i is bijective, $f_k\circ f_{k-1}\circ\ldots\circ f_1$ is bijective and thus we obtain $\lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(x) = \lim\limits_{l\to\infty}\omega_{n_{r_l}}^k(y)$ or (x,y) is proximal for $(X,\mathbb{F}_k)$. □

Remark 5.2.2 The above proof establishes the equivalence of proximality for the two systems when the family $\mathbb{F}$ is commutative and bijective. While proximality is preserved from $(X,\mathbb{F}_k)$ to $(X,\mathbb{F})$ when the family $\mathbb{F}$ is commutative, the converse is proved under additional assumption of bijectivity of the family $\mathbb{F}$. However, the proof uses only injectivity of the maps f_k and hence the result is true even when $\mathbb{F}$ is a commutative family of injective maps. Further, both commutativity and bijectivity (injectivity) are needed to establish the result and the result does not hold good when either of the conditions imposed is dropped. We now give an example to establish our claim.

Example 5.2.1 *Let I be the unit interval and let $f : I \to I$ be piecewise continuous linear map such that $f(0) = 0, f(\frac{1}{3}) = 1, f(\frac{2}{3}) = 0$ and $f(1) = \frac{2}{3}$. Let $g : I \to I$ be the defined as*

$$g(x) = \begin{cases} 2x & \text{for } x\in[0,\frac{1}{2}] \\ 2-2x & \text{for } x\in[\frac{1}{2},1] \end{cases}$$

Let $(X,\mathbb{F})$ be the non-autonomous system generated by $\mathbb{F} = \{f,g,g,\ldots\}$. It may be noted that f and g do not commute and hence non-autonomous system generated is non-commutative in nature. As $g(0) = g(1)$, $\{0,1\}$ is a proximal set for $(X,\mathbb{F}_k)$ for any $k\in\mathbb{N}$. However, as $f(0) = 0$ and $f(1) = \frac{2}{3}$ are fixed *for g, the pair is not proximal for $(X,\mathbb{F})$. Thus, commutativity is an essential condition to preserve proximal pairs from $(X,\mathbb{F}_k)$ to $(X,\mathbb{F})$.*

Example 5.2.2 *Let S^1 be the unit circle and let $f : S^1 \to S^1$ be defined as $f(\theta) = \theta + \pi$. Let $g : S^1 \to S^1$ be defined as*

$$g(\theta) = \begin{cases} \theta & \text{for } x \in [0,\pi] \\ \frac{\theta^2}{\pi} - 2\theta + 2\pi & \text{for } x \in [\pi, 2\pi] \end{cases}$$

Let $(X,\mathbb{F})$ be the non-autonomous system generated by $\mathbb{F} = \{f, g, g, \ldots\}$. It may be noted that both f and g are bijective and hence the non-autonomous system generated is bijective (but non-commutative) in nature. Further, as $f([0,\pi]) = [\pi, 2\pi]$ and π is fixed point (attracting from the right) for g, any two points in $[0,\pi]$ are proximal for $(X,\mathbb{F})$. However, as g fixes every point in $[0,\pi]$, the truncated system $(X,\mathbb{F}_k)$ $(k \geq 1)$ does not exhibit any proximal pair in $[0,\pi]$.

Example 5.2.3 *Let I be the unit interval and let $f, g : I \to I$ be defined as*

$$f(x) = \begin{cases} x & \text{for } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{1}{6} & \text{for } x \in [\frac{1}{2}, \frac{7}{8}] \\ 1 & \text{for } x \in [\frac{7}{8}, 1] \end{cases}$$

$$g(x) = \begin{cases} -2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ 2x - \frac{1}{2} & \text{for } x \in [\frac{1}{4}, \frac{1}{2}] \\ x & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Let $(X,\mathbb{F})$ be the non-autonomous system generated by $F = \{f, g, g, \ldots\}$. Then, f and g commute and hence non-autonomous system generated is commutative (but not bijective) in nature. As $f([\frac{7}{8}, 1]) = 1$ and $g(x) = x$ for $x \in [\frac{7}{8}, 1]$, any pair of distinct points in $[\frac{7}{8}, 1])$ is proximal for $(X,\mathbb{F})$ but fails to be proximal for any truncated system.

Remark 5.2.3 The above examples validate the necessity of the conditions imposed in proposition 5.2.3. While Example 5.2.1 establishes the necessity of the commutativity condition for the proposition to hold good, Examples 5.2.2 and 5.2.3 prove that commutativity or injectivity alone cannot preserve the proximal pairs in the converse direction. Consequently, both commutativity and injectivity of the maps f_k are needed for the converse to hold good and hence cannot be dropped.

Proposition 5.2.4 *$(X,\mathbb{F})$ is transitive $\Rightarrow$ $(X,\mathbb{F}_k)$ is transitive. If the family $\mathbb{F}$ is feeble open then $(X,\mathbb{F}_k)$ is transitive $\Rightarrow$ $(X,\mathbb{F})$ is transitive.*

Proof. Let $(X,\mathbb{F})$ be transitive and let U, V be any pair of non-empty open subsets in X. As $(X,\mathbb{F})$ is transitive, for the pair $U' = \omega_k^{-1}(U), V$ of non-empty open sets in X, there exists $r \in \mathbb{N}$ such that $\omega_r(U') \cap V \neq \phi$. Also, transitivity of $(X,\mathbb{F})$ ensures that the set $\{r \in \mathbb{N} : \omega_r(U') \cap V \neq \phi\}$ is infinite. Consequently there exists $r > k$ such that $\omega_r(U') \cap V \neq \phi$ or $\omega_r^k(U) \cap V \neq \phi$ and hence $(X,\mathbb{F}_k)$ is transitive.

Let $(X,\mathbb{F}_k)$ be transitive and let U, V be any pair of non-empty open subsets in X. As the family $\mathbb{F}$ is feeble open $\omega_k(U)$ has a non-empty interior. Thus, for open sets $U' = int(\omega_k(U)), V$ in X, there exists $r \in \mathbb{N}$ such that $\omega_r^k(U') \cap V \neq \phi$. Consequently, $\omega_r^k(\omega_k(U)) \cap V \neq \phi$ or $\omega_r(U) \cap V \neq \phi$ and hence $(X,\mathbb{F})$ is transitive. □

Remark 5.2.4 The above proof establishes the equivalence of transitivity for the two systems $(X,\mathbb{F})$ and $(X,\mathbb{F}_k)$. Though the property is preserved from $(X,\mathbb{F})$ to $(X,\mathbb{F}_k)$ unconditionally, the proof of the converse holds good when the family $\mathbb{F}$ is feeble open. As absence of feeble openness destroys the topological structure of an open set over iterations, feeble openness is needed for the converse to hold good. Further, as the proof does not use the structure of open sets explicitly, if U_1, U_2 interact with V_1, V_2 for $(X, \mathbb{F})$ (or $(X, \mathbb{F}_k)$) at r-th iterate then U_1, U_2 and V_1, V_2 interact at $r-k$-th (or $r+k$-th) iterate for $(X, \mathbb{F}_k)$ (or $(X, \mathbb{F})$) and hence weakly mixing is equivalent for the two systems under identical conditions. Further, as the set of times of interaction between open sets U and V for the two systems $(X, \mathbb{F})$ and $(X, \mathbb{F}_k)$ are translate of each other (by constant k), the similar proof gives equivalence of topological mixing under identical conditions. Hence we get the following corollary.

Corollary 5.2.1 *$(X, \mathbb{F})$ is weakly mixing (topological mixing) $\Rightarrow$ $(X, \mathbb{F}_k)$ is weakly mixing (topological mixing). If the family $\mathbb{F}$ is feeble open then $(X, \mathbb{F}_k)$ is weakly mixing (topological mixing) $\Rightarrow$ $(X, \mathbb{F})$ is weakly mixing (topological mixing).*

Proof. The proof follows from discussions in Remark 5.2.4 and proposition 5.2.4. □

Example 5.2.4 *Let I be the unit interval and let $f, g : I \to I$ be defined as*

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{2}] \\ 2x - 1 & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$g(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Let $(X, \mathbb{F})$ be the non-autonomous system generated by $\mathbb{F} = \{f, g, g, \ldots\}$. For any $k \in \mathbb{N}$, $(X, \mathbb{F}_k)$ is the autonomous system generated by tent map and hence exhibits all forms of mixing and sensitivities. However for any open set U, $U \subset [0, \frac{1}{2}]$, $\omega_r(U) = \{0\}$ for any $r \in \mathbb{N}$. Thus the non-autonomous system does not exhibit any form of mixing or sensitivity and hence feeble openness is needed to preserve any form of mixing or sensitivity (from $(X, \mathbb{F}_k)$ to $(X, \mathbb{F})$).

Remark 5.2.5 The above example establishes the necessity of feeble openness of the family $\mathbb{F}$ to exhibit any of the dynamical property like topological transitivity, weakly mixing (topological mixing) and sensitivity. Now we show that sensitivity is also equivalent for two systems, when the family $\mathbb{F}$ is feeble open.

Proposition 5.2.5 *$(X, \mathbb{F})$ is sensitive $\Rightarrow$ $(X, \mathbb{F}_k)$ is sensitive. If the family $\mathbb{F}$ is feeble open then $(X, \mathbb{F}_k)$ is sensitive $\Rightarrow$ $(X, \mathbb{F})$ is sensitive.*

Proof. Let $(X, \mathbb{F})$ be sensitive with δ as constant of sensitivity. For any open set U, continuity of each f_i implies $U' = \omega_k^{-1}(U)$ is open and hence there exists $r \in \mathbb{N}$ such that $diam(\omega_r(U')) > \delta$. As the set of times of expansion is infinite for a sensitive system, there exists $m > k$ such that $diam(\omega_m(U')) > \delta$ which implies $diam(\omega_m^k(U)) > \delta$ and hence $(X, \mathbb{F}_k)$ is sensitive.

Conversely let $(X, \mathbb{F}_k)$ be sensitive with δ as constant of sensitivity and let U be a non-empty open set in X. As the family $\mathbb{F}$ is feeble open, $U' = int(\omega_k(U))$ is non-empty and hence sensitivity of $(X, \mathbb{F}_k)$ yields $m \in \mathbb{N}$ such that $diam(\omega_m^k(U')) > \delta$. Consequently, $diam(\omega_m^k(\omega_k(U))) > \delta$ or $diam(\omega_m(U)) > \delta$ and hence $(X, \mathbb{F})$ is sensitive. □

Remark 5.2.6 The above proof establishes equivalence of sensitivity for the two systems $(X, \mathbb{F})$ and $(X, \mathbb{F}_k)$ under stated conditions. Once again, while sensitivity of $(X, \mathbb{F})$ implies sensitivity of $(X, \mathbb{F}_k)$ unconditionally, the converse is true when the family $\mathbb{F}$ is feeble open. As noted in Example 5.2.4, feeble openness is required for the converse to hold good and hence cannot be dropped. It may be noted that if one of the systems is sensitive with sensitivity constant δ, then the proof establishes the sensitivity of the other system with the same constant of sensitivity and hence the two systems are sensitive with same sensitivity constant. Further as the times of expansion (of an open set U) for the two systems are translate (by constant k) of each other, a similar proof establishes the equivalence of cofinite sensitivity for the two systems. Hence we get the following corollary.

Corollary 5.2.2 $(X,\mathbb{F})$ *is cofinitely sensitive* $\Rightarrow$ $(X,\mathbb{F}_k)$ *is confinitely sensitive. If the family* $\mathbb{F}$ *is feeble open then* $(X,\mathbb{F}_k)$ *is cofinitely sensitive* $\Rightarrow$ $(X,\mathbb{F})$ *is cofinitely sensitive.*

5.3 ALTERATIONS AND REARRANGEMENTS

We say $\mathbb{G}$ to be an alteration of $\mathbb{F}$ if $\mathbb{G}$ can be obtained by inserting or deleting finitely many maps from the family $\mathbb{F}$. In the previous section, it is established that for a feeble open family $\mathbb{F}$, $(X,\mathbb{F})$ exhibits any form of mixing (sensitivity) if and only if $(X,\mathbb{F}_k)$ also exhibits the same. It may be noted that if $(X,\mathbb{G})$ is an alteration of $(X,\mathbb{F})$, there exist $k\in\mathbb{N}$ such that $\mathbb{G}_k=\mathbb{F}_k$. Consequently, for a feeble open family $\mathbb{F}$, as $(X,\mathbb{F})$ and $(X,\mathbb{F}_k)$ (and similarly $(X,\mathbb{G})$ and $(X,\mathbb{G}_k)$) exhibit identical notions of mixing (sensitivity), $(X,\mathbb{F})$ exhibits any form of mixing (sensitivity) if and only if $(X,\mathbb{G})$ exhibits similar form of mixing (sensitivity) and hence different notions of mixing (sensitivity) are preserved under alterations. Also, as minimality and equicontinuity are equivalent for the two systems unconditionally, the properties are also preserved under alterations. Further, as any finite rearrangement of a system $(X,\mathbb{F})$ can be viewed as an alteration of $(X,\mathbb{F})$, the derived results hold good for any finite rearrangement of the system $(X,\mathbb{F})$. Thus we obtain the following results.

Proposition 5.3.1 *Let* $(X,\mathbb{F})$ *be a non-autonomous system and let* $\mathbb{G}$ *be an alteration (finite rearrangement) of* $\mathbb{F}$. *Then,* $(X,\mathbb{F})$ *is minimal (equicontinuous)* $\Leftrightarrow$ $(X,\mathbb{G})$ *is minimal (equicontinuous).*

Proposition 5.3.2 *Let* $\mathbb{F}$ *be feeble open and let* $\mathbb{G}$ *be an alteration (finite rearrangement) of* $\mathbb{F}$. *Then,* $(X,\mathbb{F})$ *exhibits any form of mixing (sensitivity) if and only if* $(X,\mathbb{G})$ *exhibits identical form of mixing (sensitivity).*

Proposition 5.3.3 *Let* $\mathbb{F}$ *be commutative family of bijective maps and let* $\mathbb{G}$ *be an alteration (finite rearrangement) of* $\mathbb{F}$. *Then,* (x,y) *is proximal for* $(X,\mathbb{F})$ *if and only if* (x,y) *is proximal for* $(X,\mathbb{G})$.

Remark 5.3.1 The above results provide sufficient conditions under which a dynamical notion is preserved under a finite rearrangement. Consequently, while minimality and equicontinuity are preserved unconditionally, various notions of mixing (sensitivity) are preserved when the family $\mathbb{F}$ is feeble open. However, the results derived hold good when $\mathbb{G}$ is a finite rearrangement of $\mathbb{F}$ and the dynamical behaviour of the system need not be preserved (under the stated conditions) when the rearrangement $\mathbb{G}$ is an infinite rearrangement. We now an give example in support of our claim.

Example 5.3.1 *Let* $X=\{0,1\}^{\mathbb{Z}}$ *be the collection of two-sided sequences of* 0 *and* 1 *endowed with the product topology. Let* $\sigma: X\to X$ *be defined as* $\sigma(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots)=(\ldots x_{-2}x_{-1}x_0.x_1x_2\ldots)$. *The map* σ *is the shift operator and is continuous with respect to the product topology. Let* $\mathbb{F}=\{\sigma,\sigma^{-1},\sigma,\sigma,\sigma^{-1},\sigma^{-1},\ldots\}$. *Thus, the family* $\mathbb{F}$ *is defined by defining* $f_i=\sigma$ *when* $n(n+1)+1\leq i\leq(n+1)^2$ *and* $f_i=\sigma^{-1}$ *when* $(n+1)^2+1\leq i\leq(n+1)(n+2)$. *Then, as* $\omega_{n(n+1)}(x)=x$ *and* $\omega_{n(n+1)+r}(x)=\sigma^r(x)$ *for* $1\leq r\leq n+1$, *for any open set* U *we obtain,* $\omega_{n(n+1)+r}(U)=\sigma^r(U)$ *for* $1\leq r\leq n+1$ *and hence the system* $(X,\mathbb{F})$ *exhibits all forms of mixing and sensitivity. However, as there are equal number of* σ *and* σ^{-1} *between* $f_{n(n+1)}$ *and* $f_{(n+1)(n+2)}$ *(*$n+1$ *each), the family* $\mathbb{F}$ *can be rearranged to obtain* $\mathbb{G}=\{\sigma,\sigma^{-1},\sigma,\sigma^{-1},\ldots\}$. *As* $(X,\mathbb{G})$ *does not exhibit any form of mixing or sensitivity, any form of mixing or sensitivity need not be preserved under infinite rearrangement. Further, it may be noted that* $(X,\mathbb{F})$ *is strongly sensitive and hence is not equicontinuous. However, as orbit of any* x *in* $(X,\mathbb{G})$ *is* $\{x,\sigma(x)\}$, *the system* $(X,\mathbb{G})$ *is equicontinuous and hence equicontinuity is not preserved under infinite rearrangements even when the maps* f_i *are bijective. Hence the conditions under which the dynamical behaviour is preserved for finite rearrangements strictly work when* $\mathbb{G}$ *is a finite rearrangement and need not preserve the dynamics when the family* $\mathbb{F}$ *is infinitely rearranged.*

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