1
Introduction

Historically, dynamical systems were first used in astronomical investigations by Johannes Kepler and Galileo Galilei in early seventeenth century, where they performed the qualitative analysis of planetary motions. Later, Newton formalised the studies mathematically using ordinary differential equations. Since then, mathematical analysis has been used extensively to investigate various natural and physical occurring systems. The theory in a topological setting was first investigated by Poincaré towards the end of nineteenth century where he used the theory of dynamical systems to investigate the three body problem in celestial mechanics. Birkhoff, in early twentieth century, used the qualitative theory of dynamical systems to investigate several problems in ergodic theory. The investigations have motivated further work in this area and the theory has been used to approximate various natural and physical systems in several branches of science and engineering.

By a dynamical system, we mean any time evolving system. If the governing rule for the system (with respect to time) is constant, the system is referred as an autonomous dynamical system, otherwise the system is referred as a non-autonomous dynamical system. While an autonomous system can be described by a pair (X, f) (where X is the phase space and f is a continuous self map on X), a non-autonomous system can be described by a pair $(X, \mathbb{F})$ where X is the phase space and $\mathbb{F}$ is a sequence of continuous self maps on X. In recent times, the theory of dynamical systems has been used extensively to predict the long term behaviour of various physical and natural systems occurring around us. Some early studies in population dynamics used the logistic map to predict population growth of certain species [Elaydi, 2007]. In [Beer, 2000], the author used the theory of dynamical systems to study the agent environment interaction in the cognitive setting. In [Hamill *et al.*, 1999], authors used the dynamical systems approach to lower extremity running injuries. In [Oono and Kohmoto, 1965], authors used the theory of discrete systems to model the chemical turbulence in a system. Although such studies have resulted in good approximations of the underlying systems, the theory of autonomous system has been used to obtain the respective approximations. However, as a time variant governing rule provides a better insight into the problem, approximating a given system by non-autonomous dynamical system is expected to provide a better approximation to the original problem. As a result, some of the studies have used the theory of non-autonomous dynamical system to model their respective systems. In [Kloeden and Pötzsche, 2012], authors provide a survey of different techniques used in approximating various problems in medical sciences using non-autonomous dynamical systems. As any simple chemostat model works under the assumption of fixed availability of the nutrient and its supply rate, availability of the nutrient in a system is a function of nutrient consumption rate and input nutrient concentration, which results in approximation of the system by a non-autonomous dynamical system. In [Caraballo and Han, 2016], authors discuss the Chemostat Model using non-autonomous system. To study and predict the behaviour of weather and climate is difficult due to the complexity of atmospheric evolution. In 1984, Lorenz introduced a model (Lorenz-84) to study the atmospheric circulation. The behaviour of this model has been studied extensively since its introduction. In [Anguiano and Caraballo, 2014], authors discuss the Lorenz-84 model in non-autonomous settings.

These illustrations emphasize the importance of investigating the dynamics of a general non-autonomous dynamical system. Although the topic has caught attention and some interesting results have been obtained, some natural questions for the dynamics of the non-autonomous system need to be answered. For example, what is the relation between the dynamics of a general non-autonomous dynamical system and its generating functions? If the non-autonomous dynamical system is generated by finite family $\mathbb{F}$, what is relation between the dynamics of non-autonomous system $(X,\mathbb{F})$ and autonomous system $(X, f_k\circ\cdots\circ f_2\circ f_1)$? In case of non-autonomous system generated by a uniformly convergent sequence, when can the dynamics of non-autonomous system be approximated by the limiting (autonomous) system? How does the dynamics of the system change when generating family is replaced by a rearrangement of the original family? Such questions arise naturally while investigating the dynamics of a general non-autonomous dynamical system and need to be answered. In this work, we give answers to some of the questions raised above. Before proceeding further, we give some basic concepts and definitions required.

Let (X,d) be a compact metric space and let $\mathbb{F}=\{f_n : n\in\mathbb{N}\}$ be a family of continuous self maps on X. For a fixed initial seed $x_0\in X$, any such family $\mathbb{F}$ generates a non-autonomous dynamical system via the relation $x_n = f_n(x_{n-1})$. Throughout this thesis, such a dynamical system will be denoted by $(X,\mathbb{F})$. For any $x\in X$, $\{f_n\circ f_{n-1}\circ\ldots\circ f_1(x) : n\in\mathbb{N}\}$ defines the orbit of x. For notational convenience, let $\omega^n_{n+k} = f_{n+k}\circ f_{n+k-1}\circ\ldots\circ f_{n+1}$ and $\omega_n^{-1}(y) = f_1^{-1}\circ f_2^{-1}\circ\ldots\circ f_n^{-1}(y)$. It may be noted that $\omega_n^{-1}(y)$ traces the point n units back in time and $\omega_n(x) = f_n\circ f_{n-1}\circ\ldots\circ f_1(x)$ is the state of the system after n iterations.

A family $\mathbb{F}=\{f_n : n\in\mathbb{N}\}$ is said to be *commutative* if $f_i\circ f_j = f_j\circ f_i$ for all $i,j\in\mathbb{N}$. A family $\mathbb{F}$ is *bijective* if each map $f\in\mathbb{F}$ is bijective. A map f is said to be *feeble open* if $int(f(U))\neq\emptyset$ for any non-empty open set U (in X). The family $\mathbb{F}=\{f_n : n\in\mathbb{N}\}$ is said to be *feeble open* if each $f\in\mathbb{F}$ is feeble open. A point x is called *periodic* for $(X,\mathbb{F})$ if there exists $n\in\mathbb{N}$ such that $\omega_{nk}(x) = x$ for all $k\in\mathbb{N}$. The least such n is known as the period of the point x. The system $(X,\mathbb{F})$ is *transitive* (or $\mathbb{F}$ is transitive) if for each pair of open sets U,V in X, there exists $n\in\mathbb{N}$ such that $\omega_n(U)\bigcap V\neq\phi$. The system $(X,\mathbb{F})$ is said to be *weakly mixing* if for any collection of non-empty open sets U_1,U_2,V_1,V_2, there exists a natural number n such that $\omega_n(U_i)\bigcap V_i\neq\phi$, $i=1,2$. Equivalently, we say that the system is weakly mixing if $\mathbb{F}\times\mathbb{F}$ is transitive. The system is said to be *topologically mixing* if for every pair of non-empty open sets U,V there exists a natural number K such that $\omega_n(U)\bigcap V\neq\phi$ for all $n\geq K$. The system is said to be *sensitive* if there exists a $\delta>0$ such that for each $x\in X$ and each neighbourhood U of x, there exists $n\in\mathbb{N}$ such that $diam(\omega_n(U))>\delta$. If there exists $K>0$ such that $diam(\omega_n(U))>\delta$ $\forall n\geq K$, then the system is *cofinitely sensitive*. A set S is said to be *scrambled* if for any distinct $x,y\in S$, $\limsup\limits_{n\to\infty} d(\omega_n(x),\omega_n(y))>0$ but $\liminf\limits_{n\to\infty} d(\omega_n(x),\omega_n(y))=0$. A system $(X,\mathbb{F})$ is said to be *Li-Yorke chaotic* if it contains an uncountable scrambled set. The system $(X,\mathbb{F})$ is equicontinuous if for each $\varepsilon>0$, there exists $\delta>0$ such that $d(x,y)<\delta$ implies $d(\omega_n(x),\omega_n(y))<\varepsilon$ for all $n\in\mathbb{N}$, $x,y\in X$. The system $(X,\mathbb{F})$ is said to be *minimal* if orbit of each x in X is dense in X. A pair $(x,y)\in X\times X$ is *proximal* for $(X,\mathbb{F})$ if $\liminf\limits_{n\to\infty} d(\omega_n(x),\omega_n(y))=0$. A pair $(x,y)\in X\times X$ is *distal* if it is not proximal. For any $x\in X$, the set $Prox_{\mathbb{F}}(x)=\{y\in X : (x,y) \text{ is proximal for } (X,\mathbb{F})\}$ is called the *proximal cell* of x in $(X,\mathbb{F})$. A system $(X,\mathbb{F})$ is said to exhibit dense set of proximal pairs if the set of pairs proximal for $(X,\mathbb{F})$ is dense in $X\times X$. A set S is said to be δ-*scrambled* in $(X,\mathbb{F})$ if for any distinct $x,y\in S$, $\limsup\limits_{n\to\infty} d(\omega_n(x),\omega_n(y))>\delta$ but $\liminf\limits_{n\to\infty} d(\omega_n(x),\omega_n(y))=0$. A system $(X,\mathbb{F})$ is *Li-Yorke sensitive* if there exists $\delta>0$ such that for each $x\in X$ and each neighbourhood U of x there exists $y\in U$ such that $\{x,y\}$ is a δ-scrambled set. For any $x\in X$, let $LY_{\mathbb{F}}(x)=\{y\in X : (x,y) \text{ is a Li-Yorke pair for } (X,\mathbb{F})\}$ is called the *Li-Yorke cell* of x.

We now define the notion of *topological entropy* for a non-autonomous system $(X,\mathbb{F})$.

Let X be a compact space and let $\mathscr{U}$ be an open cover of X. Then $\mathscr{U}$ has a finite subcover. Let

$\mathscr{L}$ be the collection of all finite subcovers and let $\mathscr{U}^*$ be the subcover with minimum cardinality, say $N_{\mathscr{U}}$. Define $H(\mathscr{U}) = logN_{\mathscr{U}}$. Then $H(\mathscr{U})$ is defined as the *entropy* associated with the open cover $\mathscr{U}$. If $\mathscr{U}$ and $\mathscr{V}$ are two open covers of X, define, $\mathscr{U} \vee \mathscr{V} = \{U \bigcap V : U \in \mathscr{U}, V \in \mathscr{V}\}$. An open cover β is said to be refinement of open cover α i.e. $\alpha \prec \beta$, if every open set in β is contained in some open set in α. It can be seen that if $\alpha \prec \beta$ then $H(\alpha) \leq H(\beta)$. For a self map f on X, $f^{-1}(\mathscr{U}) = \{f^{-1}(U) : U \in \mathscr{U}\}$ is also an open cover of X. Define,

$$h_{\mathbb{F},\mathscr{U}} = \limsup_{k\to\infty} \frac{H(\mathscr{U} \vee \omega_1^{-1}(\mathscr{U}) \vee \omega_2^{-1}(\mathscr{U}) \vee \ldots \vee \omega_{k-1}^{-1}(\mathscr{U}))}{k}$$

Then $\sup h_{\mathbb{F},\mathscr{U}}$, where $\mathscr{U}$ runs over all possible open covers of X is known as the *topological entropy of the system* $(X,\mathbb{F})$ and is denoted by $h(\mathbb{F})$. In case the f_n's coincide, the above definitions coincide with the known notions of an autonomous dynamical system. See [Block and Coppel, 1992; Brin and Stuck, 2002; Devaney, 1986; Elaydi, 2007] for details.

Let X be a compact metric space and let $\mathscr{K}(X)$ denote the collection of all non-empty compact subsets of X. For any $A,B \in \mathscr{K}(X)$ define $D_H(A,B) = \inf\{\varepsilon > 0 : A \subset S(B,\varepsilon) \text{ and } B \subset S(A,\varepsilon)\}$ where $S(A,\varepsilon) = \bigcup_{x\in A} S(x,\varepsilon)$ is the ε-ball around A. Then D_H defines a metric on $\mathscr{K}(X)$ and is known as the Hausdorff metric. It is known that a system (X, f) is a weakly mixing (topological mixing) if and only if for any compact set K with non-empty interior $\limsup_{n\to\infty} f^n(K) = X$ ($\lim_{n\to\infty} f^n(K) = X$) with respect to the metric D_H.

Let (X,d) be a compact metric space and let $C(X)$ denote the collection of continuous self maps on X. For any $f,g \in C(X)$, define,

$$D(f,g) = \sup_{x\in X} d(f(x),g(x))$$

It can be easily seen that D defined above is a metric on $C(X)$ and is known as the *Supremum metric*. It can be seen that a sequence (f_n) in $C(X)$ converges to f in $(C(X),D)$ if and only if (f_n) converges to f uniformly on X and hence the topology generated by the metric defined above is known as the topology of uniform convergence. A collection of sequences $\{(f_j^i) : i \in \mathfrak{I}\}$ converges collectively to $\{g^i : i \in \mathfrak{I}\}$ with respect to the metric D if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $D(f_j^i, g^i) < \varepsilon \;\; \forall j \geq n_0, \;\; i \in \mathfrak{I}$.

The above definitions generalize the known concepts for the autonomous system to a more general non-autonomous setting. Some investigations for such a setting in the discrete case have been made and interesting results have been obtained. In [Kolyada and Snoha, 1996], authors investigate the topological entropy of a general non-autonomous dynamical system generated by a equicontinuous family of continuous self maps on compact topological space. They also investigate topological entropy when the non-autonomous system generated by a finite family. In [Kolyada *et al.*, 2004], authors discuss minimality conditions for a non-autonomous system on a compact Hausdorff space, while focussing on the case when the non-autonomous system is defined on a compact interval of the real line. In this work, authors derive conditions ensuring non-minimality for a non-autonomous system. In [Dvořáková, 2012], the author proved that if a sequence (f_n) of surjective continuous self maps on interval converges uniformly, in general there is no relation between chaotic behaviour of the non-autonomous system generated by sequence (f_n) and the chaotic behaviour of limit map. Moreover, it is shown that even the full Lebesgue measure of a distributionally scrambled set of the non-autonomous system does not guarantee the existence of distributional chaos of the limit map. Conversely, the author proves the existence of a non-autonomous system with arbitrarily small distributionally scrambled set which converges to a map distributionally chaotic almost everywhere. In [Balibrea and Oprocha, 2012] authors investigate properties like weakly mixing, topological mixing, topological entropy and Li-Yorke chaos for the non-autonomous system. In particular, they prove that positive topological entropy does not imply the chaos in sense of Li and Yorke. They also give a few techniques to study the qualitative behaviour of a non-autonomous system.

Before moving further we give some of the terminologies used in literature. In recent studies, while $(X, f_{1,\infty})$, is used to denote the non-autonomous discrete dynamical systems $(X, \mathbb{F})$, $\omega_{i+(n-1)}^{i-1}$ is denoted by f_i^n. Also $f_{n,\infty}$ is used to denote the family $\mathbb{F}_{n-1} = \{f_n, f_{n+1} \ldots\}$ in literature. We now give some of the known results for the non-autonomous dynamical systems.

Proposition 1.0.1 *[Kolyada and Snoha, 1996] Let $f_{1,\infty}$ be a sequence of continuous selfmaps of a compact topological space X. Let $f_{1,\infty}$ be periodic with period n then $h(f_{1,\infty}^n) = n.h(f_{1,\infty})$.*

Proposition 1.0.2 *[Kolyada and Snoha, 1996] If $f_{1,\infty}$ is a sequence of equicontinuous selfmaps of a compact metric space (X, ρ) then $h(f_{1,\infty}^n) = n.h(f_{1,\infty})$ for all $n \geq 1$.*

Proposition 1.0.3 *[Kolyada* et al.*, 2004] Let (X, ρ) be a compact metric space and let $(X, f_{1,\infty})$ be a non-autonomous dynamical system. Then the following assertion are equivalent*
1. *$(X, f_{1,\infty})$ is not minimal*
2. *there is a non-empty open set $B \subset X$ such that $(X, f_{1,\infty})$ has arbitrarily long finite trajectories disjoint with B.*

Proposition 1.0.4 *[Dvořáková, 2012] There is a surjective non-autonomous system $(I, f_{1,\infty})$ such that for every $n \in \mathbb{N}$, $(I, f_{n,\infty})$ is distributionally chaotic almost everywhere(the scrambled set is whole $(0,1)$) and such that $(I, f_{1,\infty})$ uniformly converges to a non-chaotic map $f \in C(I)$.*

Proposition 1.0.5 *[Balibrea and Oprocha, 2012] There is NDS $([0,1], f_{1,\infty})$ such that*
1. *$h_{top}(f_{1,\infty}) \geq \log 2$*
2. *points $0, 1$ are fixed points and all others are asymptotic to 0.*

A brief summary of work done in subsequent chapters of the thesis is included below.

In chapter 2, we investigate the topological dynamics of non-autonomous generated by finite family. In the process, we compare the dynamics of the non-autonomous system $(X, \mathbb{F})$ $(\mathbb{F} = \{f_1, f_2, \cdots f_k\})$ with the dynamics of autonomous system $(X, f_k \circ \cdots f_2 \circ f_1)$. We prove that, while topological transitivity is not equivalent for two systems, weakly mixing is equivalent for two systems for the commutative family $\mathbb{F}$. We also establish the equivalence of topological mixing for two systems. We also derive necessary and sufficient conditions for a system to exhibit strong forms of mixing.

In chapter 3, we study the metric related dynamical properties of the non-autonomous dynamical system generated by finite family $\mathbb{F} = \{f_1, f_2, \cdots f_k\}$. In the process, we relate the dynamical behaviour of non-autonomous system $(X, \mathbb{F})$ with the dynamics of autonomous system $(X, f_k \circ \cdots f_2 \circ f_1)$. We prove that minimality is equivalent for two systems, when the space X is connected. We also derive conditions under which properties like equicontinuity, proximality and various forms of sensitivities are equivalent for two systems.

In chapter 4, we investigate the non-autonomous dynamical system generated by a uniformly convergent sequence of continuous self maps on a compact metric space. We relate the dynamical properties of such a non-autonomous dynamical system with the dynamics of its limiting system. We prove that the collective convergence of $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$ is sufficient to establish the equivalence of properties like minimality and equicontinuity for the system $(X, \mathbb{F})$ and limiting system (X, f). However, feeble openness is essential along with collective convergence to prove the equivalence of various notion of mixing and sensitivities for two systems. We derive conditions under which uniform convergence of (f_n) ensures the collective convergence of $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$. We prove that if f is an isometry, convergence (of (f_n)) at a sufficiently fast rate ensures the collective convergence of $\{(\omega_{n+k}^n) : k \in \mathbb{N}\}$. We show that if (f_n) commutes with f and (f_n) converges to f at a "sufficiently fast rate", many of the dynamical properties for the systems $(X, \mathbb{F})$ and (X, f) coincide. In particular, we prove that the proximal pairs (cells) are equivalent for two systems (under stated conditions).

In chapter 5, we relate the dynamical behaviour of the non-autonomous dynamical system with the dynamics of its generating functions. In the process, we show that dynamics of the generating functions, in general, is not carried over to the non-autonomous dynamical system $(X, \mathbb{F})$ (and vice-versa). Further, we discuss the dynamical behaviour of various possible rearrangements of a non-autonomous system. We prove that if the non-autonomous system is generated by feeble open maps then, any finite rearrangement of the system preserves the dynamics of the original system. We also prove that the dynamics need not be preserved under infinite rearrangements. We extend our investigations to properties like equicontinuity, minimality and proximality for the two systems.

…