2

Topological Dynamics of a Finite family

In this chapter, we investigate topological dynamics of a general non-autonomous system generated by finite family. We relate the dynamics of the non-autonomous system generated by $\mathbb{F} = \{f_1, f_2, \cdots f_k\}$ with the dynamics of the autonomous system $(X, f_k \circ \ldots f_2 \circ f_1)$. In particular, we compare properties like topological transitivity, weakly mixing, topological mixing and topological entropy for two systems. We show that, while topological transitivity is not equivalent for two systems, $(X,\mathbb{F})$ is topological mixing if and only if $(X, f_k \circ \ldots f_2 \circ f_1)$ is topological mixing. We prove that, if the generating family $\mathbb{F}$ is commutative, weakly mixing is equivalent for two systems. We also derive necessary and sufficient conditions for a non-autonomous dynamical system to exhibit stronger forms of mixing. Before we move forward, it is worth mentioning that when the dynamical system generated by a finite family $\mathbb{F} = \{f_1, f_2, \ldots, f_k\}$, the non-autonomous system is generated by the relation $x_n = g_n(x_{n-1})$ where $g_n = f_{(1+(n-1) \mod k)}$ and $n \in \mathbb{N}$.

2.1 PERIODIC POINTS AND TRANSITIVITY

Proposition 2.1.1 *For any point $x_0 \in X$, x_0 is periodic for $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1) \Leftrightarrow x_0$ is periodic for $(X,\mathbb{F})$.*

Proof. Let x_0 be a periodic point for the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$. As x_0 is periodic, there exist $n_0 \in \mathbb{N}$ such that $(f_k \circ f_{k-1} \circ \ldots \circ f_1)^{n_0}(x_0) = x_0$ and hence $\omega_{n_0 k}(x_0) = x_0$. Consequently $\omega_{rn_0 k}(x_0) = x_0 \ \ \forall r \geq 1$ and hence x_0 is periodic for $(X,\mathbb{F})$.

Conversely, let x_0 be a periodic point for the system $(X,\mathbb{F})$ of period $n \in \mathbb{N}$. Hence, $\omega_{rn}(x_0) = x_0$ for all $r \in \mathbb{N}$. In particular, we have $\omega_{nk}(x_0) = x_0$ or $(f_k \circ f_{k-1} \circ \ldots \circ f_1)^n(x_0) = x_0$ and, hence x_0 is periodic point for the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$. □

Remark 2.1.1 The above result establishes that any point x_0 is periodic for the system $(X,\mathbb{F})$ if and only if it is periodic for the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$. However, the proof establishes only the periodic behaviour of the point x_0 and the period of the point may not be preserved. Further, as periodic points are preserved between the two systems $(X,\mathbb{F})$ and $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$, $(X,\mathbb{F})$ has dense set of periodic points if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ has dense set of periodic points. Hence we get the following corollary.

Corollary 2.1.1 *The system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ exhibits dense set of periodic points if and only if $(X,\mathbb{F})$ exhibits dense set of periodic points.*

Proof. The proof follows from the discussions in Remark 2.1.1. □

Proposition 2.1.2 *If $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ is transitive, then $(X,\mathbb{F})$ is transitive.*

Proof. Let U, V be any pair of non-empty open subsets of X. As $f_k \circ f_{k-1} \circ \ldots \circ f_1$ is transitive, there exists $n \in \mathbb{N}$ such that $(f_k \circ f_{k-1} \circ \ldots \circ f_1)^n(U) \cap V \neq \phi$. Consequently $\omega_{nk}(U) \cap V \neq \phi$ and hence $(X,\mathbb{F})$ is transitive. □

Remark 2.1.2 The above result establishes the transitivity of the non-autonomous system, in case the corresponding autonomous system is transitive. However, the correspondence is one-sided and the converse of the above result does not hold good. We now give an example in support of our statement.

Example 2.1.1 *Let I be the unit interval and let f_1, f_2 be defined as*

$$f_1(x) = \begin{cases} 2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}] \\ -4x + 3 & \text{for } x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

Let $\mathbb{F} = \{f_1, f_2\}$ and $(X, \mathbb{F})$ be the corresponding non-autonomous dynamical system. As $(X, f_2 \circ f_1)$ has an invariant set $U = [\frac{1}{2}, 1]$, $f_2 \circ f_1$ is not transitive. However, as f_1 expands every open set U in $[0,1]$ and f_2 expands the right half of the unit interval with $f_2([0, \frac{1}{2}]) = [\frac{1}{2}, 1]$, the non-autonomous system generated by $\mathbb{F}$ is transitive.

It is known that transitivity of $\mathbb{F} \times \mathbb{F}$ does not imply the transitivity of $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$ [Balibrea and Oprocha, 2012]. However, if the non-autonomous system is generated by commutative family $\mathbb{F}$ then the transitivity of $\mathbb{F} \times \mathbb{F}$ implies the transitivity of $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{n \ \ times}$ for any ≥ 2 hence a result analogous to the autonomous case, holds good for the non-autonomous system. We now establish our claim below.

Proposition 2.1.3 *If $\mathbb{F}$ is a commutative family, then, $\mathbb{F} \times \mathbb{F}$ is transitive if and only if $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{n \ \ times}$ is transitive $\forall n \geq 2$.*

Proof. Let $\mathbb{F} \times \mathbb{F}$ be transitive. We prove the forward part with the help of mathematical induction. Let $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{k \ \ times}$ be transitive and let $U_1, U_2, \ldots, U_{k+1}$ and $V_1, V_2, \ldots, V_{k+1}$ be a pair of $k+1$ non-empty open sets in X. As $\mathbb{F} \times \mathbb{F}$ is transitive, there exists $r > 0$ such that $\omega_r(U_k) \cap U_{k+1} \neq \phi$ and $\omega_r(V_k) \cap V_{k+1} \neq \phi$. Let $U = U_k \cap \omega_r^{-1}(U_{k+1})$ and $V = V_k \cap \omega_r^{-1}(V_{k+1})$. Then U and V are non-empty open sets in X. Also as $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{k \ \ times}$ is transitive, there exists $t > 0$ such that $\omega_t(U_i) \cap V_i \neq \phi$ for $i = 1, 2, \ldots, k-1$ and $\omega_t(U) \cap V \neq \phi$.

As $U \subset U_k$ and $V \subset V_k$, we have $\omega_t(U_k) \cap V_k \neq \phi$. Also $\omega_t(U) \cap V \neq \phi$ implies $\omega_r(\omega_t(U)) \cap \omega_r(V) \neq \phi$. As f_i commute with each other, we have $\omega_t(\omega_r(U)) \cap \omega_r(V) \neq \phi$. As $\omega_r(U) \subseteq U_{k+1}$ and $\omega_r(V) \subset V_{k+1}$, we have $\omega_t(U_{k+1}) \cap V_{k+1} \neq \phi$. Consequently $\omega_t(U_i) \cap V_i \neq \phi$ for $i = 1, 2, \ldots, k+1$ and hence $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{k+1 \ \ times}$ is transitive.

Proof of converse is trivial as if $\underbrace{\mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}}_{n \ \ times}$ is transitive $\forall n \geq 2$, in particular taking $n = 2$ yields $\mathbb{F} \times \mathbb{F}$ is transitive. □

Remark 2.1.3 For autonomous systems, it is known that $f \times f$ is transitive, then $\underbrace{f \times f \times \ldots \times f}_{n \ \ times}$ is transitive for all $n \geq 2$ [Banks, 2005] and hence the result established above is an analogous extension of the autonomous case. It may be noted that the proof uses the commutative property of the members of the family $\mathbb{F}$ and hence is not true for a non-autonomous system generated by any general family $\mathbb{F}$. However, the proof does not use the finiteness of the family $\mathbb{F}$ and hence the result holds even when the generating family $\mathbb{F}$ is infinite.

2.2 STRONGER NOTIONS OF MIXING

Proposition 2.2.1 *If $\mathbb{F}$ is a commutative family, then $(X,\mathbb{F})$ is weakly mixing if and only if for any finite collection of non-empty open sets $\{U_1,U_2,\ldots,U_m\}$, there exists a subsequence (r_n) of positive integers such that $\lim\limits_{n\to\infty} \omega_{r_n}(U_i) = X, \quad \forall i = 1,2,\ldots,m$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and let $\{U_1,U_2,\ldots,U_m\}$ be any finite collection of non-empty open sets of X. As X is compact, there exist $x_1,x_2,\ldots x_{k_n}$ such that $X = \bigcup\limits_{i=1}^{k_n} S(x_i,\frac{1}{2n})$. As $(X,\mathbb{F})$ is weakly mixing, by proposition 2.1.3, there exists $r_n > 0$ such that $\omega_{r_n}(U_i) \cap S(x_j,\frac{1}{2n}) \neq \phi \quad \forall i,j$ and hence for any i, $D_H(\omega_{r_n}(U_i),X) \leq \frac{1}{n}$. As $n \in \mathbb{N}$ is arbitrary, $\lim\limits_{n\to\infty} \omega_{r_n}(U_i) = X \ \ \forall i$ and the proof for the forward part is complete.

Conversely, let U_1,U_2 and V_1,V_2 be two pairs of non-empty open subsets of X. For $i = 1,2$, let $v_i \in V_i$ and let $\varepsilon > 0$ such that $S(v_i,\varepsilon) \subset V_i$. By given condition, there exists a subsequence (r_n) of natural numbers such that $\lim\limits_{n\to\infty} \omega_{r_n}(U_i) = X$ for $i = 1,2$. Thus, there exists r_k such that $D_H(\omega_{r_k}(U_i),X) < \frac{\varepsilon}{2}, \quad i = 1,2$. Consequently $\omega_{r_k}(U_i) \cap V_i \neq \phi$ and, hence, $(X,\mathbb{F})$ is weakly mixing. □

Remark 2.2.1 It may be noted that the proof of converse does not need commutativity of the family $\mathbb{F}$. However, to establish the forward part, we use proposition 2.1.3 and hence use the commutativity of the family $\mathbb{F}$. Thus, the result may not hold good when considered for a general non-autonomous system. Also, the result does not use finiteness condition on $\mathbb{F}$ and hence is valid even when the system is generated by an infinite family $\mathbb{F}$.

Remark 2.2.2 It is known that an autonomous system is weakly mixing if and only if for any non-empty open set U, there exists a subsequence (r_n) of positive integers such that $\lim\limits_{n\to\infty} f^{r_n}(U) = X$ [Kwietniak and Oprocha, 2007]. Thus for non-autonomous case, the result above establishes a stronger extension of the result proved in the autonomous case. However, the above result also holds when the maps f_n coincide and, hence, a stronger version of the result in [Kwietniak and Oprocha, 2007] is true for the autonomous case. For the sake of completeness, we mention the obtained result below.

Corollary 2.2.1 *A continuous self map f is weakly mixing if and only if for any finite collection of non-empty open sets $\{U_1,U_2,\ldots,U_m\}$, there exists a subsequence (r_n) of positive integers such that $\lim\limits_{n\to\infty} f^{r_n}(U_i) = X, \quad \forall i = 1,2,\ldots,m$.*

Proof. The proof is a direct consequence of proposition 2.2.1, applied to the case when $\mathbb{F} = \{f,f,\cdots f\}$. □

Proposition 2.2.2 *$(X,\mathbb{F})$ is topologically mixing if and only if for each non-empty open set U, $\lim\limits_{n\to\infty} \omega_n(U) = X$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and let U be any non-empty open subset of X. As X is compact, there exist $x_1,x_2,\ldots x_{k_n}$ such that $X = \bigcup\limits_{i=1}^{k_n} S(x_i,\frac{1}{2n})$. As $\mathbb{F}$ is topologically mixing, there exists $M_i, \quad i =$

$1,2,\ldots,k_n$ such that $\omega_k(U) \cap S(x_i, \frac{1}{2n}) \neq \phi \quad \forall k \geq M_i$. Let $M = \max\{M_i : 1 \leq i \leq k_n\}$. Then $\omega_k(U) \cap S(x_i, \frac{1}{2n}) \neq \phi \quad \forall k \geq M$. Consequently $D_H(\omega_k(U), X) < \frac{1}{n} \quad \forall k \geq M$. As $n \in \mathbb{N}$ is arbitrary, $\lim_{n\to\infty} \omega_n(U) = X$ and the proof of forward part is complete.

Conversely, let U, V be a any pair of non-empty open subsets of X. Let $v \in V$ and let $\varepsilon > 0$ be such that $S(v, \varepsilon) \subset V$. By given condition, $\lim_{n\to\infty} \omega_n(U) = X$. Thus, there exists $K > 0$ such that $D_H(\omega_k(U), X) < \frac{\varepsilon}{2} \quad \forall k \geq K$. Consequently, $\omega_k(U) \cap V \neq \phi \quad \forall k \geq K$ and hence $(X, \mathbb{F})$ is topologically mixing. □

Remark 2.2.3 In [Kwietniak and Oprocha, 2007], the authors establish that an autonomous system (X, f) is topologically mixing if and only if for each non-empty open set U, $\lim_{n\to\infty} f^n(U) = X$. Once again, we prove that an analogous result does hold when considered for a general non-autonomous system. However, it may be noted that commutativity or finiteness of the family $\mathbb{F}$ were not needed to establish the above result and, hence, the result holds for a general non-autonomous dynamical system.

Proposition 2.2.3 *If $\mathbb{F} = \{f_1, f_2, \ldots, f_k\}$ is a finite commutative family, then, $(X, \mathbb{F})$ is weakly mixing if and only if $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ is weakly mixing.*

Proof. Let U be a non-empty open subset of X. We will equivalently prove that there exists a sequence (z_n) of natural numbers such that $\lim_{n\to\infty} (f_k \circ f_{k-1} \circ \ldots \circ f_1)^{z_n}(U) = X$. As $(X, \mathbb{F})$ is weakly mixing, by proposition 2.2.1, there exists sequence (s_n) such that $\lim_{n\to\infty} \omega_{s_n}(U) = X$. Also the family $\mathbb{F}$ is finite and hence there exists $l \in \{1, 2, \ldots, k\}$ and a subsequence (m_n) of (s_n), $m_n = l + r_n k$ such that $\lim_{n\to\infty} f_l \circ f_{l-1} \circ \ldots \circ f_1 \circ \omega_{r_n k}(U) = X$. As each f_i are surjective, $\lim_{n\to\infty} \omega_{(r_n+1)k}(U) = X$. Consequently $\lim_{n\to\infty} (f_k \circ f_{k-1} \circ \ldots \circ f_1)^{r_n+1}(U) = X$ and $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ is weakly mixing.

Conversely, let U_1, U_2 and V_1, V_2 be any two pairs of non-empty open subsets of X. As $f_k \circ f_{k-1} \circ \ldots \circ f_1$ is weakly mixing, there exists $n \in \mathbb{N}$ such that $(f_k \circ f_{k-1} \circ \ldots \circ f_1)^n(U_i) \cap V_i \neq \phi$ for $i = 1, 2$. Consequently, $\omega_{nk}(U_i) \cap V_i \neq \phi$ for $i = 1, 2$ and hence $(X, \mathbb{F})$ is weakly mixing. □

Remark 2.2.4 The result establishes the equivalence of the weakly mixing of the non-autonomous system $(X, \mathbb{F})$ and the autonomous system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$. It may be noted that as the proof uses the proposition 2.2.1 proved earlier, commutativity of the family $\mathbb{F}$ cannot be relaxed. Further, it may be noted that the above result uses the surjectivity of the maps f_i. Thus, if the maps are not surjective, the above result does not hold, i.e., the non-autonomous system may exhibit weakly mixing even if the system $(X, f_k \circ f_{k-1} \circ \ldots \circ f_1)$ is not weakly mixing. We now give an example in support of our statement.

Example 2.2.1 *Let I be the unit interval and let f_1, f_2 be defined as*

$$f_1(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ -x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} -2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ 2x - \frac{1}{2} & \text{for } x \in [\frac{1}{4}, \frac{1}{2}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

Let $\mathbb{F}$ be a finite family of maps f_1 and f_2 defined above. As $[0, \frac{1}{2}]$ is invariant for $f_2 \circ f_1$, the map $f_2 \circ f_1$ does not exhibit any of the mixing properties. However, for any open set U in $[0, 1]$, there exists $k \in \mathbb{N}$

such that $(f_2\circ f_1)^k(U)=[0,\frac{1}{2}]$. Consequently, $\omega_{2k+1}(U)=[0,1]$. As the argument holds for any odd integer greater than k, the non-autonomous system is weakly mixing.

Proposition 2.2.4 *If $\mathbb{F}=\{f_1,f_2,\dots,f_k\}$ is a finite family, then, $(X,\mathbb{F})$ is topologically mixing if and only if $(X,f_k\circ f_{k-1}\circ\ldots\circ f_1)$ is topologically mixing.*

Proof. Let U be a non-empty open subset of X. We will equivalently prove that $\lim_{n\to\infty}(f_k\circ f_{k-1}\circ\ldots\circ f_1)^n(U)=X$. As $(X,\mathbb{F})$ is topologically mixing, by proposition 2.2.2, $\lim_{n\to\infty}\omega_n(U)=X$. In particular $\lim_{n\to\infty}\omega_{kn}(U)=X$ or $\lim_{n\to\infty}(f_k\circ f_{k-1}\circ\ldots\circ f_1)^n(U)=X$ and hence $(X,f_k\circ f_{k-1}\circ\ldots\circ f_1)$ is topologically mixing.

Conversely, let U be a non-empty open subset of X. We will equivalently prove that $\lim_{n\to\infty}\omega_n(U)=X$. As $f_k\circ f_{k-1}\circ\ldots\circ f_1$ is topologically mixing, $\lim_{n\to\infty}(f_k\circ f_{k-1}\circ\ldots\circ f_1)^n(U)=X$. Consequently, $\lim_{n\to\infty}\omega_{kn}(U)=X$. As each f_i are surjective, by continuity we have for each $l\in\{1,2,\dots,k\}$, $f_l\circ f_{l-1}\circ\ldots\circ f_1(\lim_{n\to\infty}\omega_{kn}(U))=\lim_{n\to\infty}(f_l\circ f_{l-1}\circ\ldots\circ f_1\circ\omega_{kn}(U))=X$. Consequently $\lim_{n\to\infty}\omega_n(U)=X$ and $(X,\mathbb{F})$ is topologically mixing. □

Remark 2.2.5 The result once again is an analogous extension of the autonomous case. The result proves that the identical conclusion can be made for the non-autonomous case without strengthening the hypothesis. It is worth noting that the result does not use commutativity of $\mathbb{F}$ and hence asserts the complex nature of topological mixing in a general dynamical system.

2.3 TOPOLOGICAL ENTROPY

In [Kolyada and Snoha, 1996], authors prove that if $\mathbb{F}=\{f_1,f_2,\dots,f_k\}$ is a finite family, then, $h(\mathbb{F})=\frac{1}{k}h(f_k\circ f_{k-1}\circ\ldots\circ f_1)$. However, as the authors were not aware of the result while addressing the problem, for the sake of completion, proof is included here.

Proposition 2.3.1 *If $\mathbb{F}=\{f_1,f_2,\dots,f_k\}$ is a finite family, then, $h(\mathbb{F})\geq\frac{1}{k}h(f_k\circ f_{k-1}\circ\ldots\circ f_1)$. Consequently if the associated autonomous system has positive topological entropy, the non-autonomous system also has a positive topological entropy.*

Proof. For any open cover $\mathscr{U}$ of X, the entropy of the system with respect to the open cover $\mathscr{U}$ is defined as

$$h_{\mathbb{F},\mathscr{U}}=\limsup_{n\to\infty}\frac{H(\mathscr{U}\vee\omega_1^{-1}(\mathscr{U})\vee\omega_2^{-1}(\mathscr{U})\vee\ldots\vee\omega_{n-1}^{-1}(\mathscr{U}))}{n}=\limsup_{n\to\infty}\frac{H(\mathscr{U}\vee\omega_1^{-1}(\mathscr{U})\vee\omega_2^{-1}(\mathscr{U})\vee\ldots\vee\omega_{nk-1}^{-1}(\mathscr{U}))}{nk}$$

Also as $\mathscr{U}\vee\omega_k^{-1}(\mathscr{U})\vee\omega_{2k}^{-1}(\mathscr{U})\vee\ldots\vee\omega_{k(n-1)}^{-1}(\mathscr{U})\prec\mathscr{U}\vee\omega_1^{-1}(\mathscr{U})\vee\omega_2^{-1}(\mathscr{U})\vee\ldots\vee\omega_{nk-1}^{-1}(\mathscr{U})$, we have

$$H(\mathscr{U}\vee\omega_k^{-1}(\mathscr{U})\vee\omega_{2k}^{-1}(\mathscr{U})\vee\ldots\vee\omega_{k(n-1)}^{-1}(\mathscr{U}))\leq H(\mathscr{U}\vee\omega_1^{-1}(\mathscr{U})\vee\omega_2^{-1}(\mathscr{U})\vee\ldots\vee\omega_{nk-1}^{-1}(\mathscr{U}))$$

Therefore,

$$\limsup_{n\to\infty}\frac{H(\mathscr{U}\vee\omega_k^{-1}(\mathscr{U})\vee\omega_{2k}^{-1}(\mathscr{U})\vee\ldots\vee\omega_{k(n-1)}^{-1}(\mathscr{U}))}{nk}\leq\limsup_{n\to\infty}\frac{H(\mathscr{U}\vee\omega_1^{-1}(\mathscr{U})\vee\omega_2^{-1}(\mathscr{U})\vee\ldots\vee\omega_{nk-1}^{-1}(\mathscr{U}))}{nk}$$

Consequently,

$$\frac{1}{k}\limsup_{n\to\infty}\frac{H(\mathscr{U}\vee(f_k\circ f_{k-1}\circ\ldots\circ f_1)^{-1}(\mathscr{U})\vee(f_k\circ f_{k-1}\circ\ldots\circ f_1)^{-2}(\mathscr{U})\vee\ldots\vee(f_k\circ f_{k-1}\circ\ldots\circ f_1)^{(-n+1)}(\mathscr{U}))}{n}$$

$$\leq \limsup_{n\to\infty} \frac{H(\mathscr{U} \vee \omega_1^{-1}(\mathscr{U}) \vee \omega_2^{-1}(\mathscr{U}) \vee \ldots \vee \omega_{nk-1}^{-1}(\mathscr{U}))}{nk}$$

or $\frac{1}{k}H(f_k \circ f_{k-1} \circ \ldots \circ f_1, \mathscr{U}) \leq H(\mathbb{F}, \mathscr{U})$. As $\mathscr{U}$ was arbitrary, $h(\mathbb{F}) \geq \frac{1}{k}h(f_k \circ f_{k-1} \circ \ldots \circ f_1)$ and the proof is complete. □

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