1
Introduction

1.1 Introduction

As the title of the thesis suggests, in this thesis, we aim to study the nonclassical and phase properties of a family of engineered quantum states. Before, we introduce such states and properties, it would be apt to lucidly introduce the notion of nonclassical and engineered quantum states. By nonclassical state we refer to a quantum state having no classical analogue. Such states are characterized by the negative values of Glauber-Sudarshan P-function or P-function more singular than Dirac delta function Sudarshan [1963]; Glauber [1963a] and witnessed by various operational criteria (to be described in Section 1.5.1). To visualize the relevance of nonclassical states we may first note that quantum supremacy refers to the ability of performing a task using quantum resources in such a manner that either the task itself cannot be performed using classical resources or the speed/efficiency achieved using quantum resources cannot be achieved in the classical world Grover [1997]. A recent experiment performed by Google aimed at estabilishing quantum supremacy has drawn much of public attention Courtland [2017]. The relevance of the present study lies in the fact that to establish quantum supremacy or to perform a fundamental test of quantum mechanics, we would require a state having some features that would not be present in any classical state. As we have already mentioned, such a state having no classical analogue is referred to as the nonclassical state. Frequently used examples of nonclassical states include squeezed, antibunched, entangled, steered, and Bell nonlocal states. The relevance of the states having nonclassical features has already been established in the various domains of physics. For example, we may mention, teleportation of coherent states Furusawa et al. [1998], continuous variable quantum cryptography Hillery [2000], quantum radar Lanzagorta [2011], and many more. Further, we may note that the art of generating and manipulating quantum states as per need is referred to as the "quantum state engineering" Dakna et al. [1998]; Sperling et al. [2014]; Vogel et al. [1993]; Miranowicz and Leonski [2004]; DellAnno et al. [2006]. Particularly interesting examples of such engineered quantum states are Fock state, photon added/subtracted coherent state Agarwal and Tara [1991], displaced Fock state (DFS) which is also referred to as generalized coherent state and displaced number state Satyanarayana [1985]; Wunsche [1991]; Ziesel et al. [2013]; Zavatta et al. [2004]; De Oliveira et al. [1990]; Malpani et al. [2019a], photon added DFS (PADFS) Malpani et al. [2019a], and photon subtracted DFS (PSDFS) Malpani et al. [2019a]. In what follows, we will state the relevance of such engineered quantum states in the implementation of different tasks exploiting their nonclassical and phase properties. The relatively new area of research on quantum state engineering has drawn much attention of the scientific community because of its success in experimentally producing various quantum states Zavatta et al. [2004]; Torres et al. [2003]; Rauschenbeutel et al. [2000]; Gao et al. [2010]; Lu et al. [2007] having nonclassical properties and applications in realizing quantum information processing tasks, like quantum key distribution Bennett and Brassard [1984] and quantum teleportation Brassard et al.

[1998]; Chen [2015]. Engineered quantum states, such as cat states, Fock state and superposition of Fock states, are known to play a crucial role in performing fundamental tests of quantum mechanics and in establishing quantum supremacy in the context of quantum computation and communication (Kues et al. [2017] and references therein).

As mentioned in the previous paragraph, with the advent of quantum state engineering Vogel et al. [1993]; Sperling et al. [2014]; Miranowicz and Leonski [2004]; Marchiolli and José [2004] and quantum information processing (Pathak [2013] and references therein), the study of nonclassical properties of engineered quantum states have become a very important field. This is so because only the presence of nonclassical features in a quantum state can provide quantum supremacy Grover [1997]. In the recent past, various techniques for quantum state engineering have been developed Vogel et al. [1993]; Sperling et al. [2014]; Miranowicz and Leonski [2004]; Agarwal and Tara [1991]; Lee and Nha [2010]; Marchiolli and José [2004]. If we restrict ourselves to optics, these techniques are primarily based on the clever use of beam splitters, detectors, and measurements with post selection, etc. Such techniques are useful in creating holes in the photon number distribution Escher et al. [2004] and in generating finite dimensional quantum states Miranowicz and Leonski [2004], both of which are nonclassical Pathak and Ghatak [2018]. The above said techniques are also useful in realizing non-Gaussianity inducing operations, like photon addition and subtraction Zavatta et al. [2004]; Podoshvedov [2014]. Motivated by the above, in this thesis, we aim to study the nonclassical properties of a set of engineered quantum states such as photon added, photon subtracted, and photon added then subtracted displaced Fock states which can be produced by using the above mentioned techniques. In the present thesis, we also wish to investigate the phase properties of the above mentioned engineered quantum states for the reasons explained below.

The impossibility of writing a Hermitian operator for quantum phase is a longstanding problem (see Peřinová et al. [1998]; Carruthers and Nieto [1968]; Lynch [1987] for review). Early efforts of Dirac Dirac [1927] to introduce a Hermitian quantum phase operator were not successful, but led to many interesting proposals Susskind and Glogower [1964]; Pegg and Barnett [1989]; Barnett and Pegg [1986]. Specifically, Susskind-Glogower Susskind and Glogower [1964], Pegg-Barnett Pegg and Barnett [1988, 1989]; Barnett and Pegg [1990], and Barnett-Pegg Barnett and Pegg [1986] formalisms played very important role in the studies of phase properties and the phase fluctuation Imry [1971]. Thereafter, phase properties of various quantum states have been reported using these formalisms Sanders et al. [1986]; Gerry [1987]; Yao [1987]; Carruthers and Nieto [1968]; Vaccaro and Pegg [1989]; Pathak and Mandal [2000]; Alam and Mandal [2016b]; Alam et al. [2017a]; Verma and Pathak [2009]. Other approaches have also been used for the study of the phase properties. For example, quantum phase distribution is defined using phase states Agarwal et al. [1992], while Wigner Garraway and Knight [1992] and Q Leonhardt and Paul [1993]; Leonhardt et al. [1995] phase distributions are obtained by integrating over radial parameter of the corresponding quasidistribution function. In experiments, the phase measurement is performed by averaging the field amplitudes of the Q function Noh et al. [1991, 1992]; Pegg-Barnett and Wigner phase distributions are also reported with the help of reconstructed density matrix Smithey et al. [1993]. Further, quantum phase distribution under the effect of the environment was also studied in the past leading to phase diffusion Banerjee and Srikanth [2007]; Banerjee et al. [2007]; Abdel-Aty et al. [2010]; Banerjee [2018]. A measure of phase fluctuation named phase dispersion using quantum phase distribution has also been proposed in the past Peřinová et al. [1998]; Banerjee and Srikanth [2007]. Recently, quantum phase fluctuation Zheng-Feng [1992] and Pancharatnam phase Mendas and Popovic [1993] have been studied for DFS. The quantum phase fluctuation in parametric down-conversion Gantsog et al. [1991] and its revival Gantsog [1992] are also reported. Experiments on phase super-resolution without using entanglement Resch et al. [2007] and role of photon subtraction in concentration of phase information Usuga et al. [2010] are also performed. Optimal phase estimation Sanders and Milburn [1995] using different quantum states Higgins et al. [2007] (including NOON and other entangled states and unentangled single-photon states) has long been the focus of quantum metrology Giovannetti et al. [2006, 2011]. Nonclassicality measure based on the shortening of the regular distribution defined on phase difference interval broadbands due to nonclassicality is also proposed in the recent past Peřina and Křepelka [2019]; Thapliyal and Perina [2019]. In brief, quantum phase properties are of intense interest of the community since long (see Pathak [2002]; Peřinová et al. [1998] and references therein), and the interest in it has been further enlightened in the recent past as many new applications of quantum phase distribution and quantum phase fluctuation have been realized.

To be specific, this work is also motivated by the fact that recently several applications of nonclassical states and quantum phase properties have been reported. Specifically, squeezed states have played an important role in the studies related to phase diffusion Banerjee and Srikanth [2007]; Banerjee et al. [2007], the detection of gravitational waves in LIGO experiments Abbasi and Golshan [2013]; Abbott et al. [2016b,a]. The rising demand for a single photon source can be fulfilled by an antibunched light source Yuan et al. [2002]. The study of quantum correlations is important both from the perspective of pure and mixed states Chakrabarty et al. [2011]; Dhar et al. [2013]; Banerjee et al. [2010a,b]. Entangled states are found to be useful in both secure Ekert [1991] and insecure Bennett and Wiesner [1992]; Bennett et al. [1993] quantum communication schemes. Stronger quantum correlation present in the steerable states are used to ensure the security against all the side-channel attacks on devices used in one-side (i.e., either preparation or detector side) for quantum cryptography Branciard et al. [2012]. Quantum supremacy in computation is established due to quantum algorithms for unsorted database search Grover [1997], factorization and discrete logarithm problems Shor [1999], and machine learning Biamonte et al. [2017] using essentially nonclassical states. We may further stress on the recently reported applications of quantum phase distribution and quantum phase fluctuation by noting that these have applications in quantum random number generation Xu et al. [2012]; Raffaelli et al. [2018], cryptanalysis of squeezed state based continuous variable quantum cryptography Horak [2004], generation of solitons in a Bose-Einstein condensate Denschlag et al. [2000], storage and retrieval of information from Rydberg atom Ahn et al. [2000], in phase encoding quantum cryptography Gisin et al. [2002], phase imaging of cells and tissues for biomedical application Park et al. [2018]; as well as have importance in determining the value of transition temperature for superconductors Emery and Kivelson [1995].

Now to achieve the above advantages of the nonclassical states, we need to produce these states via the schemes of quantum state engineering. For the same, there are some distinct theoretical tools, like quantum scissoring Miranowicz et al. [2001], hole-burning Escher et al. [2004]; Gerry and Benmoussa [2002]; Malpani et al. [2020a] or filtering out a particular Fock state from the photon number distribution Meher and Sivakumar [2018], applying non-Gaussianity inducing operations Agarwal [2013]. However, these distinct mechanisms are experimentally realized primarily by appropriately using beam splitters, mirrors, and single photon detectors or single photon counting module. Without going into finer details of the optical realization of quantum state engineering tools, we may note that these tools can be used to generate various nonclassical states, e.g., DFS De Oliveira et al. [1990], PADFS Malpani et al. [2019a], PSDFS Malpani et al. [2019a], photon added squeezed coherent state Thapliyal et al. [2017b], photon subtracted squeezed coherent state Thapliyal et al. [2017b], number state filtered coherent state Meher and Sivakumar [2018]. Some of these states, like photon added coherent state, have already been realized experimentally Zavatta et al. [2004].

Many of the above mentioned engineered quantum states have already been studied in detail. Primarily, three types of investigations have been performed on the engineered quantum states- (i) study of various nonclassical features of these states (and their variation with the state parameters) as reflected through different witnesses of nonclassicality. Initially, such studies were restricted to the lower-order nonclassical features. In the recent past, various higher-order nonclassical features have been predicted theoretically Alam et al. [2018a,b]; Pathak and Garcia [2006]; Pathak and Verma [2010]; Verma et al. [2008]; Thapliyal et al. [2017b] and confirmed experimentally (Hamar et al. [2014]; Peřina Jr et al. [2017] and references therein) in quantum states generated in nonlinear optical processes. (ii) Phase properties of the nonclassical states have been studied El-Orany et al. [2000] by computing quantum phase fluctuations, phase dispersion, phase distribution functions, etc., under various formalisms, like Susskind and Glogower Susskind and Glogower [1964], Pegg-Barnett Pegg and Barnett [1989] and Barnett-Pegg Barnett and Pegg [1986] formalisms. (iii) Various applications of the engineered quantum states have been designed. Some of them have already been mentioned.

Motivated by the above observations, in this thesis, we would like to perform an investigation on nonclassical and phase properties of a particularly interesting set of engineered quantum states which would have the flavor of the first two facets of the studies mentioned above. Applications of the engineered quantum states will also be discussed briefly, but will not be investigated in detail. To begin with we would like to briefly describe the physical and mathematical concepts used in this thesis, and that will be the focus of the rest of this chapter.

The rest of this chapter is organized as follows. In Section 1.2, we will briefly discuss quantum theory of radiation and introduce annihilation, creation, and number operators as well as Fock and coherent states. In Section 1.3 this will be followed by an introduction to a set of quantum states which will be studied in this thesis. After this, the notion of nonclassicality will be introduced mathematically in Section 1.4, and a set of operational criteria for observing nonclassical properties will be introduced in Section 1.5. Subsequently, the parameters used for the study of quantum phase properties will be introduced in Section 1.6. These witnesses of nonclassicality and the parameters for the study of phase properties will be used in the subsequent chapters, to investigate the nonclassical and phase properties of the quantum states discussed in Section 1.3. Finally, the structure of the rest of the thesis will be provided in Section 1.7.

1.2 Quantum theory of radiation field

Historically, quantum physics started with the ideas related to quanta of radiation. To be precise, Planck's work on black body radiation Planck [1901] and Einstein's explanation of photoelectric effect Einstein [1905] involved a notion of quantized radiation field. In Planck's work, light was considered to be emitted from and absorbed by a black body in quanta; and in Einstein's work, it was also considered that the radiation field propagates from one point to another as quanta. These initial works contributed a lot in the development of quantum mechanics, but after the introduction of quantum mechanics, in 1920s, in the initial days, most of the attention was given to the quantization of matter. A quantum theory of radiation was introduced by Dirac in 1927 Dirac [1927]. In what follows, we will describe it briefly as this would form the backbone of the present thesis.

1.2.1 Creation and annihilation operator

Maxwell gave a classical description of electromagnetic field. But here the objective is to study light and its properties apart from Maxwell's equations. So, to begin with, it would be reasonable to write Maxwell's equations in free space:

$$\nabla . E = 0, \tag{1.1}$$

$$\nabla . B = 0, \tag{1.2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{1.3}$$

and

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t}, \tag{1.4}$$

where $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the speed of light without any medium (i.e. vacuum). Using the set of above equations one can express magnetic and electric fields in the form of the solution of wave equations, like

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0. \tag{1.5}$$

The quantization of radiation field can be done by assuming a cavity of length L having a linearly polarized electric field whose direction of propagation is in z direction. Because of the linearity of the wave equation (1.5), we are allowed to write the electric field in the form of linear combination of all the normal modes as

$$E_x(z,t) = \sum_n A_n q_n(t) \sin(k_n z), \tag{1.6}$$

where q_n being the amplitude of the nth normal mode with $k_n = \frac{n\pi}{L}$,V is the volume of the resonator, $A_n = \frac{2m_n \nu_n^2}{\varepsilon_0 V}$ with $\nu_n = ck_n$, and m_n is a constant (in the units of mass). With the help of this, we try to form an analogy of radiation with mechanical oscillator. In analogy to Eq. (1.6) we are able to write the corresponding magnetic field equation in cavity as

$$B_y(z,t) = \sum_n A_n \left(\frac{\dot{q}_n}{c^2 k_n}\right) \cos(k_n z). \tag{1.7}$$

So, the total energy of the field can be written as a classical Hamiltonian

$$H = \frac{1}{2}\int_V d\tau \left(\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2\right), \tag{1.8}$$

$$H = \frac{1}{2}\sum_n \left(m_n \nu_n^2 q_n^2 + m_n \dot{q}_n^2\right) = \frac{1}{2}\sum_n \left(m_n \nu_n^2 q_n^2 + \frac{p_n^2}{m_n}\right), \tag{1.9}$$

Substituting position and momentum variables by corresponding operators to obtain the quantum mechanical Hamiltonian, where $p_n = m_n \dot{q}_n$. The position (q_n) and momentum (p_n) operators follow the commutation relations

$$[q_n, p_m] = \imath\hbar\delta_{nm}, \qquad [q_n, q_m] = [p_n, p_m] = 0,$$

where $\hbar$ is the reduced Planck's constant. Using these one may define a new set of operators which can be analytically written as

$$\hat{a}_n \exp[-\imath\nu_\mathrm{n}\mathrm{t}] = \frac{1}{\sqrt{2\hbar m_n \nu_\mathrm{n}}}\left(m_n \nu_n q_n + \imath p_n\right) \tag{1.10}$$

and

$$\hat{a}_n^\dagger \exp[\imath\nu_\mathrm{n}\mathrm{t}] = \frac{1}{\sqrt{2\hbar m_n \nu_\mathrm{n}}}\left(m_n \nu_n q_n - \imath p_n\right). \tag{1.11}$$

Thus, the Hamiltonian can be written as

$$H = \sum_n \hbar\nu_n \left(\hat{a}_n{}^\dagger \hat{a}_n + \frac{1}{2}\right), \tag{1.12}$$

and the commutation relations

$$\left[\hat{a}_n, \hat{a}_m^\dagger\right] = \delta_{nm}, \qquad [\hat{a}_n, \hat{a}_m] = \left[\hat{a}_n^\dagger, \hat{a}_m^\dagger\right] = 0,$$

with corresponding electric and magnetic fields, as given by Eq. 1.1.27 in Scully and Zubairy [1997]

$$E(\overrightarrow{r}, t) = \sum_k \hat{\varepsilon}_k \xi_k \hat{a}_k \exp[-\imath\nu_k t + \imath k.\overrightarrow{r}] + \mathrm{H.c.}$$

and

$$B(\overrightarrow{r},t)=\sum_k \frac{k\times\hat{\varepsilon}_k}{\nu_k}\xi_k\hat{a}_k \exp\left[-\imath\nu_k t+\imath k.\overrightarrow{r}\right]+\text{H.c.},$$

where $\xi_k=\left(\frac{\hbar\nu_k}{\varepsilon_0 V}\right)^{1/2}$ is a constant, and $\hat{\varepsilon}_k$ is a unit polarization vector with the wave vector k.

The above analysis shows that a single-mode field is identical to harmonic oscillator. So, in the domain of quantum optics, harmonic oscillator system plays an important role.

Notice that the quantum treatment of electromagnetic radiation hinges on annihilation $\hat{a}$ (depletes photon) and creation $\hat{a}^\dagger$ (creates photon) operators. The annihilation operator $\hat{a}$ depletes one quantum of energy and thus lowers down the system from harmonic oscillator level $|n\rangle$ to $|n-1\rangle$, given by

$$\hat{a}|n\rangle=\sqrt{n}|n-1\rangle. \tag{1.13}$$

Here, $|n\rangle$ is called Fock or number state. Further, an application of annihilation operator on vacuum leads to 0, i.e., $\hat{a}|0\rangle=0$. The creation operator $\hat{a}^\dagger$ creates one quantum of energy by raising the state from $|n\rangle$ to $|n+1\rangle$. Therefore, the creation operator in the number state can be represented as

$$\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle. \tag{1.14}$$

If creation operator is applied to vacuum it creates a photon so these operators enables one to write a Fock state ($|n\rangle$) in terms of the vacuum state as

$$|n\rangle=\frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle.$$

In the above, we have seen that annihilation and creation operators are the important field operators and are required for the quantum description of radiation. These operators can induce nonclassicality and non-Gaussianity when applied on classical states Zavatta et al. [2004]; Agarwal [2013]. In the present thesis, we study the role of these non-Gaussianity inducing operations in controlling the nonclassicality of the quantum states which are often already nonclassical. For instance, enhacement in squeezing in a nonclassical state does not ensure advantage with respect to use as a single photon source and vice-versa. In the following subsection, we will introduce a set of other operators which can be expressed in terms of annihilation and creation operators and which play a crucial role in our understanding of the quantum states of radiation field.

1.2.2 Some more quantum operators of relevance

So far we have introduced some non-unitary operations (operations $\hat{O}$ which are not norm preserving and do not satisfy $\hat{O}^\dagger=\hat{O}^{-1}$, where $\hat{O}^\dagger$ and $\hat{O}^{-1}$ are the Hermitian conjugate and inverse of $\hat{O}$, respectively), namely photon addition and subtraction. We now aim to introduce some more unitary operations important in the domain of quantum state engineering in general, and in this thesis in particular. To begin with let us describe displacement operator.

(a) Displacement operator

Displacement operator is a unitary operator. The mathematical from of displacement operator is given as

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^{\dagger} - \alpha^{\star}\hat{a}\right). \tag{1.15}$$

This operator can be used as a tool to generate coherent state from vacuum. Specifically, a coherent state $|\alpha\rangle$ is defined as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$.

(b) Squeezing operator

The squeezing operator for a single mode of electromagnetic field is

$$\hat{S}(z) = \exp\left(\frac{1}{2}\left(z^{\star}\hat{a}^{2} - z\hat{a}^{\dagger 2}\right)\right). \tag{1.16}$$

The description of light is given by two quadratures namely phase (X_1) and amplitude (X_2) in the domain of quantum optics, mathematically defined as

$$\hat{X}_{\theta} = \frac{1}{\sqrt{2}}\left(i\hat{a}^{\dagger}\exp[\imath\theta] - \imath\hat{a}\exp[-\imath\theta]\right),$$

The corresponding uncertainty of these two quadratures is observed by relation $\Delta X_1 \Delta X_2 \geq \hbar/2$, where ΔX_1 (ΔX_2) is variance in the measured values of quadrature $\hat{X}_1 = \hat{X}_1(\theta = 0)$ $\left(\hat{X}_2 = \hat{X}_2\left(\theta = \frac{\pi}{2}\right)\right)$. Specifically, $\Delta X_i = \sqrt{\langle X_i^2\rangle - \langle X_i\rangle^2}$,where $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ is the expectation value of the operator A with respect to the quantum state $|\psi\rangle$. Coherent state has an equal uncertainty in both quadratures so they form a circle in the phase picture (shown in Fig. 1.1). Least value of the variance for suitable θ is studied as principle squeezing. With the advent of nonlinear optics, a very special branch of optics, the uncertainty in one of the quadratures can be reduced at the cost of increment in other quadrature's uncertainty, which means that the circle can be squeezed.

1.2.3 Eigen states of the field operators

Here, we will discuss eigen states of some of the operators we have introduced. The eigenvalue equation can be defined as $\hat{A}\lambda = a\lambda$ with eigen operator $\hat{A}$, eigenvalue a, and eigen function λ. For example, Schrodinger equation $H\psi_i = E_i\psi_i$ has Hamiltonian H as eigen operator with eigen functions ψ_i and eigenvalues as allowed energy levels.

(a) Fock state: Eigen state of the number operator

In case of quantum optics or quantized light picture, photon number state is known as number state. The single-mode photon number states are known Fock states, and its ground state is defined as vacuum state. As the set of number states are a full set of orthonormal basis so any quantum state can be written in terms of these basis. The method of representing a quantum state as superposition of number states is known as number state representation. Now using Eq. (1.13) and (1.14), we can introduce an operator

$$\hat{N} = \hat{a}^{\dagger}\hat{a}.$$

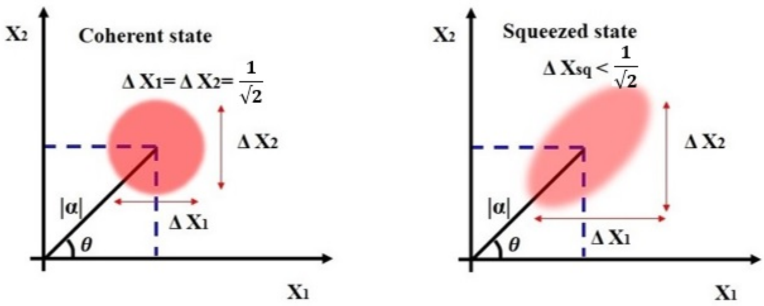

Figure 1.1: Phase picture for coherent state and squeezed state.

which would satisfy the following eigen value equation

$$\hat{N}\,|n\rangle = n\,|n\rangle\,.$$

Clearly Fock states are the eigen states of the number operators and in consistency with what has already been told, a Fock state $|n\rangle$ represent a n photon state.

(b) Coherent state: Eigen state of the annihilation operator

Coherent state Fox [2006] is considered as a state of the quantized electromagnetic field which shows classical behaviour (specifically, beavior closest to classical states). According to Erwin Schrodinger it is a minimum uncertainty state, having same uncertainity in position and momentum Schrödinger [1926]. According to Glauber, any of three mathematical definitions described below can define coherent state:

(i) Eigen vectors of annihilation operator $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, α being a complex number.

(ii) Quantum states having minimum uncertainty $\Delta X_1 = \Delta X_2 = 1/\sqrt{2}$, with X_2 and X_1 as momentum and position operators.

(iii) States realized by the application of the displacement operator $D(\alpha)$ on the vacuum state. Thus, is also known as displaced vacuum state and can be expressed as

$$|\alpha\rangle = D(\alpha)\,|0\rangle\,.$$

In Fock basis, it is expressed as infinite superposition of Fock state as

$$|\alpha\rangle = \exp\left[-\frac{|\,\alpha\,|^2}{2}\right]\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{1.17}$$

where α is a complex number. Experimentally established state very close to this coherent state was possible only after the successful development of laser. Finally, one can easily see that $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ implies $\langle\alpha|\,\hat{a}^{\dagger} = \langle\alpha|\,\alpha^{\star}$ and consequently $\langle\alpha|\,\hat{a}^{\dagger}\hat{a}\,|\alpha\rangle = \langle\alpha|\,\hat{N}\,|\alpha\rangle = N = |\alpha|^2$ or average photon number in a coherent state is $|\alpha|^2$.

1.3 Quantum states of our interest

In this section, we provide basic mathematical details of the set of engineered quantum states studied in the present thesis.

1.3.1 Displaced Fock state

Displaced Fock state Satyanarayana [1985] are formed by applying displacement operator on Fock state and thus a DFS is defined as

$$|\phi\rangle = D(\alpha)|n\rangle .$$

Analytically it is given as

$$|\phi(n,\alpha)\rangle = \frac{1}{\sqrt{n!}}\sum_{p=0}^{n}\binom{n}{p}(-\alpha^{\star})^{(n-p)}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\sqrt{(m+p)!}|m+p\rangle. \tag{1.18}$$

Various nonclassical properties of DFS are reported in literature De Oliveira et al. [1990]; El-Orany et al. [2000]; Lvovsky and Babichev [2002]; Mendas and Popovic [1993].

1.3.2 Photon added and photon subtracted displaced Fock state

Using DFS, we can define a u photon added DFS (i.e., a PADFS) as

$$\begin{aligned}|\psi_{+}(u,n,\alpha)\rangle &= N_{+}\hat{a}^{\dagger u}|\phi(n,\alpha)\rangle = \frac{N_{+}}{\sqrt{n!}}\sum_{p=0}^{n}\binom{n}{p}(-\alpha^{\star})^{(n-p)}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\\ &\times \sqrt{(m+p+u)!}|m+p+u\rangle. \end{aligned}\tag{1.19}$$

Similarly, a v photon subtracted DFS (i.e., a PSDFS) can be expressed as

$$\begin{aligned}|\psi_{-}(v,n,\alpha)\rangle &= N_{-}\hat{a}^{v}|\phi(n,\alpha)\rangle = \frac{N_{-}}{\sqrt{n!}}\sum_{p=0}^{n}\binom{n}{p}(-\alpha^{\star})^{(n-p)}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\\ &\times \frac{(m+p)!}{\sqrt{(m+p-v)!}}|m+p-v\rangle, \end{aligned}\tag{1.20}$$

where m and p are the real integers. Here,

$$N_{+} = \left[\frac{1}{n!}\sum_{p,p'=0}^{n}\binom{n}{p}\binom{n}{p'}(-\alpha^{\star})^{(n-p)}(-\alpha)^{(n-p')}\exp\left[-|\alpha|^2\right]\sum_{m=0}^{\infty}\frac{\alpha^m(\alpha^{\star})^{m+p-p'}(m+p+u)!}{m!(m+p-p')!}\right]^{-\frac{1}{2}} \tag{1.21}$$

and

$$N_{-} = \left[\frac{1}{n!}\sum_{p,p'=0}^{n}\binom{n}{p}\binom{n}{p'}(\alpha^{\star})^{(n-p)}(-\alpha)^{(n-p')}\exp\left[-|\alpha|^2\right]\sum_{m=0}^{\infty}\frac{\alpha^m(\alpha^{\star})^{m+p-p'}(m+p)!}{m!(m+p-p')!(m+p-v)!}\right]^{-\frac{1}{2}}. \tag{1.22}$$

are the normalization constants, and subscripts $+$ and $-$ correspond to photon addition and subtraction. Thus, $|\psi_+(u,n,\alpha)\rangle$ and $|\psi_-(v,n,\alpha)\rangle$ represent u photon added DFS and v photon subtracted DFS, respectively, for the DFS which has been produced by displacing the Fock state $|n\rangle$ by a displacement operator $D(\alpha)$ characterized by the complex parameter α. Clearly, the addition and the subtraction of photons on the DFS can be mathematically viewed as application of the creation and annihilation operators from the left on the Eq. (1.18). Here, it may be noted that different well-known states can be obtained as special cases of these two states. For example, using the notation introduced above to define PADFS and PSDFS, we can describe a coherent state $|\alpha\rangle$ as $|\alpha\rangle = |\psi_+(0,\alpha,0)\rangle = |\psi_-(0,\alpha,0)\rangle$, naturally, coherent state can be viewed as a special case of both PADFS and PSDFS. Similarly, we can describe a single photon added coherent state as $|\psi\rangle_{+1} = |\psi_+(1,\alpha,0)\rangle$, a Fock state as $|n\rangle = |\psi_+(0,0,n)\rangle = |\psi_-(0,0,n)\rangle$ and a DFS as $|\psi\rangle_{\rm DFS} = |\psi_+(0,\alpha,n)\rangle = |\psi_-(0,\alpha,n)\rangle$.

1.3.3 Photon added then subtracted displaced Fock state

A PASDFS can be obtained by sequentially applying appropriate number of annihilation (photon subtraction) and creation (photon addition) operators on a DFS. Analytical expression for PASDFS (specifically, a k photon added and then q photon subtracted DFS) in Fock basis can be shown to be

$$\begin{aligned} |\psi(k,q,n,\alpha)\rangle &= N\hat{a}^q\hat{a}^{\dagger k}|\psi(n,\alpha)\rangle = \frac{N}{\sqrt{n!}}\sum_{p=0}^{n}\binom{n}{p}(-\alpha^\star)^{(n-p)}\exp\left(-\frac{|\alpha|^2}{2}\right) \\ &\times \sum_{m=0}^{\infty}\frac{\alpha^m\,(m+p+k)!}{m!\sqrt{(m+p+k-q)!}}|m+p+k-q\rangle, \end{aligned} \tag{1.23}$$

where

$$N=\left[\frac{1}{n!}\sum_{p,p'=0}^{n}\binom{n}{p}\binom{n}{p'}(-\alpha^\star)^{(n-p)}(-\alpha)^{(n-p')}\exp\left[-|\alpha|^2\right]\right]^{-\frac{1}{2}}$$

is the normalization factor.

1.3.4 Even coherent state and states generated by holeburning on it

Even coherent state can be defined as the superposition of two coherent states having opposite phase ($|\phi(\alpha)\rangle \propto |\alpha\rangle + |-\alpha\rangle$). The analytical expression for ECS in number basis can be written as

$$|\phi(\alpha)\rangle = \frac{\exp\left[-\frac{|\alpha|^2}{2}\right]}{\sqrt{2\left(1+\exp\left[-2|\alpha|^2\right]\right)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(1+(-1)^n\right)|n\rangle. \tag{1.24}$$

The parameter $\alpha = |\alpha|\exp(i\theta)$, in Eq. (1.24), is complex in general and θ is phase angle in the complex plane. Various schemes to generate ECS are reported in Brune et al. [1992]; Ourjoumtsev et al. [2007]; Gerry [1993]. The nonclassical properties (witnessed through the antibunching and squeezing criteria, Q function, Wigner function, and photon number distribution, etc.) of ECS have been studied in the recent past Gerry [1993].

(a) Vacuum filtered even coherent state

As mentioned above, experimentally, an ECS or a cat state can be generated in various ways, and the same can be further engineered to produce a hole at vacuum in its photon number distribution. Specifically, filtration of vacuum will burn a hole at $n = 0$ and produce VFECS, which can be described in Fock basis as

$$|\phi_1(\alpha)\rangle = N_{\text{VFECS}} \sum_{n=0, n\neq 0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)|n\rangle, \tag{1.25}$$

where

$$N_{\text{VFECS}} = \left\{4\cosh\left(|\alpha|^2\right) - 1\right\}^{-1/2} \tag{1.26}$$

is the normalization constant. For simplicity, we may expand Eq. (1.25) as a superposition of a standard ECS and a vacuum state as follows

$$|\phi_1(\alpha)\rangle = N_{\text{VFECS}} \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)|n\rangle - 2|0\rangle \right). \tag{1.27}$$

In what follows, Eq. (1.27) will be used to explore various nonclassical features that may exist in VFECS.

(b) Photon added even coherent state

One can define a single photon added ECS as

$$|\phi_2(\alpha)\rangle = N_{\text{PAECS}}\hat{a}^{\dagger}|\phi(\alpha)\rangle = N_{\text{PAECS}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(1+(-1)^n\right)\sqrt{n+1}|n+1\rangle, \tag{1.28}$$

where

$$N_{\text{PAECS}} = \left\{\cosh\left(|\alpha|^2\right) + |\alpha|^2 \sinh\left(|\alpha|^2\right)\right\}^{-1/2}/2 \tag{1.29}$$

is the normalization constant for PAECS.

1.3.5 Binomial state and the states generated by holeburning on it

Binomial state is a finite superposition of Fock states having binomial photon number distribution. It is quite similar to the coherent state which is the linear combination of Fock states having the Poissonian photon number distribution Stoler et al. [1985]. BS can be defined as

$$|p, M\rangle = \sum_{n=0}^{M} \left[\frac{M!}{n!(M-n)!} p^n (1-p)^{M-n}\right]^{1/2} |n\rangle. \tag{1.30}$$

The binomial coefficient describes the presence of n photons with probability p in M number of ways. Recently, the study of nonclassical properties of BS, specifically, antibunching, squeezing, HOSPS Verma et al. [2008]; Verma and Pathak [2010]; Bazrafkan and Man'ko [2004], etc., have been studied very extensively. However, no effort has yet been made to study the nonclassical properties of VFBS and PABS.

(a) Vacuum filtered binomial state

The vacuum filtration of BS can be obtained by simply eliminating vacuum state from the BS as

$$|p,M\rangle_1 \quad = \quad N_{\mathrm{VFBS}} \sum_{n=0}^{M} \left[\tfrac{M!}{n!(M-n)!} p^n (1-p)^{M-n}\right]^{1/2} |n\rangle - N_{VFBS} \left[(1-p)^M\right]^{1/2} |0\rangle, \quad (1.31)$$

where

$$N_{\mathrm{VFBS}} \quad = \quad \{1-(1-p)^M\}^{-1/2} \quad (1.32)$$

is the normalization constant for the VFBS.

(b) Photon added binomial state

A hole at $n=0$ at a BS can also be introduced by the addition of a single photon on the BS. A few steps of computation yield the desired expression for PABS as

$$|p,M\rangle_2 \quad = N_{\mathrm{PABS}} \quad \sum_{n=0}^{M} \left[\tfrac{M!(n+1)!}{(n!)^2(M-n)!} p^n (1-p)^{M-n}\right]^{1/2} |n+1\rangle, \quad (1.33)$$

where

$$N_{\mathrm{PABS}} \quad = \quad (1+Mp)^{-1/2} \quad (1.34)$$

is the normalization constant for single photon added BS.

1.3.6 Kerr state and the states generated by holeburning on it

A KS can be obtained when electromagnetic field in a coherent state interacts with nonlinear medium with Kerr type nonlinearity Gerry and Grobe [1994]. This interaction generates phase shifts which depend on the intensity. The Hamiltonian involved in this process is given as

$$H = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\chi\left(\hat{a}^\dagger\right)^2(\hat{a})^2, \quad (1.35)$$

where χ depends on the third-order susceptibility of Kerr medium. Explicit contribution of H is $\exp\left[-i\chi n(n-1)\right]$. Thus, the compact analytic form of the KS in the Fock basis can be given as

$$|\psi_K(n)\rangle \quad = \quad \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{|\alpha|^2}{2}\right) \exp\left(-\imath\chi n(n-1)\right)|n\rangle. \quad (1.36)$$

(a) Vacuum filtered Kerr state

Similarly, a VFKS, which can be obtained using the same quantum state engineering process that leads to VFECS and VFBS, is given by

$$|\psi_K(n)\rangle_1 \quad = \quad N_{\mathrm{VFKS}} \left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\imath\chi n(n-1)\right)|n\rangle - |0\rangle\right], \quad (1.37)$$

where

$$N_{\mathrm{VFKS}} \quad = \quad \left(\exp\left[|\alpha|^2\right]-1\right)^{-1/2} \quad (1.38)$$

is the normalization constant for the VFKS.

(b) Photon added Kerr state

An addition of a photon to Kerr state would yield PAKS which can be expanded in Fock basis as

$$|\psi_K(n)\rangle_2 = N_{\text{PAKS}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\imath\chi n(n-1)\right)\sqrt{(n+1)}|n+1\rangle, \tag{1.39}$$

where

$$N_{\text{PAKS}} = \left(\exp\left[|\alpha|^2\right]\left(1+|\alpha|^2\right)\right)^{-1/2} \tag{1.40}$$

is the normalization constant for the PAKS.

1.4 The notion of nonclassical states

Quantum states which do not have any classical analogue have been referred to as nonclassical states Agarwal [2013]. In other words, states having their P-distribution more singular than delta function or having negative values are referred to as nonclassical states Dodonov and Renó [2003]. This idea was possible only when Glauber and Sudarshan published papers in 1963 Sudarshan [1963]; Glauber [1963b,a]. Sudarshan found a mathematical form to represent any state in the coherent basis, mathematically given as

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha,$$

where $P(\alpha)$ is known as Glauber-Sudarshan P-function, which follows normalization condition as $\int dP(\alpha) = 1$, but it may have negative values. Thus, it is defined as quasidistribution function or quasiprobability distribution. When $P(\alpha)$ attains a positive probability density function, immediately it indicates that the state is classical. This leads to the definition of nonclassicality. If an arbitrary quantum state is failed to represent as mixture of coherent states, that is known as nonclassical state. To establish quantum supremacy, these nonclassical states play very essential role, for instance in theses states are useful in establishing quantum supremacy of quantum information processing, quantum communication, etc. Although P-function is not reconstructable for any arbitrary state yet it has been of major interest as it provides an important signature of nonclassicality. The negativity (positivity or non-negativity) of the P-function essentially provides the nonclassical (classical) behavior of the state under consideration. The experimental difficulty associated with the easurement of P-function in its reconstruction led to various feasible substitutes as nonclassicality witnesses. Here, we list some of those nonclassicalty witnesses. These witnesses can be viewed as operational criterion of nonclassicality.

1.5 Nonclassical states: witnesses and measures

Using nonclassical states, the essence of quantum theory of light can be understood. There are various tools for characterization of nonclassical states. In this section, some tools are described which are used to characterize such states. If historically seen, the first such approach was aimed to check the deviation from Poissonian photon statistics, the second is to evaluate the volume of the

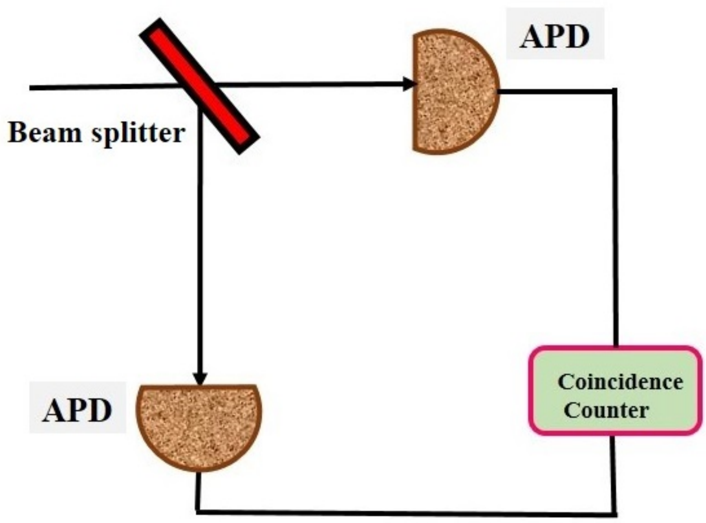

Figure 1.2: Hanbury Brown and Twiss setup. Here, APD is avalanche photo diode.

negative part of the quasiprobability distribution in the phase space, etc. An infinite set of moments based criteria is available in literature which is used as witness of nonclassicality equivalent to P-function Shchukin and Vogel [2005b]. Any subset of this infinite set may detect nonclassicality or fail to do so. Example of these witnesses are lower- and higher-order antibunching, sub-Poissonian photon statistics, squeezing as well as Mandel Q_M parameter, Klyshko's, Vogel's, and Agarwal-Tara's criteria, Q function, etc. To quantify the amount of nonclassicality, a number of measures have been proposed, like linear entropy, Wigner volume, concurrence and many more. A small description of these criteria is given here.

1.5.1 Witnesses of nonclassicality

(a) Lower- and higher-order antibunching

In this section, we study lower- and higher-order antibunching. To do so, we use the following criterion of $(l-1)$th order antibunching (Pathak and Garcia [2006] and references therein) in terms of nonclassicality witness $(d(l-1))$ as

$$d(l-1) = \langle \hat{a}^{\dagger l}\hat{a}^{l}\rangle - \langle \hat{a}^{\dagger}\hat{a}\rangle^{l} < 0. \tag{1.41}$$

This nonclassical feature characterizes suitability of the quantum state to be used as single photon source as the negative values of $d(l-1)$ parameter show that the probability of photons coming bunched is less compared to that of coming independently. The signature of lower-order antibunching can be obtained as a special case of Eq. (1.41) for $l=2$, and that for $l \geq 3$, the negative values of $d(l-1)$ correspond to higher-order antibunching of $(l-1)$th order. Figure 1.2 illustrates the scheme for studying antibunching experimentally (corresponds to $l=2$). For higher values of l, we require more beamsplitters and APDs. On these cascaded beamplitters signal is mixed with vacuum and measured higher-order correlation Avenhaus et al. [2010].

(b) Lower- and higher-order sub-Poissionian photon statistics

The lower-order counterparts of antibunching and sub-Poissonian photon statistics are closely associated as the presence of latter ensures the possibility of observing former (see Thapliyal et al. [2014a, 2017b] for a detailed discussion). However, these two nonclassical features were shown to be independent phenomena in the past (Thapliyal et al. [2014a, 2017b] and references therein). Higher-order counterpart of sub-Poissonian photon statistics can be introduced as

$$\mathscr{D}_h(l-1) \quad = \quad \sum_{e=0}^{l}\sum_{f=1}^{e} S_2(e,f)\,{}^lC_e\,(-1)^e\,d(f-1)\langle N\rangle^{l-e} \quad < \quad 0, \tag{1.42}$$

where $S_2(e,f)$ is the Stirling number of second kind, and ${}^lC_e$ is the usual binomial coefficient.

(c) Higher-order squeezing

As mentioned beforehand, the squeezing of quadrature is defined in terms of variance in the measured values of the quadrature (say, position or momentum) below the corresponding value for the coherent state, i.e., minimum uncertainty state. The higher-order counterpart of squeezing is studied in two ways, namely Hong-Mandel and Hillery-type squeezing Hong and Mandel [1985b,a]; Hillery [1987a]. Specifically, the idea of the higher-order squeezing originated from the pioneering work of Hong and Mandel Hong and Mandel [1985b,a], who generalized the lower-order squeezing using the higher-order moments of field quadrature. According to the Hong-Mandel criterion, the lth order squeezing can be observed if the lth moment (for even values of $l>2$) of a field quadrature operator is less than the corresponding coherent state value. The condition of Hong-Mandel type higher-order squeezing is given as follows Hong and Mandel [1985b,a]

$$S(l) = \frac{\langle(\Delta X)^l\rangle - \left(\frac{1}{2}\right)_{\frac{l}{2}}}{\left(\frac{1}{2}\right)_{\frac{l}{2}}} < 0, \tag{1.43}$$

Here, $S_{(l)}$ is higher-order squeezing, ΔX is the quadrature (ΔX_1) as defined in Section 1.2.2. Further, $(x)_l$ is conventional Pochhammer symbol. The inequality in Eq. (1.43) can also be rewritten as

$$\langle(\Delta X)^l\rangle \quad < \quad \left(\tfrac{1}{2}\right)_{\frac{l}{2}} = \frac{1}{2^{\frac{l}{2}}}(l-1)!! \tag{1.44}$$

with

$$\langle(\Delta \mathrm{X})^l\rangle = \sum_{r=0}^{l}\sum_{i=0}^{\frac{r}{2}}\sum_{k=0}^{r-2i}(-1)^r\,\frac{1}{2^{\frac{l}{2}}}\,(2i-1)!^{2i}C_k{}^lC_r{}^rC_{2i}\langle\hat{a}^\dagger+\hat{a}\rangle^{l-r}\langle\hat{a}^{\dagger k}\hat{a}^{r-2i-k}\rangle. \tag{1.45}$$

(d) Klyshko's criterion

This criterion is relatively simpler as to calculate this witness of nonclssicality, only three consecutive probability terms are required rather than all the terms. Negative values of $B(m)$ are symbol of nonclassicality present in the state. Klyshko introduced this criterion Klyshko [1996] to investigate the nonclassical property using only three successive photon-number probabilities. In terms of the photon-number probability $p_m = \langle m|\rho|m\rangle$ of the state with density matrix ρ, the

Klyshko's criterion in the form of an inequality can be written as

$$B(m) = (m+2)p_m p_{m+2} - (m+1)\left(p_{m+1}\right)^2 < 0. \tag{1.46}$$

(e) Vogel's criterion

The moments-based nonclassicality criterion of the previous subsection was later extended to Vogel's nonclassicality criterion Shchukin and Vogel [2005b] in terms of matrix of moments as

$$v = \begin{bmatrix} 1 & \langle\hat{a}\rangle & \langle\hat{a}^\dagger\rangle \\ \langle\hat{a}^\dagger\rangle & \langle\hat{a}^\dagger\hat{a}\rangle & \langle\hat{a}^{\dagger 2}\rangle \\ \langle\hat{a}\rangle & \langle\hat{a}^2\rangle & \langle\hat{a}^\dagger\hat{a}\rangle \end{bmatrix}. \tag{1.47}$$

The negative value of the determinant dv of matrix v in Eq. (1.47) is signature of nonclassicality.

(f) Agarwal-Tara's criterion

There were certain quantum states having negative P-function yet showing no squeezing and sub-Poissonian behavior, to witness the nonclassicality residing in those particular types of states Agarwal and Tara Agarwal and Tara [1992] introduced this criterion which is again a moments based criterion. This can be written in a matrix form and expressed as

$$A_3 = \frac{\det m^{(3)}}{\det \mu^{(3)} - \det m^{(3)}} < 0, \tag{1.48}$$

where

$$m^{(3)} = \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix}$$

and

$$\mu^{(3)} = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix}.$$

The matrix elements are defined as $m_i = \langle\hat{a}^{\dagger i}\hat{a}^i\rangle$ and $\mu_j = (\langle\hat{a}^\dagger\hat{a}\rangle)^j = (m_1)^j$.

(g) Mandel Q_M parameter

The Mandel Q_M parameter Mandel [1979] illustrates the nonclassicality through photon number distribution in a quantum state. The Mandel Q_M parameter is defined as

$$Q_M = \frac{\langle(\hat{a}^\dagger\hat{a})^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2 - \langle\hat{a}^\dagger\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle}. \tag{1.49}$$

The negative values of Q_M parameter essentially indicate the negativity for P-function and so it gives a witness for nonclassicality. For the Poissonian statistics it becomes 0, while for the sub-Poissonian (super-Poissonian) photon statistics it has negative (positive) values.

1.5.2 Other quasiprobability distributions

Inability to give a phase space description of quantum mechanics is exploited in terms of quasiprobability distributions (Thapliyal et al. [2015] and references therein). Later, it was found that they are useful as witnesses of nonclassicality. These real and normalized quasiprobability distributions allow to calculate the expectation value of an operator as any classical probability distribution. One such quasiprobability distributions is Q function Husimi [1940], and zeros of this function are signature of nonclassicality. Another example is Wigner Wigner [1932] function whose negative values corresponds to the nonclassicality.

(a) Q function

Q function Husimi [1940] is defined as

$$Q = \frac{1}{\pi}\langle\beta|\rho|\beta\rangle, \tag{1.50}$$

where $|\beta\rangle$ is the coherent state (1.17).

(b) Wigner function

Another quasiprobability distribution is Wigner function formulated by Wigner in 1932 Wigner [1932] in the early stage of quantum mechanics, the motive was to connect the wavefunction approach to a probability distribution in phase space. Negativity of Wigner function represents the nonclassicality present in an arbitrary quantum state. Also the ability to reconstruct the Wigner function experimentally makes this approach more impactful than any other approach. Specifically, Wigner function obtained through optical tomography can be used to obtain other quasidistributions, however, Wigner function is stronger witness of nonclassicality than Q function while is not singular like P-function. Mathematically, it is expressed as

$$W(\gamma,\gamma^{\star}) = A\exp\left[-2|\gamma|^{2}\right]\int d^{2}\lambda\langle-\lambda|\rho|\lambda\rangle\exp\left[-2\left(\gamma^{\star}\lambda-\gamma\lambda^{\star}\right)\right]. \tag{1.51}$$

The zeros of Q function while the negativity of P-function and Wigner function correspond to the nonclassical behavior of any arbitrary quantum state. It is worth stressing here that only P-function is both necessary and sufficient criterion of nonclassicality, while rest of the quasidistribution functions are only sufficient.

1.5.3 Measures of nonclassicality

In the above section, we have seen that there exist numerous criteria of nonclassicality. However, most of these criteria only witness the nonclassicality. They do not provide any quantification of the nonclassicality. Except P-function and infinite set of vogel's criteria, all other criteria are sufficient but not necessary. However, many efforts have been made for the quantification of nonclassicality, e.g., in 1987, a distance-based measure of nonclassicality was introduced by Hillery Hillery [1987b]. A trace norm based measure Mari et al. [2011] was introduced by Mari et al., for the set of all states having the positive Wigner function. In 1991, Lee gave a measure of nonclassicality known as nonclassical depth Lee [1991]. However, in this work, we will not study these measures. There are certain measures those can be exploited in terms of entanglement, like linear entropy Wei et al. [2003], which we will use for our calculations and the same is described below.

(a) Linear entropy

In 2005, a measure of nonclassicality was proposed as entanglement potential, which is the amount of entanglement in two output ports of a beam splitter with the quantum state ρ_{in} and vacuum $|0\rangle\langle 0|$ sent through two input ports Asbóth et al. [2005]. The amount of entanglement quantifies the amount of nonclassicality in the input quantum state as classical state can not generate entanglement in the output. The post beam splitter state can be obtained as $\rho_{out} = U\left(\rho_{in} \otimes |0\rangle\langle 0|\right) U^{\dagger}$ with $U = \exp[-iH\theta]$, where $H = (\hat{a}^{\dagger}\hat{b} + \hat{a}\hat{b}^{\dagger})/2$, and $\hat{a}^{\dagger}\,(\hat{a})$, $\hat{b}^{\dagger}\,(\hat{b})$ are the creation (annihilation) operators of the input modes. For example, considering quantum state ($|\psi\rangle = \sum\limits_{n=0}^{\infty} c_n|n\rangle$) and a vacuum state $|0\rangle$ as input states, we can write the analytic expression of the two-mode output state as

$$|\phi\rangle = U\left(|\psi\rangle \otimes |0\rangle\right) \equiv U|\psi, 0\rangle = \sum_{n=0}^{\infty} \frac{c_n}{2^{n/2}} \sum_{j=0}^{n} \sqrt{{}^{n}C_j}\, |j, n-j\rangle. \tag{1.52}$$

We can measure the amount of entanglement in the output state to quantify the amount of input nonclassicality in $|\psi\rangle$. Here, we use linear entropy of single mode subsystem (obtained after tracing over the other subsystem) as entanglement potential. The linear entropy for an arbitrary bipartite state ρ_{AB} is defined as Wei et al. [2003]

$$\mathscr{L} = 1 - \mathrm{Tr}\left(\rho_B^2\right), \tag{1.53}$$

where ρ_B is obtained by tracing over subsystem A.

1.6 Analytic tools for the study of phase properties of nonclassical states

In this section, we aim to introduce the parameters that are used to study phase properties of a given quantum state under consideration in this section.

1.6.1 Phase distribution function

A distribution function allows us to calculate expectation values of an operator analogous to that from the corresponding density matrix. Phase distribution function for a given density operator Banerjee and Srikanth [2007]; Agarwal et al. [1992] can be defined as

$$P_\theta = \frac{1}{2\pi}\langle\theta|\rho|\theta\rangle, \tag{1.54}$$

where the phase state $|\theta\rangle$, complementary to the number state $|n\rangle$, is defined Agarwal et al. [1992] as

$$|\theta\rangle = \sum_{n=0}^{\infty} e^{\imath n\theta}|n\rangle. \tag{1.55}$$

1.6.2 Phase dispersion

A known application of phase distribution function (1.54) is that it can be used to quantify the quantum phase fluctuation. Although the variance is also used occasionally as a measure of phase fluctuation, it has a drawback that it depends on the origin of phase integration Banerjee and Srikanth [2007]. A measure of phase fluctuation, free from this problem, is phase dispersion Peřinová et al.

[1998] defined as

$$D = 1 - \left| \int_{-\pi}^{\pi} d\theta \exp\left[-\iota\theta\right] P_\theta \right|^2 . \tag{1.56}$$

1.6.3 Angular Q function

Analogous to the phase distribution P_θ, phase distributions are also defined as radius integrated quasidistribution functions which are used as the witnesses for quantumness Thapliyal et al. [2015]. One such phase distribution function based on the angular part of the Q function is studied in Leonhardt and Paul [1993]; Leonhardt et al. [1995]. Specifically, the angular Q function is defined as

$$Q_{\theta_1} = \int_0^\infty Q(\beta, \beta^\star) |\beta| d|\beta|, \tag{1.57}$$

where the Q function Husimi [1940] is defined in Eq. (1.50).

1.6.4 Phase fluctuation

In attempts to get rid of the limitations of the Hermitian phase operator of Dirac Dirac [1927], Louisell Louisell [1963] first mentioned that bare phase operator should be replaced by periodic functions. As a consequence, sine ($\hat{\mathscr{S}}$) and cosine ($\hat{\mathscr{C}}$) operators appeared, explicit forms of these operators were given by Susskind and Glogower Susskind and Glogower [1964], and further modified by Barnett and Pegg Barnett and Pegg [1986] as

$$\hat{\mathscr{S}} = \frac{\hat{a} - \hat{a}^\dagger}{2\iota\left(\bar{N} + \frac{1}{2}\right)^{\frac{1}{2}}} \tag{1.58}$$

and

$$\hat{\mathscr{C}} = \frac{\hat{a} + \hat{a}^\dagger}{2\left(\bar{N} + \frac{1}{2}\right)^{\frac{1}{2}}}. \tag{1.59}$$

Here, $\bar{N}$ is the average number of photons in the measured field, and here we refrain our discussion to Barnett and Pegg sine and cosine operators Barnett and Pegg [1986]. Carruthers and Nieto Carruthers and Nieto [1968] have introduced three quantum phase fluctuation parameters in terms of sine and cosine operators

$$U = (\Delta N)^2 \left[(\Delta \mathscr{S})^2 + (\Delta \mathscr{C})^2\right] / \left[\langle \hat{\mathscr{S}} \rangle^2 + \langle \hat{\mathscr{C}} \rangle^2\right], \tag{1.60}$$

$$S = (\Delta N)^2 (\Delta \mathscr{S})^2, \tag{1.61}$$

and

$$Q = S / \langle \hat{\mathscr{C}} \rangle^2. \tag{1.62}$$

These three phase fluctuation parameters U, S and Q show phase properties of PADFS and PSDFS, while U parameter is shown relevant as a witness of nonclassicality (antibunching).

1.6.5 Quantum phase estimation parameter

Quantum phase estimation is performed by sending the input state through a Mach-Zehnder interferometer and applying the phase to be determined (ϕ) on one of the arms of the interferometer. To study the phase estimation using Mach-Zehnder interferometer, angular momentum operators Sanders and Milburn [1995]; Demkowicz-Dobrzański et al. [2015], defined as

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}\right), \tag{1.63}$$

$$\hat{J}_y = \frac{\iota}{2}\left(\hat{b}^\dagger\hat{a}-\hat{a}^\dagger\hat{b}\right), \tag{1.64}$$

and

$$\hat{J}_z = \frac{1}{2}\left(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}\right), \tag{1.65}$$

are used. Here, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the modes corresponding to two input ports of the Mach-Zehnder interferometer. The average value of $\hat{J}_z$ operator in the output of the Mach-Zehnder interferometer, which is one-half of the difference of photon numbers in the two output ports (1.65), can be written as

$$\langle\hat{J}_z\rangle = \cos\phi\langle\hat{J}_z\rangle_{in} - \sin\phi\langle\hat{J}_x\rangle_{in}. \tag{1.66}$$

Therefore, variance in the measured value of operator $\hat{J}_z$ can be computed as

$$(\Delta J_z)^2 = \cos^2\phi\,(\Delta J_z)^2_{in} + \sin^2\phi\,(\Delta J_x)^2_{in} - 2\sin\phi\,\cos\phi\,\mathrm{cov}\left(\hat{J}_x,\hat{J}_z\right)_{in}, \tag{1.67}$$

where covariance of the two observables is defined as

$$\mathrm{cov}\left(\hat{J}_x,\hat{J}_z\right) = \frac{1}{2}\langle\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle - \langle\hat{J}_x\rangle\langle\hat{J}_z\rangle. \tag{1.68}$$

This allows us to quantify precision in phase estimation Demkowicz-Dobrzański et al. [2015] as

$$\Delta\phi = \frac{\Delta J_z}{\left|\frac{d\langle\hat{J}_z\rangle}{d\phi}\right|}. \tag{1.69}$$

Before we proceed further and conclude this chapter by noting the structure of the rest of the thesis, it would be apt to to note that there exist various methods of quantum state engineering (some of which have already been mentioned) and photon addition, subtraction, filteration, punching etc., which can be viewed as examples of quantum state engineering processes. In rest of the thesis these processes will be studied with detail.

1.7 Structure of the rest of the thesis

This thesis has 6 chapters. The next 4 chapters are focused on the study of nonclassical and phase properties of the engineered quantum states and the last chapter is dedicated to conclusion. These chapters and thus the rest of this thesis is organized as follows.

In Chapter 1, in Section 1.3, quantum states of our interest (i.e., PADFS and PSDFS) have been introduced in detail. In Chapter 2, in Section 2.3, the analytical expressions of various witnesses of nonclassicality are reported. Further, the existence of various lower- and higher-order nonclassical features in PADFS and PSDFS are shown through a set of plots. Finally, we conclude in Section 2.4.

In Chapter 3, in Section 3.3, we investigate the phase properties of PADFS and PSDFS from a number of perspectives. Finally, the chapter is concluded in Section 3.4.

In Chapter 4, we describe the quantum state of interest (i.e., PASDFS) in Fock basis and calculate the analytic expressions for the higher-order moments of the relevant field operators for this state. In Section 4.3, we investigate the possibilities of witnessing various nonclassical features in PASDFS and its limiting cases by using a set of moments-based criteria for nonclassicality. Variations of nonclassical features (witnessed through different criteria) with various physical parameters are also discussed here. In Section 4.4, phase properties of PASDFS are studied. Q function for PASDFS is obtained in Section 4.5. Finally, we conclude in Section 4.6.

In Chapter 5, in Section 5.2, we have introduced the quantum states of our interest which include ECS, BS, KS, VFECS, VFBS, VFKS, PAECS, PABS, and PAKS. In Section 5.3, we have investigated the nonclassical properties of these states using various witnesses of lower- and higher-order noncassicality as well as a measure of nonclassicality. Specifically, in this section, we have compared nonclassicality features found in vacuum filtered and single photon added versions of the states of our interest using the witnesses of Higher-order antibunching (HOA), Higher-order squeezing (HOS) and Higher-order sub-Poissonian photon statistics (HOSPS). Finally, in Section 5.5, the results are analyzed, and the chapter is concluded.

Finally, the thesis is concluded in Chapter 6, where we have summarized the findings reported in Chapter 2-5 and have emphasized on the main conclusion of the present thesis. We have also discussed the scopes of future work.