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Analysing the nuances of noise and weak measurements in the settings of a two-player game

4.1 INTRODUCTION

In 1991, Gisin demonstrated that any two-qubit non-product quantum state violates the Bell inequality [Gisin, 1991]. The inconsistency between Bell inequality violation and entanglement was, however, witnessed in experiments, as few mixed entangled states did not violate any Bell inequality [Popescu and Rohrlich, 1992]. This prompted discussions on the existence of any direct or indirect relation between entanglement and nonlocality [Bennett *et al.*, 1999a; Horodecki *et al.*, 2003; Brunner *et al.*, 2005]. Moreover, nonlocality as opposed to entanglement was shown to be the predominant requisite for efficient quantum computation [Vértesi and Brunner, 2012]. Further, Popescu also demonstrated the presence of stronger nonlocal correlations than the one expected from quantum theory [Popescu, 2014]. In fact, in noisy one-way quantum computation, no direct relation between the amount of entanglement and computational efficiency of the resource state was established [Chaves and de Melo, 2011]. Instead, it was observed that quantum discord [Ollivier and Zurek, 2001] which is a measure of nonlocal correlations even in the absence of entanglement, is a necessary resource for computational speed-up [Brodutch and Terno, 2011; Gu *et al.*, 2012]. Quantum discord was further shown to be an essential property for quantum remote state preparation [Dakić *et al.*, 2012]. In addition, discord also found applications in quantum algorithms [Brodutch, 2013], device independent quantum cryptography [Pirandola, 2014], and various other applications in quantum information processing [Brodutch *et al.*, 2011; Streltsov, 2014; Brodutch and Terno, 2017].

As discussed in previous chapters, one of the interesting aspects of understanding and analysing nonlocal correlations in terms of nonlocality is to characterize quantum games where these correlations assure benefits to quantum players against their classical counterparts [Meyer, 1999; Eisert *et al.*, 1999; Wiesner, 1983; Vaidman, 1999; Goldenberg *et al.*, 1999; Flitney and Abbott, 2002; Phoenix and Khan, 2013]. Therefore, in order to exploit the advantages offered by nonlocality in games, Bayesian games, which also contain the element of incompleteness, were proposed [Cheon and Iqbal, 2008]. Bayesian games were described by Harsanyi [Harsanyi, 1967a,b,c], as games where the class of at least one player is not fixed, and is also unknown to other players- due to this very reason players have incomplete knowledge about each other. Initially, Cheon and Iqbal [Cheon and Iqbal, 2008] proposed an incomplete information game and demonstrated a relation between game's payoff and Cereceda inequalities [Cereceda, 2001]. On similar lines, a general two player Bayesian game was designed with payoffs in accordance with the Bell inequality [Brunner and Linden, 2013]. They further discussed efficient utilization of non-local correlations in terms of quantum advice to players, which then leads to a clear benefit to quantum players in comparison to players using classical strategy or advice in the game. In general, non-local games emphasizing the advantages of quantum players and quantum strategies versus classical players and classical strategies are instances of common interest games [Clauser *et al.*, 1969; Mermin, 1990b,c; Peres, 1990; Bar-Jossef *et al.*, 2004; Buhrman *et al.*, 2011]. The success of quantum players in these games relies on specific constraints in the game setting and associated rules [Benjamin and Hayden, 2001; Eisert *et al.*, 2001; van Enk and Pike, 2002; Aharon and Vaidman, 2008]. Clearly, the distinct advantages to quantum players over classical players in Bayesian games are due to the presence of experimentally-proven nonlocal correlations in the entangled systems, which forms the very basic premise of a Bayesian game setting.

For ideal conditions, all Bayesian games emphasize the significance of non-local correlations in winning probability of these games. However, in real conditions, entanglement may be distributed to different users through noisy channels [Nielsen and Chuang, 2011]. Evidently, the noisy channels adversely affect the quantum state by degrading spatial correlations existing between qubits, in turn affecting the payoff and winning possibility of quantum games [Situ and Huang, 2016; Huang *et al.*, 2017b; Situ *et al.*, 2017; Gawron, 2010; Gawron *et al.*, 2008; Dajka *et al.*, 2015]. The interactions of the underlying state with environment, thus decreases the overall efficiency of players to utilize nonlocal correlations in winning the game. In order to protect the lost coherence, one can use models such as entanglement distillation [Bennett *et al.*, 1996b,a; Pan *et al.*, 2003], quantum error correcting codes [Shor, 1995; Calderbank and Shor, 1996; Knill and Laflamme, 1997; Steane, 1996b], decoherence free subspace [Kwiat *et al.*, 2000; Lidar *et al.*, 1998], and quantum zeno effect [Facchi *et al.*, 2004; Maniscalco *et al.*, 2008]. One such model that stands out, is the application of weak measurement and its reversal operations [Korotkov and Keane, 2010; Korotkov and Jordan, 2006; Kim *et al.*, 2009; Xiao and Li, 2013; Cheong and Lee, 2012; Sun *et al.*, 2009]. More precisely, before distributing the entangled qubits through noisy channels, one performs non unitary weak measurements on individual qubits locally. Once the qubits pass through noisy channels, another set of local operations are performed on individual qubits, i.e., weak measurement reversal operations, to retrieve nonlocal correlations. This concept of protecting nonlocality is based on the fundamental possibility of reversing the partial measurement operations. For maximizing nonlocal correlations, an optimal relation between different parameters, i.e., noise, weak measurement and its reversal operations, is required or different combination of values of these parameters may also suffice [Singh and Kumar, 2018c,a,b]. The experimental implementation of weak measurements [Lee *et al.*, 2011; Kim *et al.*, 2012; Korotkov and Jordan, 2006; Kim *et al.*, 2009; Xu *et al.*, 2013; Katz *et al.*, 2006; Groen *et al.*, 2013] for quantum systems further strengthens the evidence to the employment of this mechanism in preserving correlations and entanglement in the system.

In this chapter, we study the effects of noise and weak measurement operations, under the settings of a game. For this, we club weak measurement operations with the state parameter, analyse a game scenario where noise acts as one player and weak measurement reversal operation acts as another player. The game illustrates the design of payoffs of players as a difference of Bell correlations before and after application of weak measurement operations. Alternately, in another game setting, we analyse geometric discord [Dakić *et al.*, 2010] to express payoffs in the game which captures a wider range of quantum correlations. Moreover, our results also describe NE [Nash, 1950, 1951] strategy sets for both game settings, with the aid of elimination of strategies, weakly dominated by at least one strategy or a mixture of remaining strategies. The analysis thus performed describes the strategy sets which yield maximum payoff for two players, leading to comparison of these strategies with the NE strategy sets, thus observing a contrast from the point of view of our game setting and any protocol setting. Although the depolarizing noise is considered as highly destructive, our results interestingly show that weak measurement reversal player attains the maximum payoff for this noise as opposed to the cases where the noise player is represented by an amplitude damping or a phase damping noise in the first game setting where the Bell-CHSH operator is considered as a measure of payoffs of players. Similarly the depolarizing noise player achieves lesser maximum payoff as compared to other two noises for a large range of state parameter. Therefore, in such a game setting, one can effectively utilize the destructive nature of depolarizing noise for the benefit of weak measurement reversal player. Considering the first game setting at NE, we further find that the player represented by weak measurement reversal always wins the game against the phase damping noise, and also against the depolarizing noise for a large range of state parameters. On the other hand, amplitude damping noise wins the game against weak measurement strategies at the equilibrium.

In contrast to the first game setting, on analysing the second game setting at NE, our analysis show that the player employing the weak measurement technique is the winner as opposed to the noise player for amplitude damping as well as phase damping noise, but not for depolarizing noise. For maximum payoff, unlike the previous case, weak measurement reversal player achieves higher payoff if the noise player is represented by an amplitude damping noise instead of a depolarizing noise. Surprisingly, the results further demonstrate that the maximum payoff for the weak measurement player is attained by sharing a non-maximally entangled state as opposed to a maximally entangled Bell state for amplitude and phase damping noises in first game setting, and for all three noises under consideration in second game setting. Moreover, we also describe different strategies opted by players leading to the equilibrium point under different situations. The striking difference in the results of both game settings is due to the varying range of values of Bell-CHSH operator and geometric discord for a given quantum state. The analysis thus becomes useful in comprehending strategies opted by the players to achieve a better payoff under the assumptions of a protocol or a game. Subsequently, this study can also be considered as another justification of stating discord as a better measure of quantum correlations.

4.2 APPLICATION OF WEAK MEASUREMENT TO PROTECT NONLOCAL CORRELATIONS DEGRADED DUE TO NOISE

As discussed earlier, when a pure two-qubit state ρ passes through a noisy channel, the initial state ρ evolves to ρ' as shown in Eq. (4.1) where $N_0(\lambda_1)$ and $N_1(\lambda_2)$ are single qubit Kraus operators for noise with noise parameters λ_1 and λ_2 for first and second qubit, respectively. For simplicity and convenience, it is assumed that both qubits pass through the same noisy channel having the noise parameter λ, such that

$$\rho' = \varepsilon(\rho, \lambda) = \sum_{i,j=0}^{1} \{N_i(\lambda) \otimes N_j(\lambda)\}\rho\{N_i(\lambda) \otimes N_j(\lambda)\}^\dagger \tag{4.1}$$

Clearly the underlying two-qubit pure state passing through noisy channels undergoes decoherence and evolves as a mixed state. This further results in degradation of nonlocal correlations between the qubits [Carvalho *et al.*, 2004; Hein *et al.*, 2005; Almeida *et al.*, 2007; Liu *et al.*, 2010; Mahdian *et al.*, 2012]. Recent studies have shown that the mechanism of weak measurement can be used to preserve and enhance the correlations in the quantum state passing through noisy channels [Korotkov and Keane, 2010; Korotkov and Jordan, 2006; Kim *et al.*, 2009; Xiao and Li, 2013; Cheong and Lee, 2012; Sun *et al.*, 2009; Singh and Kumar, 2018c; Lee *et al.*, 2011; Kim *et al.*, 2012; Xu *et al.*, 2013; Katz *et al.*, 2006; Groen *et al.*, 2013; Guan *et al.*, 2017; Pramanik and Majumdar, 2013; Singh and Kumar, 2018a,b]. The single-qubit operators associated with weak measurement (strength=w) and its reversal (strength=w_r) operations can be expressed as

$$\begin{aligned} wk(w) &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-w} \end{bmatrix} \\ wkrev(w_r) &= \begin{bmatrix} \sqrt{1-w_r} & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{4.2}$$

In general, each qubit of the initial state ρ undergoes a local non-unitary weak measurement transformation given by $wk(w)$ where w is the weak measurement strength. The quantum state is then allowed to pass through a noisy channel. Once the qubits pass through the noisy channel, they are locally subjected to weak measurement reversal operations given by $wkrev(w_r)$ where w_r is the weak measurement reversal strength. Therefore the joint state of two qubits, after applications of noise and weak measurement (with weak measurement strength=w, weak measurement reversal strength=w_r), evolves as

$$\rho'' = \left(wkrev(w_r)\otimes wkrev(w_r)\right)\varepsilon\left[\{wk(w)\otimes wk(w)\}\rho\{wk(w)\otimes wk(w)\}^{\dagger},\lambda\right]\left(wkrev(w_r)\otimes wkrev(w_r)\right)^{\dagger} \tag{4.3}$$

The mechanism of weak measurement and its reversal operations has generated intrigue among the quantum theorists to reduce the effect of noise. Since its inception, the technique has been used for vast applications in protecting entanglement, nonlocality, and efficiency of a protocol under noisy conditions [Korotkov and Keane, 2010; Korotkov and Jordan, 2006; Kim *et al.*, 2009; Lee *et al.*, 2011; Kim *et al.*, 2012; Xu *et al.*, 2013; Katz *et al.*, 2006; Groen *et al.*, 2013; Guan *et al.*, 2017; Pramanik and Majumdar, 2013; Singh and Kumar, 2018a,b]. For understanding the nuances of noise and weak measurements, in this chapter the problem of degradation and protection of nonlocal correlations is modeled in the framework of a game.

4.3 DESIGN OF A GAME USING A TWO-QUBIT PURE STATE

For a game theoretic perspective, the beginning marks by considering a general two-qubit pure state, represented as

$$|\psi\rangle = cos\theta|00\rangle + sin\theta|11\rangle \tag{4.4}$$

It is further assumed that the state is prepared by a common source, who before distributing the qubits, pre-applies weak measurement wk(w) operations locally to both qubits. Thus, the quantum state evolves from $\rho = |\psi\rangle\langle\psi|$ to ρ_{new} as

$$\rho_{new} = \{wk(w)\otimes wk(w)\}\rho\{wk(w)^{\dagger}\otimes wk(w)^{\dagger}\} \tag{4.5}$$

One can evaluate that the quantum state ρ_{new} transformed under weak measurement operations, is a pure state, and can also be expressed in form of a general two-qubit state as

$$\rho_{new} = |\psi_{new}\rangle\langle\psi_{new}| = \left[cos\eta|00\rangle + sin\eta|11\rangle\right] \tag{4.6}$$

where $\eta = tan^{-1}[(1-w)tan\theta]$. After evolution through local weak measurement operations, the state ρ_{new} is then distributed to Alice and Bob via similar noisy channels.

For a two player game, noise is considered as the strategy available to one of the players (Player 1) and weak measurement reversal parameter as the strategy available to the other player (Player 2). Player 1 applies a certain noise on qubits and transforms the state ρ_{new} to $\varepsilon(\rho_{new},\lambda)$ as described in Eq. (4.1). Followed by player 1's move, player 2 performs weak measurement reversal operations locally on both qubits, and thus change $\varepsilon(\rho_{new},\lambda)$ to $\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)$ where $\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)$ is expressed as

$$\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right) = \left(wkrev(w_r)\otimes wkrev(w_r)\right)\varepsilon(\rho_{new},\lambda)\left(wkrev(w_r)^{\dagger}\otimes wkrev(w_r)^{\dagger}\right) \tag{4.7}$$

Clearly, the aim of player 1 (Noise on qubits 1 and 2) is to reduce nonlocal correlations of the quantum state so that player 2 (Alice and Bob) is unable to achieve a higher payoff using any preventive strategy. Moreover, player 2 is aware of the mischief of player 1 and thus uses her/his best strategies to reduce the effect of noise by performing weak measurement reversal operations on her/his qubit. With the above-discussed scenario under consideration, different expressions for the payoffs were studied, among which the following (refer to Game setting 1 and Game setting 2) constituted a fair game set-up for both parties. Although there are various ways of quantifying the extent of nonlocal correlations present in a quantum system [Bell, 1964; Clauser *et al.*, 1969; Ollivier and Zurek, 2001; Giorda and Paris, 2010; Rulli and Sarandy, 2011; Dakić *et al.*, 2010; Horodecki *et al.*, 2005; Rajagopal and Rendell, 2002; Luo and Fu, 2011], here, the game is designed using two quantification techniques for nonlocal correlations.

- **Game setting 1:** For our analysis in the first game setting, nonlocality in an underlying two-qubit state is quantified using the Bell-CHSH [Bell, 1964; Clauser *et al.*, 1969] operator given by

$$\langle B\rangle = |\langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle| \leq 2 \tag{4.8}$$

Here, Alice chooses to perform a measurement A or A' on her qubit with an equal probability of $\frac{1}{2}$ each; and likewise, Bob chooses to perform a measurement B or B' on his qubit with probability $\frac{1}{2}$ each. These measurement operators are defined using spin projection operator $\sigma_1 = (\sigma_x, \sigma_y, \sigma_z)$ such as,

$$A = \vec{a}.\vec{\sigma}_1 \quad A' = \vec{a}'.\vec{\sigma}_1 \quad B = \vec{b}.\vec{\sigma}_2 \quad B' = \vec{b}'.\vec{\sigma}_2 \tag{4.9}$$

where $\vec{a}$, $\vec{a}'$, $\vec{b}$, and $\vec{b}'$ are unit vectors defined in Eq. (4.10)

$$\begin{aligned} a &= (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a) \\ a' &= (\sin\theta_{a'} \cos\phi_{a'}, \sin\theta_{a'} \sin\phi_{a'}, \cos\theta_{a'}) \\ b &= (\sin\theta_b \cos\phi_b, \sin\theta_b \sin\phi_b, \cos\theta_b) \\ b' &= (\sin\theta_{b'} \cos\phi_{b'}, \sin\theta_{b'} \sin\phi_{b'}, \cos\theta_{b'}) \end{aligned} \tag{4.10}$$

Classically, the observables A, A', B, and B' can take value ± 1, leading to the value of Bell-CHSH operator to never exceed 2. However, for a two-qubit entangled state, if Alice and Bob perform specific measurements, the value of Bell-CHSH operator exceeds 2, thus violating the Bell-CHSH inequality and confirming the existence of non-local correlations in a quantum system.

In the game setting, Player 1 assumes that Player 2 will take preventive measures to increase nonlocal correlations. Thus, she/he will opt for a noise strategy that leads to the increase in difference between nonlocal correlations in presence of noise and nonlocal correlations in presence of noise and the preventive strategy of player 2. The incentive of Player 2, on the other hand, is to increase nonlocal correlations in the shared state beyond the nonlocal correlations in presence of noise, thus maximizing the gap between the final correlations in presence of noise and weak measurement reversal operations (i.e., $\langle B\rangle_{\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)}$), and the correlations in presence of noise (i.e., $\langle B\rangle_{\varepsilon(\rho_{new},\lambda)}$). Keeping the motives of both the players intact, utility functions for both players are formulated as

$$\begin{aligned} \$_1 &= \langle B\rangle_{\varepsilon(\rho_{new},\lambda)} - \langle B\rangle_{\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)} \quad \textit{and} \\ \$_2 &= \langle B\rangle_{\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)} - \langle B\rangle_{\varepsilon(\rho_{new},\lambda)} \end{aligned} \tag{4.11}$$

, respectively. Here, λ is the noise parameter strategy chosen by Player 1 for both qubits; and w_r is the weak measurement reversal strength strategy chosen by Player 2 for both qubits.

- **Game setting 2:** In this game setting, quantum correlations are quantified using geometric discord [Dakić *et al.*, 2010] given by

$$D_G = \frac{1}{4}[\|x\|^2 + \|T\|^2 - K] \tag{4.12}$$

where K is the maximum eigenvalue of $xx^T + TT^T$; x is a 3×1 vector with elements $x_i = \langle \sigma_i \otimes I\rangle$; T is a 3×3 correlation matrix with elements $T_{ij} = \langle \sigma_i \otimes \sigma_j\rangle$; and σ_1, σ_2, and σ_3 are the Pauli spin operators. Similar to the previous game setting, Player 1 will try to decrease the quantum correlations as opposed to the Player 2. Player 2 in turn, will apply the technique of weak measurement reversal operations to preserve correlations in the quantum system. Understanding the motivation of each player, their payoffs are defined as

$$\begin{aligned} \$_1 &= D_G(\varepsilon(\rho_{new}, \lambda)) - D_G(\zeta(\varepsilon(\rho_{new}, \lambda), w_r)) \quad and \\ \$_2 &= D_G(\zeta(\varepsilon(\rho_{new}, \lambda), w_r)) - D_G(\varepsilon(\rho_{new}, \lambda)) \end{aligned} \tag{4.13}$$

The formulation of a game in terms of the noise parameter, the state parameter, and weak measurement parameters can be utilized to effectively understand the best possible strategies of players to achieve maximum payoff. When each player opts for the best move to gain maximum payoff, the game will tend towards NE. Player 1 knows that Player 2 will perform strategies that may reduce the effect of his/her actions. Similarly, Player 2 is also aware that Player 1 introduces noise in the channel which will adversely affect nonlocal correlations in the underlying state. Since each player strives for his/her own maximum utility and decides to go for his/her best strategy, their choices will constitute the NE. Since NE optimizes the payoff of both players in a game scenario, the analysis of NE thus becomes interesting and worthwhile. It is demonstrated that the tendency of players in the game is to prefer a particular strategy, using the formulated payoffs for two players in Eqs. (4.11) and (4.13), and NE [Nash, 1950, 1951]. Moreover, Eqs. (4.11) and (4.13) show that $\$_1 = -\$_2$ and thus the designed game is a zero-sum game, also termed as a conflict game, and therefore all strategies in the game are pareto-optimal.

Furthermore, analysis is done on the strategies of both players to achieve maximum payoff independent of the perspective of a game. This maximum individual payoff is nothing, but the maximum possible payoffs of one player alone, without considering the best strategy for the other player. In other words, maximum possible payoff of one player among all strategies of Player 1 and Player 2 is evaluated. This is where the NE differs from the above mentioned maximum payoff for a single player. Unlike the above case, in a game theoretic scenario, the game tends towards the NE- defining the set of strategies which optimize payoffs of both players and not the individuals- which clearly differs from the maximum payoff (with respect to only one player).

For our analysis, 50 different noise (λ) strategies of Player 1 have been considered where $\lambda \in [0.05, 0.95]$, and 50 different weak measurement reversal operation (w_r) strategies of Player 2 where $w_r \in [0.05, 0.95]$ for 40 different ρ_{new} ($\eta \in [0°, 45°]$). Therefore, for each value of η, 2500 strategies of noise and weak measurement reversal operation have been closely studied so as to reach to an equilibrium. To achieve this, all strategies which were weakly dominated by all pure strategies as well as mixed strategies were eliminated [Osborne, 2003]. In other words, the strategies which would never be preferred over any pure strategy or any mixture of remaining strategies by the players due to lesser payoffs attained by opting for them, have been eliminated. The final table now contains ≤ 2 strategies for both players, and hence the study to evaluate and obtain NE for the desired game becomes convenient. For the case of a pure quantum state preparation, a pure-strategy NE exists with certainty. However, the effect of decoherence breaks this certainty of obtaining pure strategy NE. In such cases, the analysis can be extended to study the existence of NE formed by probabilistic mixture of strategies [Khan and Humble, 2019]. For simplicity, only pure quantum strategy NE is evaluated for the game. Thus, this method of elimination of dominated strategies and evaluation of a pure strategy NE has helped us study maximum and NE payoffs of players for different types of noisy channels. For this, both the game settings are studied considering that Player 1 introduces different noises such as an amplitude damping noise, a phase damping noise, or a depolarizing noise, on both the qubits. A description of three noisy channels is given in Section 1.3.5.

4.4 THE GAME SETTING QUANTIFYING QUANTUM CORRELATIONS USING THE BELL-CHSH OPERATOR

Figure 4.1 demonstrates individual maximum payoffs achieved by players for different noisy channels, against an input state depending on the angle η. For the amplitude damping noise, the payoff of Player 2 is better than the payoff of Player 1 for $\eta < 25^\circ$. However, for $\eta \geq 25^\circ$, the payoff of Player 1 exceeds that of Player 2. Similarly for the phase damping noise, the payoff of Player 2 is more than the payoff of Player 1 for $\eta < 22^\circ$. Interestingly, for both types of noise, the payoff of Player 2 decreases with the increase in degree of entanglement, i.e., Player 2 will be benefited more if the shared state between the two players is a non-maximally entangled state instead of a maximally entangled state. In fact, the highest payoff for Player 2 can be achieved only at very small values of the angle η for both noises. In contrast to the amplitude damping and phase damping noises, the payoff of Player 2 slightly increases with increase in entanglement for a depolarizing channel. In this case thus, Player 2 gets maximum payoff on sharing a maximally entangled Bell state instead of other general Bell-type non-maximally entangled states. Further, the maximum payoff of Player 2 is always higher than Player 1, if Player 1 is represented by a depolarizing noisy channel. Our results suggest another interesting observation that Player 2 attains maximum payoff for the depolarizing noise, and not for the amplitude damping or the phase damping noise. In addition, the depolarizing noise player gets a lesser payoff as compared to amplitude damping and phase damping noise players for the most range of η. Clearly, for a depolarizing channel Player 2 attains the maximum payoff, and the payoff of Player 1 is affected the maximum in comparison to other two noises. This suggests that the designed payoff effectively utilizes the destructive effects of depolarizing noise for the benefit of Player 2. Further in general, both players achieve a better payoff if Player 1 is represented by an amplitude damping noise instead of a phase damping noise.

In case of amplitude damping and phase damping noises, the maximum value of Player 1's payoff occurs for the highest value of noise parameter (here, 0.95) and highest value of weak measurement reversal strength (here, 0.95). In case of the amplitude damping noise, the maximum payoff of Player 2 occurs for noise ranging from 0.05 to 0.5 and weak measurement reversal strength ranging from 0.95 to 0.5 for increasing η. On the other hand, for the phase damping noise, the highest payoff for Player 2 occurs for lowest value of noise parameter (here, 0.05), but not for the highest value of weak measurement reversal strength. Interestingly, payoff of Player 2 attains maximum value for a fixed value of weak measurement reversal strength (w_r) depending on angle η such that

$$w_r = 1 - tan(\eta) \tag{4.14}$$

Since depolarizing noise results in maximum payoffs (for different η) which are quite different from the maximum payoffs in case of the other two noises, the strategies which lead to such payoffs are also unusual. High values of noise parameter (0.95) and weak measurement reversal strength (0.95) yield maximum payoff of Player 2 instead of Player 1. The maximum payoff of Player 1 is seen at low noise and high weak measurement reversal strength for $\eta > 17^\circ$.

In an actual game scenario, the payoff of each player may or may not attain the above discussed individual maximum payoff. This is because the players may not opt for extremely high or low values of noise parameters and/or weak measurement reversal strengths. Moreover, in a game scenario, the best strategy is to optimize payoffs of both players (NE) and not the individual ones. In addition, the game is defined as an imperfect complete information game, where one player does not know the strategy/ move, the other player will take. Imperfection implies a situation similar to a simultaneous move game where one player does not know the strategy opted by the other player. Therefore, personification of noisy channels and the technique of weak measurement reversal operations as players become an important mean of evaluating noise and weak measurement parameters at NE. Hence, players tend to opt for NE strategies from a game-theoretic perspective. In the game-theoretic analysis, our results show some interesting

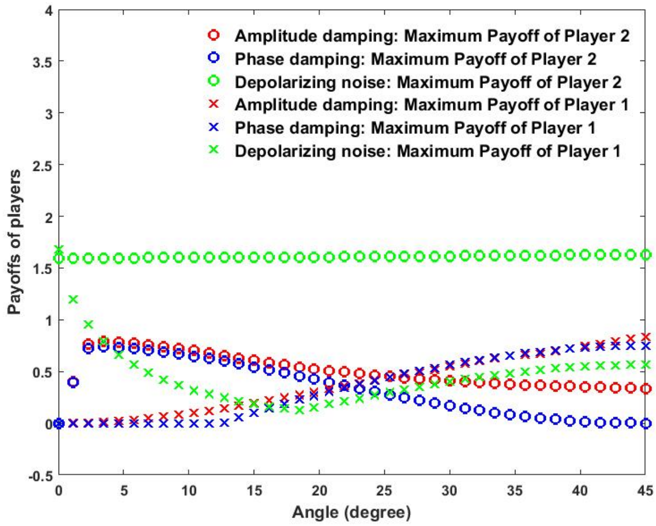

Figure 4.1 : Maximum payoffs of players vs angle η of the initial state ρ_{new} in the game setting 1

observations as against the analysis presented above. Figure 4.2 demonstrates the NE payoff and corresponding strategies of Player 1 (noise) and Player 2 (Weak measurement reversal), in case of an amplitude damping noise. For certain angles, a pure strategy NE could not be found [Khan and Humble, 2019] as depicted in Figure 4.2. In comparison to the previous case (Figure 4.1) where Player 2 gets a higher payoff for a larger set of η for the amplitude damping channel, the game theoretic analysis shows Player 1 to be a winner at equilibrium with a higher payoff as compared to Player 2. In addition, for a large range of values of η, the NE payoff of Player 1 increases and the NE payoff of Player 2 decreases with the increase in η. From the strategic point of view, a very high value of noise parameter strategy (roughly between 0.8 and 0.95) and a low value of weak measurement reversal strategy (0.05) contribute to the NE payoff. Surprisingly, the NE payoff of Player 2 is always negative and less than that of Player 1. The analysis thus shows that even the best attempts of both players to maximize their respective payoffs defined in Eq. (4.11) never yield $\langle B \rangle_{\zeta\left(\varepsilon(\rho_{new},\lambda),w_r\right)} \geq \langle B \rangle_{\varepsilon(\rho_{new},\lambda)}$. A comparison between Figure 4.1 and Figure 4.2, and NE analysis for the amplitude damping channel therefore raise questions over Bell-CHSH operator being a good measure for quantifying nonlocality in noisy conditions under the assumptions of this game.

In the other case, where Player 1 applies the phase damping noise on qubits, Player 2 wins the game at equilibrium. However, the payoffs are very small (of the order of 10^{-3}) at NE, as shown in Figure 4.3. The figure also shows that the NE payoff is achieved at maximum value of noise parameter (0.95) with weak measurement reversal strength varying as represented by Eq. (4.14).

For the depolarizing channel, for $\eta \geq 10^\circ$, Player 2 stands as a winner at equilibrium as represented in Figure 4.4. The NE payoff of Player 2 attains maximum value for a non-maximally entangled state at $\eta = 16.1538^\circ$. Moreover, the NE payoff of Player 2 lies between -0.038 to $+0.2975$ indicating that the depolarizing noise is much more advantageous for Player 2 as the obtained NE with this noise is mostly a win for Player 2, and the payoff thus attained is much more than the one obtained in the phase damping case. Figure 4.4 further demonstrates the strategies of both players which lead to NE. It depicts that there is no fixed pattern for the NE noise parameter strategies of Player 1 and the NE weak measurement reversal strategies of Player 2.

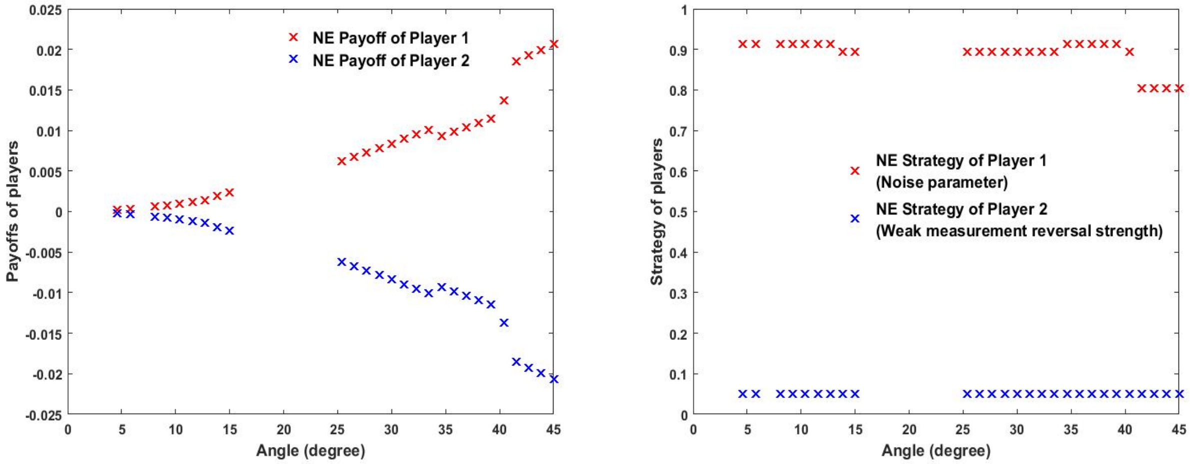

Figure 4.2 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies an amplitude damping noise in the game setting 1

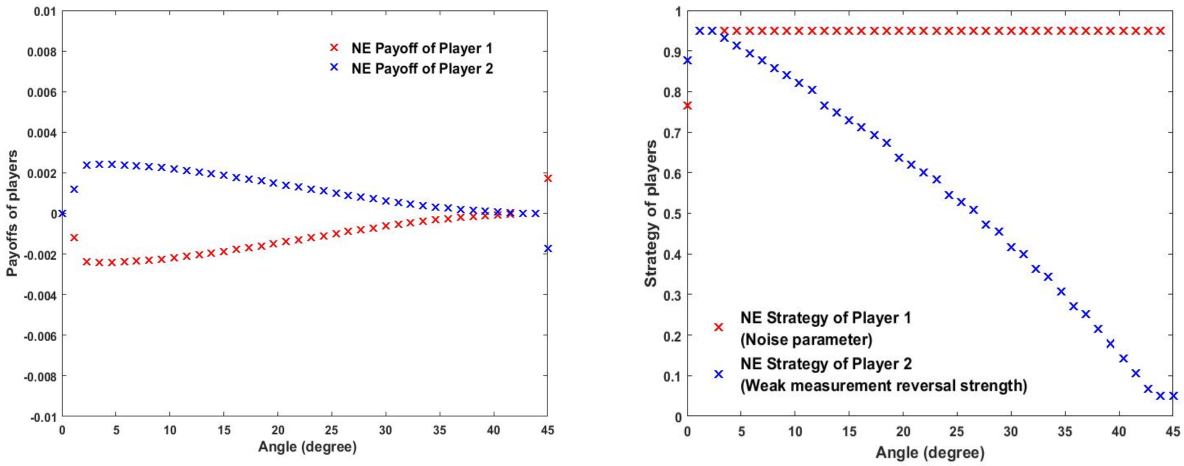

Figure 4.3 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies a phase damping noise in the game setting 1

The contrasting differences between the two analysis motivates us to discuss an alternate game scenario based on quantum discord as a measure of nonlocal correlations in the next section.

4.5 THE GAME SETTING QUANTIFYING QUANTUM CORRELATIONS USING GEOMETRIC DISCORD

In this section, the game setting 2 is analysed as defined by Eq. (4.13) considering that the noise player is represented either by an amplitude damping noise, a phase damping noise, or a depolarizing noise. Similar to the previous case, Figure 4.5 demonstrates individual maximum payoffs of players for the three noises. The maximum payoff of Player 2 is more than the maximum payoff of noise player in case of an amplitude damping noise for $\eta > 23.5^{\circ}$; in case of a phase damping noise for $\eta > 23^{\circ}$; and in case of a depolarizing noise for $\eta > 21^{\circ}$. In general, for all

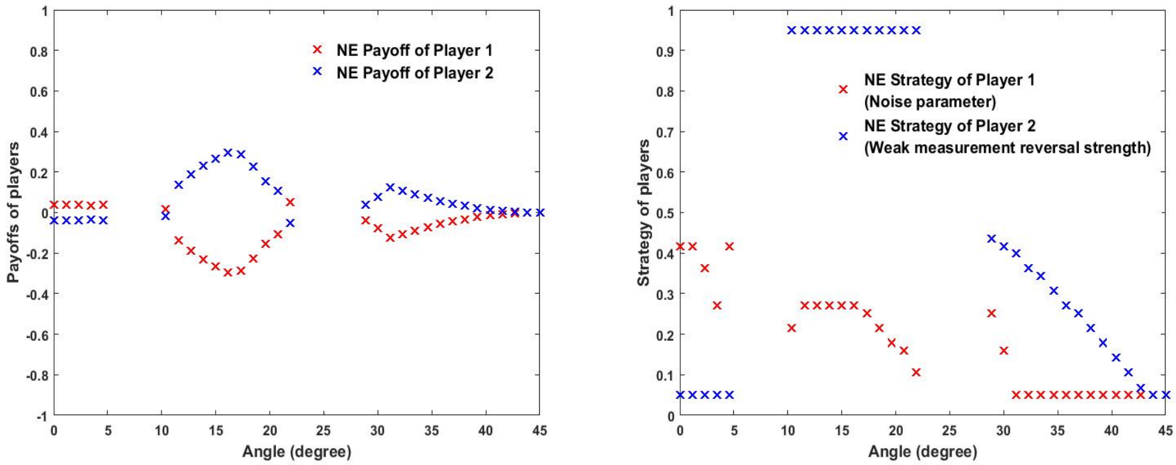

Figure 4.4 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies a depolarizing noise in the game setting 1

three noises, the maximum payoff of Player 2 decreases and the maximum payoff of Player 1 increases, with increase in the degree of entanglement of the shared state. Although noise player benefits more in case of maximally entangled shared state, Player 2 gets advantage by sharing a non-maximally entangled input state instead of a maximally entangled two-qubit state. In contrast to the game setting 1, the maximum payoff of Player 2 for the depolarizing noise is always less than that of other two noises. In addition, the maximum payoff of Player 1 is highest for depolarizing noise. Clearly, these results indicate that depolarizing noise benefits the noise player most. Moreover, the maximum payoff of Player 2 is the highest (0.5) for amplitude damping noise. However, this noise yields lower maximum payoff of Player 1 as compared to the other two noises. This value signifies the highest difference in geometric discord of the quantum state after and before application of weak measurement.

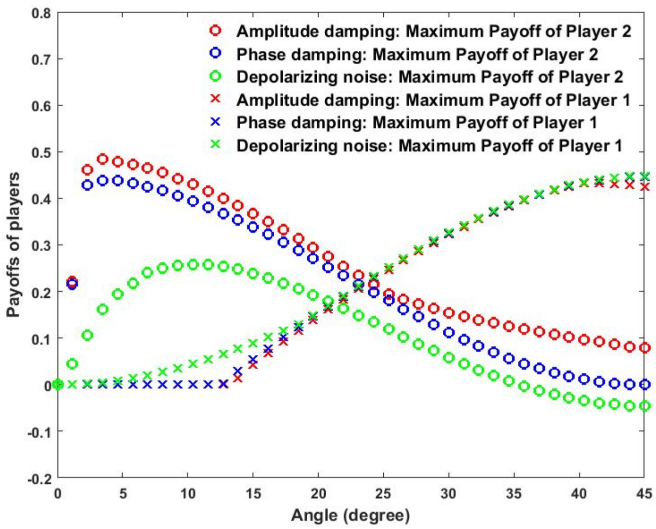

Figure 4.5 : Maximum payoffs of players vs angle η of the initial state ρ_{new} in the game setting 2

From the game theoretical analysis, the payoffs of players at equilibrium attain values different from the individual maximum payoff because at NE, each player tries to perform the best strategy to arrive at the highest payoff. For example, the NE payoff of players when the noise player applies an amplitude damping noise is given in Figure 4.6. Clearly, Player 2 is the winner at NE, and the NE payoff of Player 2 increases from 0 to the highest ($\simeq 0.12$) and later decreases with the increase in angle η. Therefore, in line with the above discussion for maximum payoff in the game setting 2, NE analysis also confirms that starting with a non-maximally entangled state instead of a maximally entangled two-qubit state is much more beneficial for Player 2 at least when the noise player is represented by the amplitude damping noise. Further, the strategies that lead to NE for this noise are also shown in Figure 4.6.

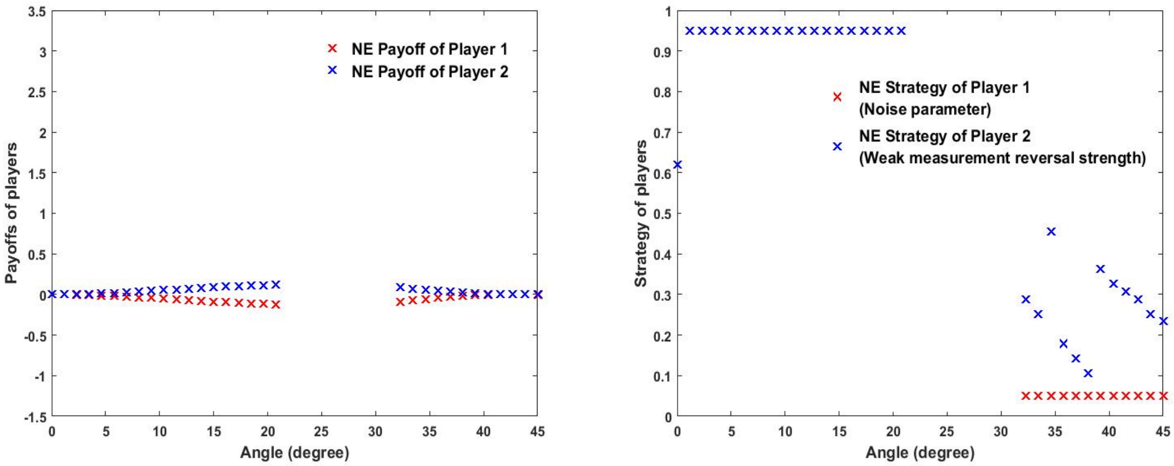

Figure 4.6 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies an amplitude damping noise in the game setting 2

Similar to the case of amplitude damping noise in the game setting 2, Player 2 wins the game for phase damping (Figure 4.7) as well, except at $\eta = \frac{\pi}{4}$. In case of a maximally entangled Bell state as initial resource, Player 2 looses the game for phase damping noise at equilibrium. However, the difference in players NE payoff values is very small, i.e., of the order of 10^{-3}. Surprisingly, strategies leading to NE in case of the phase damping noise are exactly the same as described in the game setting 1 where the Bell-CHSH operator is used as a measure of nonlocality. Interestingly, the NE weak measurement reversal strategy in both the game settings coincide with the optimal weak measurement reversal strength, given the initial state parameter, and weak measurement strength [Singh and Kumar, 2018c]. The figure suggests that even if noise player opts for the highest noise strength for the most destructive attempt, the noise player loses the game at equilibrium.

In case of depolarizing noisy channel, Player 1 wins the game at equilibrium. Though a very high value of noise parameter leads to NE for depolarizing noise, it is difficult to analyse the variation of weak measurement reversal strategy for Player 2 as is clear from Figure 4.8. Interestingly, this result is different from the one seen in game setting 1 where Player 2 won the game at NE for depolarizing noise.

4.6 CONCLUSIONS

When a quantum state is prepared and distributed, one cannot assure that the prepared state would pass through a noise-free channel. In real conditions, the system interacts with the environment and noise is inevitable. Therefore, noise clearly alters the state of a system under

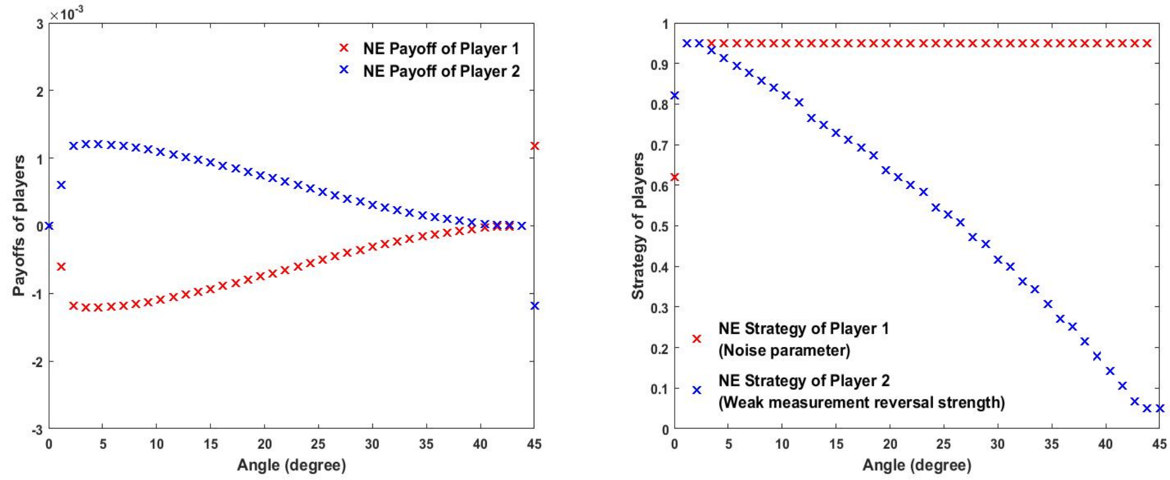

Figure 4.7 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies a phase damping noise in the game setting 2

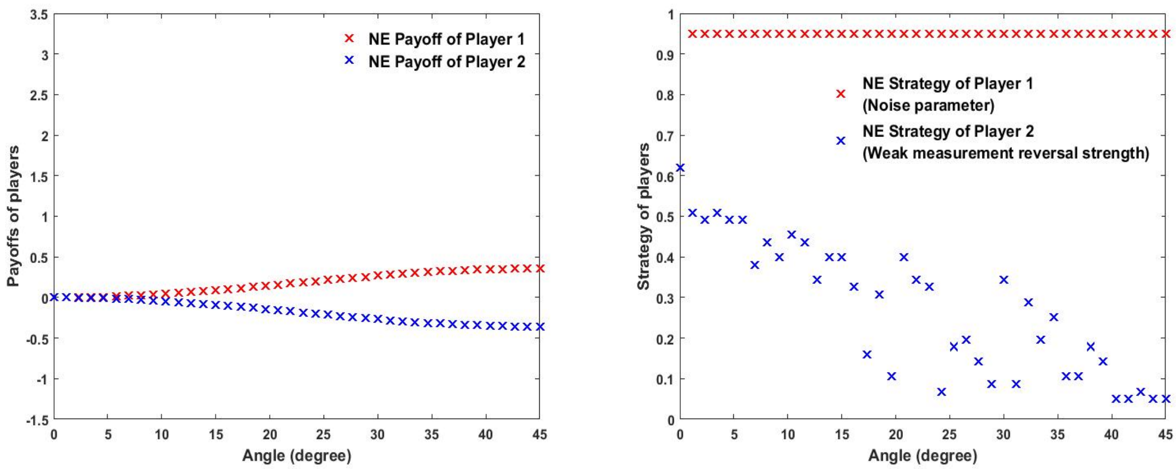

Figure 4.8 : NE payoffs and strategies of players vs angle η of the initial state ρ_{new} when Player 1 applies a depolarizing noise in the game setting 2

study in an adverse manner. In general, higher the noise, lesser are the nonlocal correlations in a quantum system. In the settings of entanglement distribution or a particular protocol, the need of a preventive measure to preserve nonlocal correlations or to reduce the degradation effect of noise is the utmost priority. A situation where one personifies noise as a player, assuming that the role of noise is to maximally reduce the nonlocal correlations between the qubits, allows one to analyse the effect of noise on nonlocal correlations with the perspective of a game. Such analysis for different game settings is imperative, considering nonlocal correlation to be a substantial resource for computational speed-up and an important requisite for quantum advantages in many quantum information processing protocols and computations. Although there exist several models to protect entanglement from noise, the applications of weak measurement and its reversal operations have been taken into consideration for the analysis. For a better understanding, the entire analysis is formulated in form of a game being played between two players, i.e., noise and weak measurement reversal operation. In order to study the effects of noise and weak measurement operations on nonlocal correlations, the amount of nonlocal correlations have been quantified using the Bell-CHSH operator and geometric discord in two different game settings. After weak elimination of the strategies dominated by any other strategy or mixture of more than one strategies, the NE for these games is evaluated.

An alternate description from the game, shows that the maximum payoff of Player 2 decreases and the maximum payoff of Player 1 increases with increase in the initial entanglement of the state in the second game setting for all three noises, for a large range of η under consideration. Nevertheless, in the first game setting also, the maximum payoff of Player 2 decreases and the maximum payoff of Player 1 increases with increase in the degree of entanglement of the shared state for amplitude damping and phase damping noise. However, in case of first game setting for depolarizing noise, the payoff of Player 2 increases with increase in entanglement of the initial state; and is always more than the maximum payoff of Player 1, which is indeed an interesting result. This outcome can be attributed to a wider range of expectation values of the Bell-CHSH operator from 0 to $2\sqrt{2}$ as compared to the value of geometric discord which varies between 0 and 0.5 only. This leads to a higher difference in nonlocal correlations in case of quantification by the Bell-CHSH operator rather than by geometric discord. On comparing the maximum payoff of Player 2 in two game settings for amplitude damping and phase damping noises, it is found that the amplitude damping noise is more beneficial in both the settings. Moreover, the maximum payoff of Player 2 in the second game setting occurs in case of the amplitude damping noise, indicating that the technique of weak measurement is highly effective in preserving nonlocal correlations to the maximum possible extent for the amplitude damping noise. Surprisingly, in the first game setting, the maximum payoff of Player 2 is the highest for a depolarizing noise channel. This exceptional result is attributed to the high range of values a Bell-CHSH operator can take for different states.

As opposed to the analysis of maximum payoff, at NE for the game setting where the Bell-CHSH operator is considered as a measure of nonlocality, Player 2 gets higher payoff than Player 1 in case of phase damping as well as depolarizing noise (for large range of η). However, when geometric discord is considered, then at NE, Player 2 gets higher payoff than Player 1 for amplitude damping as well as phase damping noises. From the strategic perspective in case of amplitude damping and depolarizing noises, one cannot evaluate the pattern of strategies to be opted by Player 2 in order to win the game at equilibrium. Interestingly for the phase damping noise, the NE win of Player 2 is attained by picking the weak measurement reversal strategy defined in Eq. (4.14). This relation is independent of noise and thus aids in getting maximum correlations irrespective of the strength of noise. Surprisingly, both game settings give contrasting results in terms of winning party at NE in case of amplitude damping and phase damping noises. For the amplitude damping noise at NE, a value of weak measurement reversal strength (= 0.05) and noise parameter (= 0.95) corresponds to a win for Player 1 in the first game setting, but another set of NE strategies correspond to a loss for the Player 1 in the second game setting. Similarly for depolarizing noise, a high value of noise strategy (= 0.95) corresponds to a NE loss for Player 2 in the second game setting, but an entirely different set of strategies are responsible for NE win of Player 2 in the first game setting. Clearly, this is due to the different structure of payoffs in both game settings. Overall, the analysis presented in the chapter gives a better insight of a two-player (noise and weak measurement) game setting, specifically highlighting the best weak measurement strategies corresponding to different noise parameters for any given state parameter of a general two-qubit state.

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