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Nonlocality, Entanglement, and Randomness in different conflicting interest Bayesian games

6.1 INTRODUCTION

Based on EPR's argument [Einstein *et al.*, 1935] of reality and localism, Bell designed an inequality [Bell, 1964; Clauser *et al.*, 1969] to delimit the boundaries between classical and quantum correlations. The Bell inequality is violated by all two-qubit pure entangled states confirming the presence of non-local correlations in an underlying entangled pure state. These correlations are very important to understand the foundational aspects of quantum theory, and its wide applicability in secure information processing, and computation [Einstein *et al.*, 1935; Batle and Casas, 2011; Bennett and Brassard, 1984; Ekert, 1991; Nielsen and Chuang, 2011]. With the advent of entanglement and nonlocality, the discussions regarding incommensurability of entanglement and quantum nonlocality continued to exist- in fact it is established that entanglement and nonlocality can be considered as distinct resources for quantum information and computation [Bennett *et al.*, 1999a; Horodecki *et al.*, 2003; Brunner *et al.*, 2005; A. Acín and Latorre, 2002; M´ethot and Scarani, 2007]. Eventually, it was observed that quantum discord which is a measure of nonlocal correlations even in the absence of entanglement, is a necessary resource for computational speed-up [Brodutch and Terno, 2011; Gu *et al.*, 2012]. Although a general intuition further suggests direct correspondence between entanglement, nonlocality, and randomness in an experiment, it was established that states with arbitrarily less entanglement and nonlocality can produce randomness close to 2 bits suggesting that maximal entanglement or nonlocality do not coincide with maximum randomness [Acin *et al.*, 2012].

In order to efficiently analyse the benefits of nonlocality in computational tasks, a special class of games, i.e., Bayesian games [Harsanyi, 1967a,b,c] serve as the best tool to represent quantum correlations as they contain the required element of incompleteness. These games contain partial information about the other player; since the type of at least one player in the game is a random variable. The first link between Bayesian games and nonlocality was proposed to demonstrate the relation between the game's payoffs and Cereceda inequalities [Cheon and Iqbal, 2008; Cereceda, 2001]. Later, Brunner and Linden showed a direct correspondence between the Bell inequality and payoffs of a general two player Bayesian game [Brunner and Linden, 2013]. They further discussed that nonlocal correlations play a substantial role in generating efficient quantum strategy for players to win and perform better than any classical strategy in Bayesian games.

In general, Bayesian games were studied and analysed with both players having common interests- either they jointly won or jointly lost the game. Howbeit, in real life scenarios, the interests of players may not always coincide, but may differ on several occasions. Conflicting interest games [Osborne, 2003] are those in which both the players have different preferences, like in the case of Battle of Sexes game [Osborne and Rubinstei, 1994]. On these lines, Pappa *et al.* [Pappa *et al.*, 2015] found that quantum correlations can also be used to win Bayesian games wherein players have conflicting interests. For this, they formulated a combination of CHSH and Battle of Sexes game, and demonstrated that for any classically correlated strategy, the sum of average payoffs of both players can never exceed $\frac{9}{8}$, and a fair classical equilibrium exists at the average payoff of each player being $\frac{9}{16}$. However, when the players share a maximally entangled two-qubit state and rely on the outcomes of various projective measurements as an advise to choose their quantum strategy, the picture turns out to be in their favour as the sum of average payoff exceeds the classical bound of $\frac{9}{8}$. The fair quantum equilibrium (where sum of average payoff of both players is 1.28) exists at the projective measurement settings which give maximum violation for the Bell-CHSH inequality, i.e., $2\sqrt{2}$ [Bell, 1964; Clauser *et al.*, 1969]. Followed by this, some interesting conflicting interest games have been proposed, in which the payoffs directly depend on Bell-type inequalities [Situ, 2016; Situ *et al.*, 2017; Rai and Pal, 2017]. Further, various three player conflicting interest games [Situ *et al.*, 2016; Bolonek-Lason, 2017] have also been presented, where the payoffs hold relation to three-qubit Bell-type inequalities [Svetlichny, 1987].

Apart from Bayesian games, there are various two-player games [Wiesner, 1983; Meyer, 1999; Vaidman, 1999; Eisert *et al.*, 1999] which demonstrate the advantages of quantum players with respect to their classical counterparts. Although most of the games describe the usefulness of maximally entangled Bell states, few also analyse the behaviour of non-maximally entangled states [Kaur and Kumar, 2017, 2019], mixed quantum states [Das and Chowdhury, 2018], and the failure of all quantum strategies [Anand and Benjamin, 2015]. Another pertinent question is the relation of nonlocality with the degree of entanglement present in the system [Bennett *et al.*, 1999a; Horodecki *et al.*, 2003; Brunner *et al.*, 2005]. Since the foundation of Bayesian games lies on the structure of nonlocal correlations, the effect of entanglement in an underlying state being shared by the players in winning quantum games is studied in this chapter. For this, the game proposed by Pappa *et al.* [Pappa *et al.*, 2015] is analysed using general two-qubit pure Bell states as resources- this allows one to analyse the behaviour of non-maximally entangled states towards playing a conflicting interest game.

For a Bayesian game comprising a conflicting interest game and a common interest game, we find that all pure entangled states quantum strategies surpass the classical limit to win the game; and the total payoffs of players increases with the increase in degree of entanglement of the shared resource. We further describe a quantum game by combining two conflicting interest games, i.e., Battle of the Sexes game [Osborne and Rubinstei, 1994] and Chicken game [Sugden, 2005]. Precisely, the players undergo Chicken game when type of both players is Type 1, or Battle of the Sexes game when type of at least one player is Type 0 [Brunner and Linden, 2013]. Since both the Battle of the Sexes and Chicken games demonstrate conflicting interests of the players involved, the analysis of quantum strategies for these games is substantial in understanding fully conflicting interest CHSH-type Bayesian games. Surprisingly, our results indicate that the players achieve a better payoff by sharing a set of non-maximally entangled pure states instead of the maximally entangled pure state. Interestingly, in both game settings the total payoff of players has a direct correspondence with the maximum expectation value of the Bell-CHSH operator for the underlying state. Clearly, for CHSH inequality involving pure two-qubit states, less entanglement always corresponds to a violation lesser than the one obtained using the maximally entangled state [Popescu and Rohrlich, 1992]. Although, in both game settings, the maximally entangled state violates the Bell-CHSH operator maximally, the setting where two conflicting interest games are merged as a Bayesian game lead to the interesting result that a team of players sharing a set of non-maximally entangled two-qubit pure states will win the game against a team of players sharing the maximally entangled two-qubit Bell state. This result will be very useful in formulation of a game where non-maximally entangled states offer more benefit over the maximally entangled state even in the settings of a CHSH inequality. Further, we also analyse two game settings for sharing mixed entangled states since mixed states are not explored much as far as game theory is concerned. For this, we consider the use of Werner [Werner, 1989] and Horodecki states [Horodecki *et al.*, 1996]. Unlike pure states, for mixed states, quantum strategies only offer an advantage over a certain range of state parameters. In fact, our results show that classical strategies may be more useful than quantum strategies for the use of mixed states even in the range where mixed states violate the Bell-CHSH inequality. Therefore, well within the limitations of the discussed game settings, mere violation of the Bell-CHSH inequality may not guarantee a team of quantum players a win over their classical opponents.

Moreover, considering that the use of tilted Bell-CHSH operator leads to high randomness when sharing a non-maximally entangled state, we further propose an efficient demonstration of a tilted version of Bell-CHSH inequality [Acin *et al.*, 2012] as common interest and conflicting interest Bayesian games. To the best of our knowledge, the analysis of tilted Bell-CHSH inequality has not been proposed under the premise of game theory even for a pure two-qubit maximally entangled Bell state. The extra term representing tilt in the expression for the tilted Bell-CHSH inequality prompted us to study the inequality under the framework of game theory. For this, we demonstrate a model to analyse the tilted Bell-CHSH inequality in the framework of a quantum game for general two-qubit entangled pure states as well as mixed states. For pure states, we observe that the quantum game where conflicting interest games are merged as a Bayesian game results in a much larger set of non-maximally entangled states offering advantage over the maximally entangled state as opposed to the quantum game where common interest games are merged. On the other hand, for mixed states the use of Horodecki states, where common interest games are merged, leads to quantum advantage for a relatively larger set of Horodecki states in comparison to the quantum game where conflicting interest games are merged. Our analysis suggests a similar observation in case of another important class of mixed states [Singh and Kumar, 2018c]; where the set of states offers a relatively better payoff than Horodecki states within the settings of the discussed game.

6.2 STRUCTURE OF BAYESIAN GAMES THAT HOLDS DIRECT RELATION WITH THE CHSH INEQUALITY

The structure of a CHSH game described in Section 1.5.3 can be utilized in the settings of a Bayesian game [Brunner and Linden, 2013], where the players are of different types ('x_A' and 'x_B') depending on the inputs ('x' and 'y') they receive from the referee. For instance, input $x = 0$ corresponds to type 0 of Alice, i.e., $x_A = 0$; input $x = 1$ corresponds to type 1 of Alice, i.e., $x_A = 1$; input $y = 0$ corresponds to type 0 of Bob, i.e., $x_B = 0$; and input $y = 1$ corresponds to type 1 of Bob, i.e., $x_B = 1$. Moreover, outputs ('a' and 'b') define the strategies ('y_A' and 'y_B') that the players opt for. Thus, in order to maintain the structure of a Bayesian game similar to the winning conditions of a CHSH game, when $x_A = x_B = 0$, or $x_A \neq x_B$, Alice and Bob get a non-zero payoff on choosing strategies (y_A and y_B) such that $y_A \oplus_2 y_B = 0$. Similarly, to satisfy the CHSH setting, when $x_A = x_B = 1$, the players get a non-zero payoff on choosing strategies $y_A(y_B) = 0(1)$ or $y_A(y_B) = 1(0)$. Hence, the overall condition of win in a CHSH game ($x_A \cdot x_B = y_A \oplus_2 y_B$) enables quantum players to exploit nonlocal correlations existing in the shared quantum system in order to win the game. Table 1.6 shows the payoffs attained by different types of Alice and Bob in a game with the above defined settings. In each cell, the first number represents the payoff of Player 1, i.e., Alice, and the second number represents the payoff of Player 2, i.e., Bob. The diagonal and off-diagonal terms appearing in the two payoff matrices are results of winning conditions of the game.

6.3 COMBINATIONS OF COORDINATION AND ANTI-COORDINATION GAMES

Coordination and anti-coordination games are those in which there is no fixed dominant strategy for a player. The players should coordinate (or anti-coordinate) in order to attain maximum payoff for themselves, in a respective game. Coordination means that players choose same or corresponding strategies at equilibria. On the other hand, anti-coordination signifies that the players choose different strategies or strategies different from the corresponding ones, at equilibria. Hence both games have multiple NE. In order to maintain the set-up of a game defined in Table 1.6, the players should coordinate and choose same strategies at equilibrium, when the logical AND of the type of players (x_A and x_B) is 0; and should anti-coordinate and choose different strategies at equilibrium, when the logical AND of the type of players (x_A and x_B) is 1.

6.3.1 The combination of a conflicting interest (Battle of the Sexes game) and a common interest game

Battle-of-the-Sexes (BoS) game is a two-player coordination game, where two players, a man and a woman wish to spend an evening together. However they have different choices about spending time together. The woman prefers to watch a movie whereas the man prefers to watch a football match. Table 1.3 shows the payoffs of man and woman in the game. One can see that there is no fixed dominating strategy for any player. Still, if man watches a football match, then woman gets a better payoff by opting for the same. Similarly, if woman watches a movie, then man gets a better payoff by opting to go for the movie. Thus, $\{Football, Football\}$ and $\{Movie, Movie\}$ are two pareto-optimal NE of the game. Further, the game is a conflicting interest one, since man prefers the $\{Football, Football\}$ equilibrium, and woman prefers the $\{Movie, Movie\}$ equilibrium. In contrast the common interest games are those in which both players benefit equally by opting for a NE strategy, and do not have any preference over the other [Osborne and Rubinstei, 1994].

For our purpose, BoS game is combined with a common interest anti-coordination game to demonstrate the role of quantum strategies in Bayesian games with CHSH-type dependence on payoffs. When the logical AND of the type of players is 0, then the players play the coordination BoS game and when the logical AND of the type of players is 1, then the players play an anti-coordination game, the payoffs of which are defined in Table 6.1. Here, the usefulness

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{16}{9}, \frac{8}{9}$	$0, 0$
$y_A = 1$	$0, 0$	$\frac{8}{9}, \frac{16}{9}$

(a) $x_A \cdot x_B = 0$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$0, 0$	$\frac{4}{3}, \frac{4}{3}$
$y_A = 1$	$\frac{4}{3}, \frac{4}{3}$	$0, 0$

(b) $x_A \cdot x_B = 1$

Table 6.1 : Payoffs of Alice and Bob when they either play a conflicting interest coordination game similar to the Battle of Sexes game or a common interest anti-coordination game

of general two-qubit pure Bell states as well as mixed quantum states (Werner and Horodecki states) is elaborated instead of a maximally entangled state [Pappa *et al.*, 2015] when the quantum strategies correspond to performing different single-parameter measurements on the qubits.

Classical scenario

There can be 16 different classical strategies since y_A and y_B can take two different (0 or 1) values, respectively, for two different (0 or 1) individual values of x_A and x_B. After analysing all possible classical strategy sets, one can conclude that for no classical strategy, the total payoff of the players exceeds 2. In addition, there are three NE for the game, i.e.,

- $y_A = 0$ irrespective of the value of x_A and $y_B = x_B$: This strategy leads to a pareto-optimal NE preferred by Alice since Alice gets a payoff of $\frac{11}{9}$ and Bob gets a payoff of $\frac{7}{9}$;
- $y_A = x_A$ and $y_B = \overline{x_B}$: This strategy also leads to a pareto-optimal NE preferred by none of the players since both players get an equal payoff of 1; and
- $y_A = \overline{x_A}$ and $y_B = 1$ irrespective of the value of x_B : This strategy further leads to a pareto-optimal NE preferred by Bob since Alice gets a payoff of $\frac{7}{9}$ and Bob gets a payoff of $\frac{11}{9}$

Quantum scenario

However, the situation becomes different when the players are allowed to share an entangled quantum state. Let us assume that the probability of Alice to be of type 0 ($x_A = 0$) be 'p' and the probability of Bob to be of type 0 ($x_B = 0$) be 'q'. By using any quantum strategy thereof, the sum of payoff of Alice ($\$_A$) and payoff of Bob ($\$_B$) is given as

$$\begin{aligned}\$_A + \$_B = \frac{8}{3}\big[pq\big(P^{00}_{00} + P^{11}_{00}\big) + p(1-q)\big(P^{00}_{01} + P^{11}_{01}\big) + (1-p)q\big(P^{00}_{10} + P^{11}_{10}\big) \\ + (1-p)(1-q)\big(P^{01}_{11} + P^{10}_{11}\big)\big]\end{aligned} \tag{6.1}$$

where probability P^{kl}_{ij} is defined as a product of two conditional probabilities, i.e., $P(y_A = k|x_A = i)P(y_B = l|x_B = j)$. In game theory, the notion '\$' represents payoff to a player in a game which may be considered as an incentive to the player for using a certain strategy. The payoff may be monetary or more spiritual such as happiness quotient. If there are two players Alice and Bob in the game then '$\$_A$' and '$\$_B$' represents payoff of Alice and Bob in the game. Further, the coefficient $\frac{8}{3}$ in Eq. (6.1) appears because of the values assumed as payoffs of players in Table 6.1. Moreover, the structure of the designed game requires that the sum of payoffs of both players in the games in Table 6.1 (a) and Table 6.1 (b) should be same, just as in our case, i.e., $\frac{16}{9} + \frac{8}{9} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$. Since the game in Table 6.1 (a) is a payoff representation of Battle-of-Sexes game, it must obey certain conditions such as, $\$_A > \$_B$ for strategies $y_A = 0$ and $y_B = 0$, and $\$_A < \$_B$ for strategies $y_A = 1$ and $y_B = 1$. The coefficient term could be any non-negative number. The payoff values are taken as an example of a Bayesian game combining a conflicting interest coordination game and a common interest anti-coordination game. Assuming the strategies $y_A = 0$ and $y_B = 0$ correspond to measurement outcomes yielding positive eigenvalue (+1), and strategies $y_A = 1$ and $y_B = 1$ correspond to measurement outcomes yielding negative eigenvalue (−1); the expectation value $E(ij) = E(x_A = i, x_B = j)$ can be defined as $P^{00}_{ij} - P^{01}_{ij} - P^{10}_{ij} + P^{11}_{ij}$. Thus the total payoff of players can be re-expressed as

$$\$_A + \$_B = \frac{4}{3}\big[1 + pqE(00) + p(1-q)E(01) + (1-p)qE(10) - (1-p)(1-q)E(11)\big] \tag{6.2}$$

For simplicity, it is assume that $p = q = \frac{1}{2}$. Under this assumption, the sum of the payoffs of Alice and Bob hold a direct relation with the Bell-CHSH operator $\langle B\rangle$ [Bell, 1964] as

$$\$_A + \$_B = \frac{4}{3}\Big[1 + \frac{\langle B\rangle}{4}\Big] \tag{6.3}$$

In order to attain a better payoff in the quantum scenario in comparison to classical scenario, it is considered that Alice and Bob share a general two-qubit entangled quantum state. If x_A and x_B are types of Alice and Bob, respectively, and y_A and y_B are their respective moves in the game, then the measurements performed by them as a team on their qubits can be represented as $A^{y_A}_{x_A}$ and $B^{y_B}_{x_B}$ [Pappa *et al.*, 2015], where

$$\begin{aligned} A^a_0 &= |\Phi_a(0)\rangle\langle\Phi_a(0)|, \\ A^a_1 &= |\Phi_a(\frac{\pi}{4})\rangle\langle\Phi_a(\frac{\pi}{4})|, \\ B^b_0 &= |\Phi_b(\lambda)\rangle\langle\Phi_b(\lambda)|, \\ B^b_1 &= |\Phi_b(-\lambda)\rangle\langle\Phi_b(-\lambda)| \end{aligned} \tag{6.4}$$

The basis of measurement $|\phi_0(\theta)\rangle$ and $|\phi_1(\theta)\rangle$ are the same as $|v_0(\theta)\rangle$ and $|v_1(\theta)\rangle$ respectively, as defined in Eq. (1.32).

Moreover, one can also relate this quantum strategy with the experimental settings in a Bell-CHSH experiment [Bell, 1964; Clauser *et al.*, 1969]. For example, in a Bell-CHSH experimental set-up, Alice randomly chooses to perform a measurement $Q = \begin{bmatrix} cos\theta_1 & sin\theta_1 e^{-i\phi_1} \\ sin\theta_1 e^{i\phi_1} & -cos\theta_1 \end{bmatrix}$ or $R = \begin{bmatrix} cos\theta_1^{'} & sin\theta_1^{'} e^{-i\phi_1^{'}} \\ sin\theta_1^{'} e^{i\phi_1^{'}} & -cos\theta_1^{'} \end{bmatrix}$ and similarly Bob also randomly chooses to perform a measurement $S = \begin{bmatrix} cos\theta_2 & sin\theta_2 e^{-i\phi_2} \\ sin\theta_2 e^{i\phi_2} & -cos\theta_2 \end{bmatrix}$ or $T = \begin{bmatrix} cos\theta_2^{'} & sin\theta_2^{'} e^{-i\phi_2^{'}} \\ sin\theta_2^{'} e^{i\phi_2^{'}} & -cos\theta_2^{'} \end{bmatrix}$, on their respective qubits. The expectation value of Bell-CHSH operator thus designed is equal to $E(QS) + E(RS) + E(RT) - E(QT)$ which is same as the expectation value of Bell-CHSH operator $\langle B \rangle$ obtained in Eq. (6.3). However, the measurements performed by Alice and Bob as quantum strategies in the game (Eq. (6.4)) correspond to the measurements in the experimental set-up when $\theta_1 = 0$, $\theta_1^{'} = \frac{\pi}{2}$, $\theta_2 = 2\lambda$, $\theta_2^{'} = -2\lambda$, and $\phi_1 = \phi_1^{'} = \phi_2 = \phi_2^{'} = 0$. This can also be termed as restricted one parameter (λ) measurements. The above set of measurements are choosen so as to achieve maximum expectation value of the Bell-CHSH operator for a general two-qubit Bell state. Nevertheless, quantum advantage can still be witnessed in a wide class of pure and mixed states.

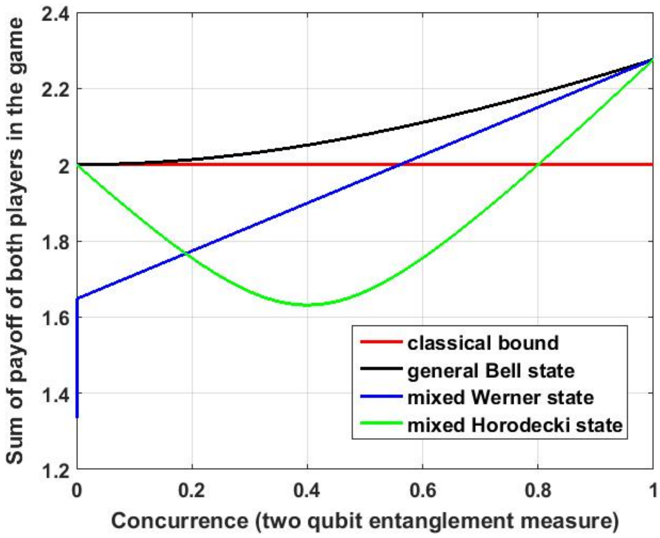


Figure 6.1 : Relation between the sum of payoffs of players with the concurrence for a combination of BoS and common interest anti-coordination games when different quantum states are shared among the players

Furthering proceeding to analyse the total payoff of players in the game, it is considered that Alice and Bob share a general two-qubit Bell state given by

$$|\phi\rangle_{Bell} = cos\theta|00\rangle + sin\theta|11\rangle \tag{6.5}$$

Clearly, the state in Eq. (6.5) violates the Bell-CHSH inequality for all values of $\theta\varepsilon(0, \frac{\pi}{4}]$, and the violation increases with the increase in degree of entanglement of the shared state, i.e, the Bell-CHSH inequality is maximally violated by the maximally entangled state, and not by a non-maximally entangled state. The total payoff of both players sharing a general two-qubit Bell state can be evaluated as

$$\$_A + \$_B = \frac{2}{3}\left[2 + sin2\theta sin2\lambda + cos2\lambda\right] \tag{6.6}$$

One can see that the same value of total payoff can be achieved when players share a two-qubit arbitrary state $|\psi_{arbitrary}\rangle = cos\theta|00\rangle + e^{i\phi} sin\theta|11\rangle$, and perform measurements given in Eq. (6.4) in an arbitrary orthogonal basis given by $|\phi_0(\theta)\rangle = cos\theta|0\rangle + e^{i\phi/2}sin\theta|0\rangle$ and $|\phi_1(\theta)\rangle = -e^{-i\phi/2}sin\theta|0\rangle + cos\theta|1\rangle$. Moreover, it can be evaluated that the maximum value of the summed payoff in Eq. (6.6) is achieved at $\lambda = \frac{1}{2}tan^{-1}\big(sin2\theta\big)$. Thus, the black line in Figure 6.1 shows that the sum of payoffs of Alice and Bob thereof always exceeds the classical bound (red line) of 2. In other words, general two-qubit pure Bell states offer quantum advantage for all values of state parameter θ in a CHSH-type Bayesian game setting including conflicting and common interest games. Therefore, although the maximally entangled Bell state gives maximum total payoff of 2.276, non-maximally entangled Bell states for $0 < \theta < \frac{\pi}{4}$ also give better total payoff than any other classical strategy.

As examples of mixed states for this scenario, Werner class states [Werner, 1989] and Horodecki states [Horodecki *et al.*, 1996] are considered. Firstly, the Werner states is described as a linear combination of a Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$ and a maximally mixed state $I_4 = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$ as given below in Eq. (6.7).

$$\rho_{werner} = \gamma|\phi^+\rangle\langle\phi^+| + \frac{1-\gamma}{4}I_4 \tag{6.7}$$

where γ is a state parameter. Acin *et al.* [Acín *et al.*, 2006] have shown that Werner states violate the Bell-CHSH inequality only for $\gamma > \frac{1}{\sqrt{2}}$, i.e., even though they possess entanglement for $\frac{\gamma}{3} < \gamma \leq \frac{1}{\sqrt{2}}$, they violate the inequality only for states having concurrence more than $\frac{3}{2\sqrt{2}} - \frac{1}{2}$. Moreover, the total payoff of both the players sharing a Werner state is

$$\$_A + \$_B = \frac{2}{3}\big[2 + \gamma sin2\lambda + \gamma cos2\lambda\big] \tag{6.8}$$

Clearly the maximum value of the summed payoff in Eq. (6.8) is achieved at $\lambda = \frac{\pi}{8}$. Thus, the blue line in Figure 6.1 shows that the sum of payoffs of Alice and Bob thereof exceeds the classical bound (red line) of 2 for $\gamma > \frac{1}{\sqrt{2}}$ or $C = (1.5\gamma - 0.5) > 0.5606$. In other words, Werner states offer quantum advantage for a fixed range of state parameters only, and for $\gamma \leq \frac{1}{\sqrt{2}}$, classically defined strategies yield a better payoff than a team of Alice and Bob equipped with quantum strategies.

As mentioned above, another class of mixed states that are analysed are Horodecki states which are defined as a linear combination of a Bell state $|\psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$ and a separable state $|00\rangle$ as shown below in Eq. (6.9)

$$\rho_{horodecki} = \mu|\psi^+\rangle\langle\psi^+| + (1-\mu)|00\rangle\langle 00| \tag{6.9}$$

where μ is the state parameter. It was established that Horodecki states violate the Bell-CHSH inequality for $\mu > \frac{1}{\sqrt{2}}$, i.e., even though they possess entanglement for all values of the state parameter, they violate the Bell-CHSH inequality only for states having concurrence greater than $\frac{1}{\sqrt{2}}$ [B. Horst and Miranowicz, 2013]. Thus, the total payoff of both the players sharing a Horodecki state is

$$\$_A + \$_B = \frac{2}{3}\big[2 + \mu sin2\lambda + \big(1-2\mu\big)cos2\lambda\big] \tag{6.10}$$

As in the previous cases, one can calculate the maximum value of the summed payoff in Eq. (6.10) which is achieved at $\lambda = \frac{1}{2}tan^{-1}\frac{\mu}{1-2\mu}$. Therefore, the green line in Figure 6.1 demonstrates that the sum of payoffs of Alice and Bob thereof exceeds the classical bound (in red) of 2 for $C(=\mu) > 0.8$. In other words, Horodecki states offer quantum advantage for a fixed range of state parameter only. Interestingly, for the range $\frac{1}{\sqrt{2}} < C(=\mu) < 0.8$, even though Horodecki states violate the Bell-CHSH inequality, they do not provide any quantum advantage in the designed set-up of the game. This can be attributed to the difference in measurement settings of our game from the Bell-CHSH experimental set-up, as explained above.

Hence, any general two-qubit pure Bell state, any Werner class for $\gamma > \frac{1}{\sqrt{2}}$, or any Horodecki state for $\mu > 0.8$, can be used as a quantum resource by Alice and Bob in order to exceed the total payoff from the classical bound of 2. In other words, a large set of pure and mixed states have been analysed for which the above defined quantum strategy gives better summed payoff than any strategy opted by classical players. Thus, it can be concluded that just like pure states, mixed quantum states are also useful for the players in the game described above. However, for a fixed value of concurrence, pure states help attain better payoff in the game than mixed states.

6.3.2 The combination of two conflicting interest games (Battle of the Sexes game and Chicken game)

In the previous subsection, the analysis was performed for the scenario where players either got engaged in a conflicting interest game or a common interest game, depending on their inputs. In this section, an attempt is made to combine two conflicting interest games in order to understand the benefits of nonlocality when the players always have conflicting interests. A simple example of conflicting interest anti-coordination game, also known as the hawk-dove game or snowdrift game, is Chicken game [Sugden, 2005]. Firstly, Chicken game is briefly described through two drivers who drive towards each other, and can suffer a head-on collision if both keep driving straight. On the other hand, if one driver swerves and the other does not, the one who swerved will be named "chicken", signifying him as a coward. Therefore, the coward gets a lower payoff than the one who drives straight. However, the lowest payoff incurs when none of the players risk to be a chicken or a coward, but rather choose to go straight, i.e., if both drivers do not swerve. Thus, there is no dominant strategy for this game as indicated in Table 6.2. One can see that if Driver 1 goes straight,

Driver 1 \ Driver 2	Swerve	Straight
Swerve	$0,0$	$-1,+1$
Straight	$+1,-1$	$-5,-5$

Table 6.2 : A payoff matrix for the Chicken game

then Driver 2 gets a better payoff by opting to swerve rather than going straight. Similarly, if Driver 2 goes straight, then Driver 1 gets a better payoff by opting to swerve instead of driving straight. Thus $\{Swerve, Straight\}$ and $\{Straight, Swerve\}$ are two pareto-optimal NE of the game. Further, the game is a conflicting interest game as Driver 1 prefers the $\{Straight, Swerve\}$ equilibrium, and Driver 2 prefers the $\{Swerve, Straight\}$ equilibrium. Due to no fixed dominating strategy, the game can be termed as an anti-coordination one. Being a conflicting interest anti-coordination game involving greed between the players, but no fear, analysing the effects of quantum strategies on such games becomes an interesting problem. Thus, BoS game is combined with Chicken game to demonstrate the role of quantum strategies in conflicting interest Bayesian games with CHSH-type dependence on payoffs. For this, it is considered that when the logical AND of the type of players is 0, then the players play the coordination BoS game and when the logical AND of the type of players is 1, then the players play the anti-coordination Chicken game; the payoffs of which are defined in Table 6.3.

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{4}{3}, \frac{2}{3}$	$0, 0$
$y_A = 1$	$0, 0$	$\frac{2}{3}, \frac{4}{3}$

(a) $x_A \cdot x_B = 0$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$0, 0$	$-\frac{1}{2}, \frac{5}{2}$
$y_A = 1$	$\frac{5}{2}, -\frac{1}{2}$	$-1, -1$

(b) $x_A \cdot x_B = 1$

Table 6.3 : Payoffs of Alice and Bob when they either play a conflicting interest coordination game similar to the Battle of Sexes game or a conflicting interest anti-coordination game similar to the Chicken game

Classical scenario

For the above defined game, the maximum total payoff achieved by opting for any of the strategies is $\frac{3}{2}$. Thus, there are three NE for the game, i.e.,

- $y_A = 0$ irrespective of the value of x_A and $y_B = x_B$: This strategy leads to a pareto-optimal NE preferred by Bob since Alice gets a payoff of $\frac{13}{24}$ and Bob gets a payoff of $\frac{23}{24}$;
- $y_A = x_A$ and $y_B = \overline{x_B}$: This strategy also leads to a pareto-optimal NE preferred by Alice since Alice gets a payoff of $\frac{9}{8}$ and Bob gets a payoff of $\frac{3}{8}$; and
- $y_A = 1$ and $y_B = 1$ irrespective of the value of x_A and x_B : This strategy leads to non pareto-optimal NE since Alice gets a payoff of $\frac{1}{4}$ and Bob gets a payoff of $\frac{3}{4}$.

Quantum scenario

Unlike the previous game setting, where a conflicting interest game is combined with a common interest game, the combination of two conflicting interest games lead to some interesting observations. Similar to the previous case, let us start with assuming the probability of Alice to be of type 0 ($x_A = 0$) be 'p' and the probability of Bob to be of type 0 ($x_B = 0$) be 'q'. By using any quantum strategy thereof, the sum of payoffs of Alice and Bob is given as

$$\begin{aligned}\$_A + \$_B = 2\big[&pq\big(P_{00}^{00} + P_{00}^{11}\big) + p(1-q)\big(P_{01}^{00} + P_{01}^{11}\big) + (1-p)q\big(P_{10}^{00} + P_{10}^{11}\big) \\ &+ (1-p)(1-q)\big(P_{11}^{01} + P_{11}^{10} - P_{11}^{11}\big)\big]\end{aligned} \tag{6.11}$$

where the probability P_{ij}^{kl} are defined earlier. The occurrence of the factor 2 in Eq. (6.11) can also be understood following the description of previous game setting. Further, considering the strategies $y_A = 0$ and $y_B = 0$ correspond to measurement outcomes yielding positive eigenvalue (+1) and strategies $y_A = 1$ and $y_B = 1$ correspond to measurement outcomes yielding negative eigenvalue (−1), the expectation value $E(ij) = E(x_A = i, x_B = j)$ can be defined as $P_{ij}^{00} - P_{ij}^{01} - P_{ij}^{10} + P_{ij}^{11}$. Thus Eq. (6.11) is re-expressed as

$$\begin{aligned}\$_A + \$_B = 2\Big[&\frac{1}{2} + \frac{1}{2}\big\{pqE(00) + p(1-q)E(01) + (1-p)qE(10) - (1-p)(1-q)E(11)\big\} \\ &- (1-p)(1-q)P_{11}^{11}\Big]\end{aligned} \tag{6.12}$$

As earlier, for simplicity, it is assumed that $p = q = \frac{1}{2}$. Therefore, Eq. (6.12) can be re-expressed in terms of the Bell-CHSH operator $\langle B \rangle$ [Bell, 1964] along with an additional conditional probability term as

$$\$_A + \$_B = 2\Big[\frac{1}{2} + \frac{\langle B\rangle}{8} - \frac{P_{11}^{11}}{4}\Big] \tag{6.13}$$

Since the game is being described under CHSH setting, it is important to note that a maximally entangled pure state will violate the Bell-CHSH inequality maximally. Furthermore, since $P_{11}^{11} > 0$ and classically $\langle B\rangle_{max} = 2$, these values can be replaced in Eq. (6.13) and verify that the classical bound for this game is $\leq \frac{3}{2}$.

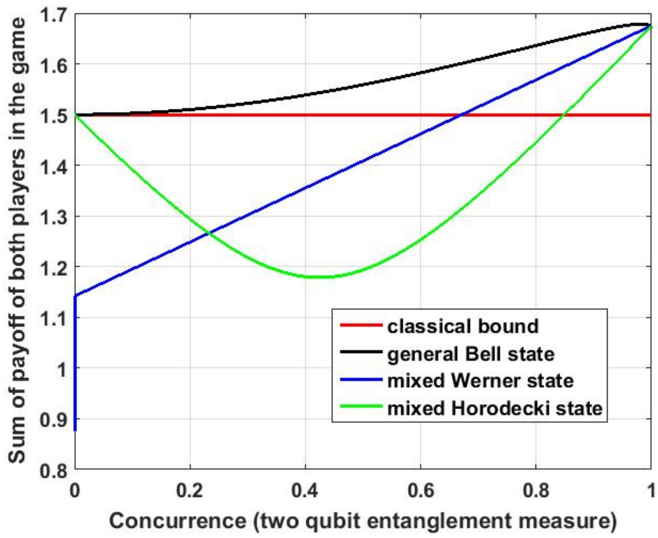


Figure 6.2 : Relation between the sum of payoffs of players with the concurrence for a combination of BoS and Chicken games when different quantum states are shared among the players

Clearly, when players share a general two-qubit Bell state as given by Eq. (6.5) and perform measurements defined in Eq. (6.4), then the total payoff of both players is

$$\$_A + \$_B = \frac{1}{8}\big[7 + 5sin2\theta sin2\lambda + \big(cos2\theta + 4\big)cos2\lambda\big] \tag{6.14}$$

As explained earlier, one can also attain the same payoff when players share a two-qubit arbitrary state $|\psi_{arbitrary}\rangle = cos\theta|00\rangle + e^{i\phi}sin\theta|11\rangle$, and perform measurements given in Eq. (6.4) in an arbitrary orthogonal basis given by $|\phi_0(\theta)\rangle = cos\theta|0\rangle + e^{i\phi/2}sin\theta|0\rangle$ and $|\phi_1(\theta)\rangle = -e^{-i\phi/2}sin\theta|0\rangle + cos\theta|1\rangle$. Therefore, the maximum value of the summed payoff in Eq. (6.14) is achieved at $\lambda = \frac{1}{2}tan^{-1}\frac{5sin2\theta}{cos2\theta + 4}$. Evidently, the black line in Figure 6.2 shows that the sum of payoffs of Alice and Bob always exceeds the classical bound (red line) of $\frac{3}{2}$. In other words, general two-qubit pure Bell states offer quantum advantage for all values of the state parameter θ. Surprisingly, the maximally entangled Bell state ($\theta = \frac{\pi}{4}$) does not give maximum total payoff. However, a non-maximally entangled state at $\theta = 40.188°$ with concurrence 0.986 gives maximum summed payoff of approx. 1.6819 for both players. This is contrary to our general belief that the maximally entangled pure state will always be more efficient than a non-maximally entangled pure state, at least under the setting of the discussed game. Although, the value of Bell-CHSH operator $\langle B\rangle$ is maximum at $\theta = 45°$, the value of $\langle B\rangle - 2P_{11}^{11}$ at $\theta = 40.188°$ is more than the value of $\langle B\rangle - 2P_{11}^{11}$ at $\theta = 45°$ while maximizing total payoff $\$_A + \$_B$. Since $P_{11}^{11} = P(y_A = 1, y_B = 1|x_A = 1, x_B = 1)$ is a very small number, an angle very close to 45° (with concurrence very close to 1) gives the maximum total payoff. Furthermore, the analysis shows that all non-maximally entangled Bell states in the range $0 < \theta \leq \frac{\pi}{4}$ give better total payoff than any classical strategy. Interestingly, non-maximally entangled states for $34.1804^\circ \leq \theta < 45^\circ$ lead to a better payoff in the game in comparison to a maximally entangled Bell state.

As examples of entangled mixed states, Werner and Horodecki states are again considered as defined in Eq. (6.7) and Eq. (6.9), respectively. When the players share a mixed Werner class state, then the total payoff of both players is

$$\$_A + \$_B = \frac{1}{8}\left[7 + 5\gamma sin2\lambda + 4\gamma cos2\lambda\right] \tag{6.15}$$

Thus, the maximum value of the summed payoff in Eq. (6.15) is achieved at $\lambda = \frac{1}{2}tan^{-1}\frac{5}{4}$. The blue line in Figure 6.2 describes that the sum of payoffs of Alice and Bob exceeds the classical bound (red line) of $\frac{3}{2}$ for $C(= 1.5\gamma - 0.5) > 0.6712$. Similar to the previous case, a mixed Werner class state offers quantum advantage for a fixed range of state parameter only. However, unlike the previous case, even though Werner states exhibit non-local correlations for $\frac{1}{\sqrt{2}} < \gamma \leq 0.7808$, i.e. $0.5606 < C < 0.6712$, they still fail to provide any advantage to quantum players in this particular game setting. The failure of Werner states to provide advantage to quantum players over classical players in the above region even though they violate the Bell-CHSH inequality in that particular region is attributed to the combination of two conflicting interest games in comparison to the previous case (combination of a conflicting interest game with a common interest game) where no such result was obtained.

Similarly, when the players share a Horodecki state, then the total payoff of both players is

$$\$_A + \$_B = \frac{1}{8}\left[7 + 5\mu sin2\lambda + (5 - 9\mu)cos2\lambda\right] \tag{6.16}$$

One can evaluate that the maximum value of the summed payoff in Eq. (6.16) is achieved at $\lambda = \frac{1}{2}tan^{-1}\frac{5\mu}{5-9\mu}$. Thus, the green line in Figure 6.2 shows that the sum of payoffs of Alice and Bob exceeds the classical bound (red line) of $\frac{3}{2}$ for $C(= \mu) > 0.85$ only. Hence, the players do not get a better payoff on sharing a Horodecki states in the range of state parameter, i.e., $\frac{1}{\sqrt{2}} < C(= \mu) \leq 0.85$, even though the quantum state possesses non-local correlations in that range. It is interesting to see that the range of state parameter where the state violates the Bell-CHSH inequality is increased in comparison to the previous game setting where $C > 0.8$ is obtained as the value of concurrence which leads to quantum advantage. Clearly, the results obtained for combination of two conflicting interest games are quite different and interesting from the case where the combination of a common interest and a conflicting interest game are considered.

This study which highlights the advantages offered by non-maximally entangled pure states in comparison to the maximally entangled pure state due to the terms such as P_{11}^{11} will definitely motivate us to study games where such contribution to total payoff may have more drastic effect towards the final output of the game. In addition the analysis also highlights that under the proposed game settings use of mixed entangled states may not always result in advantage even if the state violated the Bell-CHSH inequality. Moreover, it further paves way to study the important property of randomness in quantum states, which is discussed in the following section.

6.4 REPRESENTATION OF THE TILTED BELL-TYPE INEQUALITY IN A BAYESIAN GAME SETTING

Like nonlocal correlations, randomness is also inherent to the foundations of quantum mechanics. The outcomes of an experiment designed to test the Bell inequality formalism are always random (for violation of the inequality), i.e., measurement outcomes cannot be deterministically predicted within quantum theory. Acin *et. al* [Acin *et al.*, 2012] have shown that the maximal violation of the Bell-CHSH inequality involves the generation of only 1.23 bits of randomness instead of generating 2 bits of global randomness. Interestingly, they found that less entangled states can produce randomness close to 2 bits, showing that there is no direct relation between entanglement (or nonlocality) with randomness. In order to capture high randomness in non-maximally entangled states, a specific class of Bell-type inequality was defined, i.e.,

$$I_{\alpha}^{\beta} = \beta\langle A_0\rangle + \alpha\langle A_0B_0\rangle + \alpha\langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \tag{6.17}$$

The inequality in Eq. (6.17) depends on two parameters, i.e., $\alpha \geq 1$ and $\beta \geq 0$. The classical bound of the inequality is $\beta + 2\alpha$ and the maximum violation of the inequality is attained at $I_{\alpha}^{\beta} = 2\sqrt{(1+\alpha^2)\left(1+\frac{\beta^2}{4}\right)}$. For simplicity, the inequality I_1^{α} is considered, known as tilted Bell-CHSH inequality and its effect when two players share a two-qubit quantum state for playing a Bayesian game is demonstrated.

To the best of our knowledge, no Bayesian game representation of the tilted version of the Bell-CHSH operator as shown in Eq. (6.18) has been encountered. Therefore, the following study attempts to represent the input-output relation of the tilted counterpart of the CHSH games as type of players-reward relation in Bayesian games.

$$I_1^{\alpha} = \alpha\langle A_0\rangle + \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \tag{6.18}$$

In order to collectively represent the operator into the settings of a Bayesian game, there is need to incorporate the additional $\langle A_0\rangle$ term in the payoff table. Therefore, it is assumed that Alice and Bob play a tilted coordination game whenever Alice's input is $x_A = 0$. However, for Alice's input $x_A = 1$ and Bob's input $x_B = 0$, the players play the usual un-tilted version of the coordination game. Further, when Alice's and Bob's inputs are 1 each, i.e., $x_A = x_B = 1$, they play an untilted anti-coordination game. The players in these coordination and anti-coordination games can have varying interests- either common interest or conflicting interest. Table 1.6 shows the payoff of Alice and Bob in a general Bayesian game where the payoffs depend on the type of players in a fashion similar to the winning input-output relation in a tilted CHSH game as defined by the tilted Bell-CHSH-type operator in Eq. (6.18). Here, the total payoff of the players is given by $T = u_i^A + u_i^B$ where $i \in \{1,2,3,4\}$ and $T' = u_5^A + u_5^B$. The sum of payoffs which play role in the untilted version of the game is taken as T and the sum of payoffs which contributes to the tilted version of the game is taken as T'. Therefore, the value of T' depends on the term α of the tilted Bell-CHSH expression in Eq. (6.18). Assuming that the chances of $x_A(x_B)$ to be 0 and 1 are equiprobable, the sum of payoffs of Alice and Bob can be given as

$$\$_A + \$_B = \frac{T}{4}\left[(P_{00}^{00} + P_{00}^{11}) + (P_{01}^{00} + P_{01}^{11}) + (P_{10}^{00} + P_{10}^{11}) + (P_{11}^{01} + P_{11}^{10})\right] + \frac{T'}{2}\left[P_0^0 - P_0^1\right] \tag{6.19}$$

where $P_{ij}^{kl} = P(y_A = k, y_B = l|x_A = i, x_B = j)$ and $P_m^n = P(y_A = n|x_A = m)$. Thus, similar to the calculations in Eqs. (14) and (15), the sum of payoffs of the players can be re-expressed as

$$\$_A + \$_B = \frac{T}{2} + \frac{T}{8}\left[\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle + \frac{4T'}{T}\langle A_0\rangle\right] \tag{6.20}$$

which is equal to $\frac{T}{2} + \frac{T}{8}(I_1^{\alpha})$ at $\alpha = \frac{4T'}{T}$. In order to exemplify the payoff Table 6.4 in detail, $T = 2$ is considered and thus $T' = \frac{\alpha T}{4} = \frac{\alpha}{2}$. As described above, there can be two types of games, or simply,

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$u_1^A + \overline{x_A} u_5^A, u_1^B + \overline{x_A} u_5^B$	$\overline{x_A} u_5^A, \overline{x_A} u_5^B$
$y_A = 1$	$-\overline{x_A} u_5^A, -\overline{x_A} u_5^B$	$u_2^A - \overline{x_A} u_5^A, u_2^B - \overline{x_A} u_5^B$

(a) $x_A \cdot x_B = 0$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$0, 0$	u_3^A, u_3^B
$y_A = 1$	u_4^A, u_4^B	$0, 0$

(b) $x_A \cdot x_B = 1$

Table 6.4 : Payoffs of Alice and Bob in a general game setting where dependence of payoffs on type of players commensurate with the input-output relation in a tilted CHSH game where u_1^A, u_1^B, u_2^A, u_2^B, u_3^A, u_3^B, u_4^A, u_4^B, u_5^A, and u_5^B are non-zero

two ways of representation: common interest games and conflicting interest games. In a common interest game $u_i^A = u_i^B$ for $i \in \{1, 2, 3, 4, 5\}$, but in a conflicting interest game, $u_i^A \neq u_i^B$, thus creating a conflict in interest of the two players in preferring one strategy over the other.

6.5 A COMMON INTEREST GAME FOR TILTED CHSH OPERATOR

In a common interest games, both players have the same payoff for any strategy set. As an instance, assume the utilities as $u_1^A = u_1^B = u_2^A = u_2^B = u_3^A = u_3^B = u_4^A = u_4^B = 1$, and $u_5^A = u_5^B = \frac{\alpha}{4}$. The payoffs of Alice and Bob are given by Tables 6.5(a), 6.5(b), and 6.5(c). Clearly, Table 6.5(a) represents a tilted common interest coordination game and Table 6.5(b) represents a usual/ non-tilted common interest coordination game. Both games (Table 6.5(a) and (b)) have $\{y_A = 0, y_B = 0\}$ and $\{y_A = 1, y_B = 1\}$ as the two NE. In case of the non-tilted version, both NEs are pareto-optimal. But, in case of tilted common interest game, one of the NEs is not pareto-optimal due to extra 'tilt' factor α. Similarly, Table 6.5(c) shows a common interest anti-coordination game.

Classical scenario

Out of 16 different classical strategies, there can be four pareto-optimal NEs as follows

- $y_A = y_B = 0$ irrespective of the values of x_A and x_B;
- $y_A = 0$ irrespective of the values of x_A and $y_B = x_B$;
- $y_A = x_A$ and $y_B = 0$ irrespective of the value of x_B; and
- $y_A = x_A$ and $y_B = \overline{x_B}$.

Each of the above NE strategy is preferred by none of the players as each gives them a payoff of $\frac{6+\alpha}{8}$. In addition, there are two non-pareto optimal NE for the common interest game, i.e.,

- $y_A = \overline{x_A} = 0$ and $y_B = 1$ irrespective of the value of x_B; and

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$1+\frac{\alpha}{4}, 1+\frac{\alpha}{4}$	$\frac{\alpha}{4}, \frac{\alpha}{4}$
$y_A = 1$	$-\frac{\alpha}{4}, -\frac{\alpha}{4}$	$1-\frac{\alpha}{4}, 1-\frac{\alpha}{4}$

(a) $(x_A = 0, x_B = 0)$ or $(x_A = 0, x_B = 1)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	1,1	0,0
$y_A = 1$	0,0	1,1

(b) $(x_A = 1, x_B = 0)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	0,0	1,1
$y_A = 1$	1,1	0,0

(c) $(x_A = 1, x_B = 1)$

Table 6.5 : Payoffs of Alice and Bob in a common interest game setting where dependence of payoffs on type of players commensurate with the input-output relation in the tilted CHSH game

- $y_A = y_B = 1$ irrespective of the values of x_A and x_B.

Both NEs are again not preferred by any player, and each yields a payoff of $\frac{6-\alpha}{8}$.

The quantum strategy for this game is similar to the game where players have conflicting interests, and hence is merged with the next section.

6.6 A CONFLICTING INTEREST GAME FOR TILTED BELL-CHSH OPERATOR

In order to better understand the representation of tilted CHSH game as a conflicting interest game, the following values of utilities are assumed, $u_1^A = u_2^B = u_3^B = u_4^A = \frac{1}{2}$, $u_1^B = u_2^A = u_3^A = u_4^B = \frac{3}{2}$, and $u_5^A = u_5^B = \frac{\alpha}{4}$. The payoffs of Alice and Bob engaging in the above defined game are given in Tables 6.6(a), 6.6(b), and 6.6(c). Clearly, Table 6.6(a) represents tilted Battle-of-Sexes game, and Table 6.6(b) represents the usual Battle-of-Sexes game. Thus, for $x_A.x_B = 0$, the players engage in a BoS game, with an exception at $x_A = 0$, where the players play a tilted version of the BoS game. Still for both versions of the game, both $\{y_A = 0$ (Activity 0 by Alice), $y_B = 0$ (Activity 0 by Bob)$\}$ and $\{y_A = 1$ (Activity 1 by Alice), $y_B = 1$ (Activity 1 by Bob)$\}$ are the pareto-optimal NE of the game. However, the first NE is preferred by Bob and the second NE is preferred by Alice. Also, due to the tilt α at $x_A = 0$, the total payoff of the first NE (preferred by Bob) is more than the total payoff of the second NE (preferred by Alice).

At $x_A = x_B = 1$, the players play an anti-coordination game which we term as a lottery game, where $y_A = 0$ or $y_B = 0$ corresponds to winning a bigger/ more desired prize, and $y_A = 1$ or $y_B = 1$ corresponds to winning a smaller/ less desired prize. Apparently, the desire of both players to win the bigger prize is the same, and both win different prizes at a time on the basis of a lottery. This leads to a conflicting-interest anti-coordination game as depicted in Table 6.6(c).

Classical scenario

After analysing all possible classical strategy sets, one can conclude that for no classical strategy, the total payoff of the players exceeds $\frac{6+\alpha}{4}$. The strategy sets forming the NE, however,

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{1}{2}+\frac{\alpha}{4}, \frac{3}{2}+\frac{\alpha}{4}$	$\frac{\alpha}{4}, \frac{\alpha}{4}$
$y_A = 1$	$-\frac{\alpha}{4}, -\frac{\alpha}{4}$	$\frac{3}{2}-\frac{\alpha}{4}, \frac{1}{2}-\frac{\alpha}{4}$

(a) $(x_A = 0, x_B = 0)$ or $(x_A = 0, x_B = 1)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{1}{2}, \frac{3}{2}$	$0, 0$
$y_A = 1$	$0, 0$	$\frac{3}{2}, \frac{1}{2}$

(b) $(x_A = 1, x_B = 0)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$0, 0$	$\frac{3}{2}, \frac{1}{2}$
$y_A = 1$	$\frac{1}{2}, \frac{3}{2}$	$0, 0$

(c) $(x_A = 1, x_B = 1)$

Table 6.6 : Payoffs of Alice and Bob in a conflicting interest game setting where dependence of payoffs on type of players commensurate with the input-output relation in the tilted CHSH game

are different for the different type of interest (common or conflicting) of players. In case of the conflicting interest game discussed above, there are two pareto-optimal NE, i.e.,

- $y_A = y_B = 0$ irrespective of the values of x_A and x_B; and
- $y_A = x_A$ and $y_B = 0$ irrespective of the value of x_B.

Both strategies lead to a pareto-optimal NE. Both these NE strategies are still preferred by Bob since Alice gets a payoff of $\frac{3+\alpha}{8}$, and Bob gets a payoff of $\frac{9+\alpha}{8}$.

Furthermore, for $\alpha < 1$, there are two more pareto-optimal NE for the conflicting interest game, i.e.,

- $y_A = \overline{x_A}$ and $y_B = x_B$; and
- $y_A = 1$ irrespective of the value of x_A, and $y_B = \overline{x_B}$.

Both strategies lead to a pareto-optimal NE strategy preferred by Alice wherein Alice gets a payoff of $\frac{7-\alpha}{8}$, and Bob gets a payoff of $\frac{5-\alpha}{8}$.

6.7 ANALYSIS OF THE TILTED CHSH GAME FOR DIFFERENT QUANTUM STATES

The games represented in Table 6.5 or 6.6 refer to Bayesian games where the total sum of payoff of both players (Alice and Bob) is equal to the value of operator $1 + \frac{I_1^\alpha}{4}$ where I_1^α is the value of the tilted Bell-CHSH operator.

6.7.1 Quantum scenario using a pure state

When the players share a general two-qubit entangled pure state $|\psi\rangle = cos\theta|00\rangle + sin\theta|11\rangle$, and perform measurements as shown in Eq. (6.4) as strategies in the game, where $|\Phi_0(\theta)\rangle =$

$cos\theta|0\rangle + sin\theta|1\rangle$ and $|\Phi_1(\theta)\rangle = -sin\theta|0\rangle + cos\theta|1\rangle$ the total payoff of both players after their respective moves/measurements is estimated as

$$\$_A + \$_B = \frac{1}{4}\Big[\alpha cos2\theta + 4 + 2cos2\lambda + 2sin2\theta sin2\lambda\Big] \tag{6.21}$$

Similar to the previous cases, same payoff can be achieved when players share an arbitrary two-qubit entangled state $|\psi_{arbitrary}\rangle = cos\theta|00\rangle + e^{i\phi}sin\theta|11\rangle$. This aggregated payoff achieves its maximum at measurement parameter $\lambda = \frac{1}{2}tan^{-1}(sin2\theta)$, and can be given by

$$\$_A + \$_B = \frac{1}{4}\Big[\alpha cos2\theta + 4 + 2\sqrt{1+sin^2 2\theta}\Big] \tag{6.22}$$

Optimizing with respect to θ, shows that the optimal value of $\$_A + \$_B = 1 + \frac{1}{4}\sqrt{8+2\alpha^2}$ can be obtained for a non-maximally entangled state where $\theta = \frac{1}{2}sin^{-1}\sqrt{\frac{4-\alpha^2}{4+\alpha^2}}$. Moreover, as the value of α increases from 0 to 2, the classically attained sum of payoffs increases from 1.5 to 2. Also, with increase in α, the angle θ at which Alice and Bob benefit with highest possible payoff decreases. At $\alpha = 0$, the maximally entangled Bell state gives the maximum possible payoff to the players, as the total payoff holds correspondence with the original CHSH inequality. But at $\alpha \sim 2$, a quantum state very close to a separable state gives the highest total payoff of Alice and Bob. Apart from this, for higher α the maximally entangled state gives no benefit over classical strategies. However, non-maximally entangled states still give better payoff than the classically attained payoff. Therefore, the analysis suggests an interesting anomaly where players in this quantum game can achieve higher payoff by sharing a non-maximally entangled pure state instead of a maximally entangled pure state of two qubits in line with [Acín *et al.*, 2006]. For comparison between the two game settings, one can analyse that the quantum game where conflicting interest games are merged as a Bayesian game results in a much larger set of non-maximally entangled states offering advantage over the maximally entangled state as opposed to the quantum game where common interest games are merged. Thus, this model is a clear instance where high randomness in non-maximally entangled pure states help quantum players benefit over their classical counterparts.

For further demonstration of the dependence of total payoff of the degree of entanglement, the total payoff of the players is plotted with the entanglement measure (concurrence) in Figure 6.3. Interestingly, as the value of α increases, opting for a classical strategy pays more dividends for a larger range of degree of entanglement in comparison to opting for a quantum strategy.

6.7.2 Quantum scenario using a mixed state

In order to study the quantum scenario using mixed states, first Horodecki states are considered. Figure 6.4 clearly shows that as α increases, the classical payoff exceeds the quantum mechanically achieved sum of payoff. However, for smaller values of α, Horodecki states with concurrence close to unity leads to quantum advantage. Interestingly, Werner states lead to similar results as in the previous section as the total payoff attained does not depend on the parameter α.

The total payoff in the game is now analysed by sharing an efficient class of two-qubit mixed states [Singh and Kumar, 2018c], represented as

$$\rho = \frac{1}{N}\Big[\frac{1}{2}\gamma(1-\eta)\{\gamma(1-\eta)|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|\} + |\phi^+\rangle\langle\phi^+|\Big] \tag{6.23}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}\big[|00\rangle + |11\rangle\big]$, γ represents the amplitude-damping noise parameter, η represents the weak measurement strength parameter [Korotkov and Jordan, 2006; Kim *et al.*, 2012], and

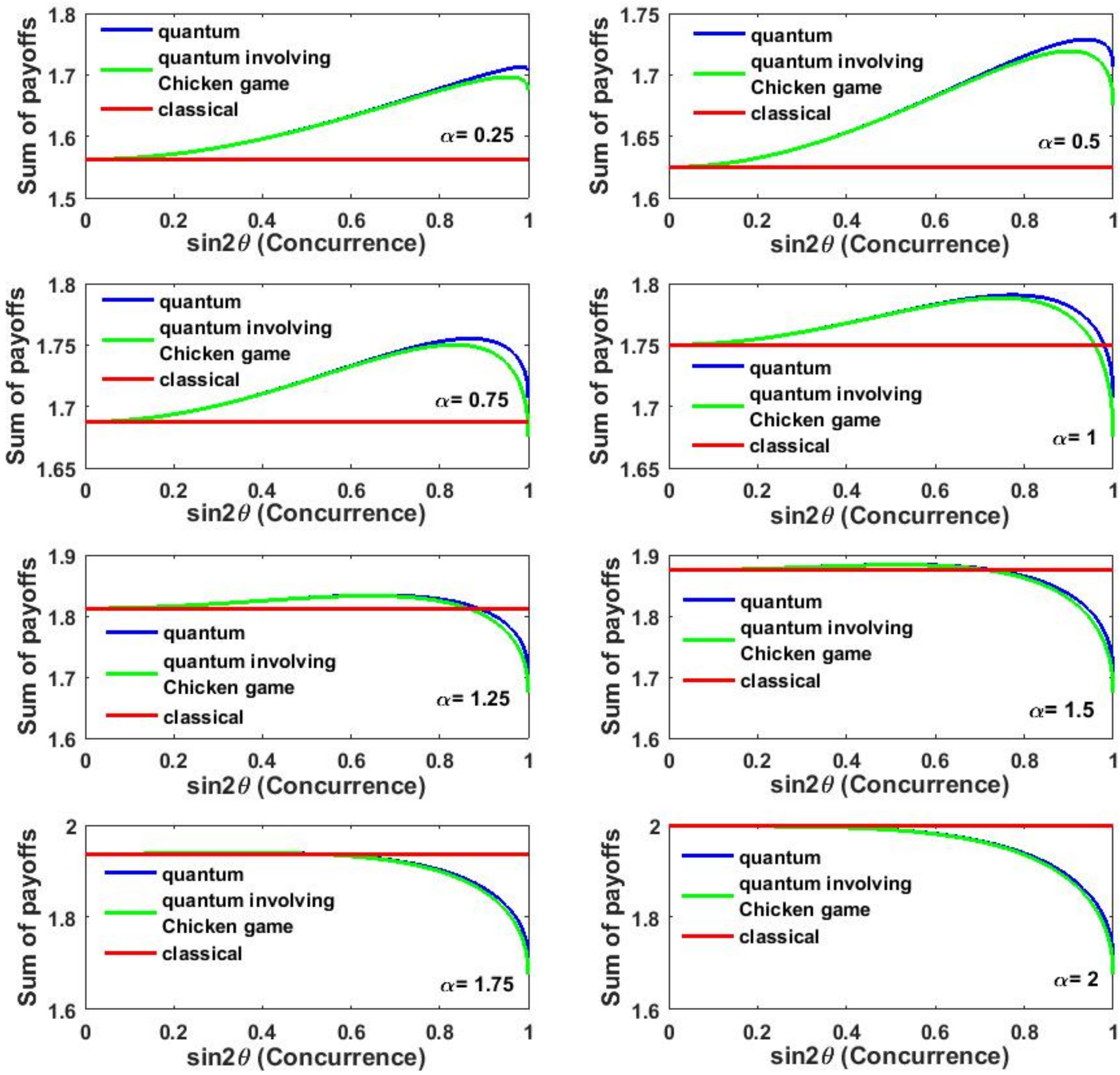
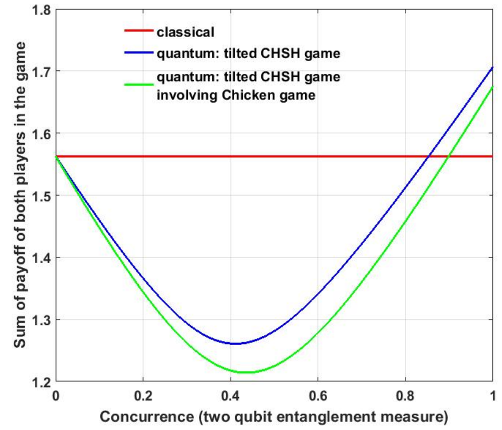
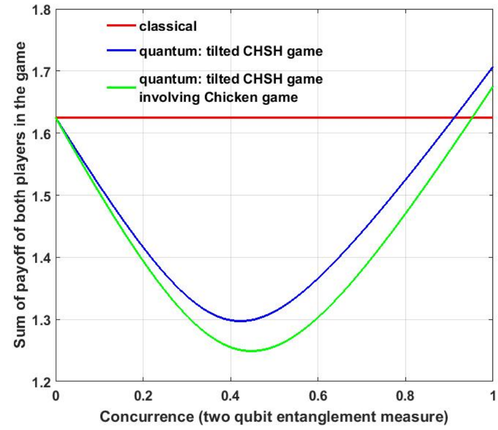


Figure 6.3 : Relation between the sum of payoffs of players with the concurrence of a general two-qubit Bell state for Bayesian game representation of the tilted Bell-CHSH operator

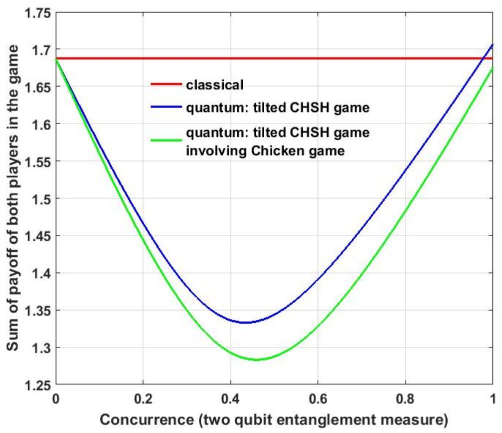
$N = \frac{1}{2}\left[2+\gamma(1-\eta)\{2+\gamma(1-\eta)\}\right]$. The proposed quantum state is entangled for all values of γ and η, but violates the Bell-CHSH inequality only for the range: $max\left\{0, 1-\frac{0.2428}{\gamma}\right\} < \eta < 1$ [Singh and Kumar, 2018c]. In comparison to the original Bell-CHSH inequality, tilted CHSH inequality is violated for a slightly bigger range of values of η. However, Figure 6.5 shows that as the value of tilt increases, Alice and Bob achieve quantum advantage for a smaller range of concurrence close to unity. Even at $\alpha = 0.25$, only the mixed state with $0.78 < C < 1$ help attain better payoff than classical strategy. In addition, at $\alpha > 0.825$, classical strategies are better than the considered mixed states as they result in higher reward to the players. Thus, a very small set of mixed states benefit the quantum players in this game setting. Nevertheless in comparison to



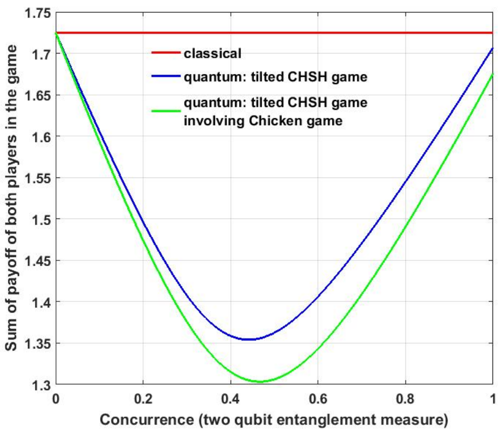
(a) $\alpha = 0.25$



(b) $\alpha = 0.5$



(c) $\alpha = 0.75$



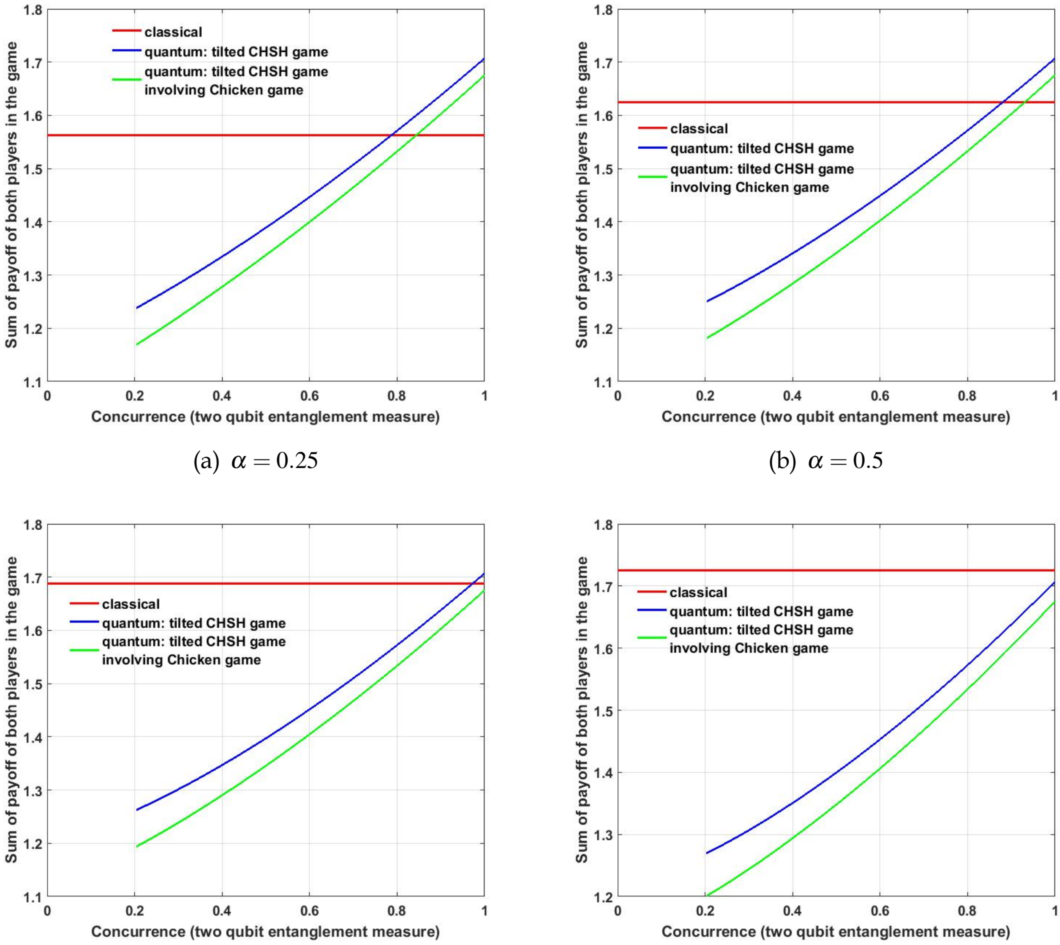
(d) $\alpha = 0.9$

Figure 6.4 : Relation between the sum of payoffs of players in tilted CHSH game with the concurrence of the Horodecki state for Bayesian game representation of the tilted Bell-CHSH operator

Horodecki states, use of these states result in attaining better payoff in the game. For measurement strategies, the orthogonal basis vectors are considered as $|\Phi_0(\theta)\rangle = cos\theta|0\rangle + e^{i\phi}sin\theta|1\rangle$ and $|\Phi_1(\theta)\rangle = -e^{-i\phi}sin\theta|0\rangle + cos\theta|1\rangle$ at $\phi = 0^\circ$ so as to attain maximum possible total payoff.

6.8 A CONFLICTING INTEREST GAME FOR TILTED BELL-CHSH OPERATOR INVOLVING CHICKEN GAME

Similar to the discussion above, this section presents a study on the tilted version of the game represented in Table 6.3. Here, the players play a tilted BoS game when the type of atleast one player is Type 0, and a Chicken game otherwise. The payoff table (Table 6.7) of the game is shown below. The payoffs of the players playing this game is shown by green line in Figure 6.3 for a general two-qubit Bell state; in Figure 6.4 for a Horodecki state; and in Figure 6.5 for the mixed state represented in Eq. 6.23. It is clear from these figures that the introduction of Chicken game in the Bayesian representation of tilted Bell-CHSH inequality, reduces the payoff for pure as well as mixed states. The behaviour of resources in this game is similar to the one discussed above (blue curve). However, the green curve in Figure 6.3 indicates that the set of non-maximally entangled states that perform better than the maximally entangled Bell state increases under this setting. On



(a) $\alpha = 0.25$ (b) $\alpha = 0.5$

(c) $\alpha = 0.75$ (d) $\alpha = 0.9$

Figure 6.5 : Relation between the sum of payoffs of players in tilted CHSH game with the concurrence of the defined mixed state for Bayesian game representation of the tilted Bell-CHSH operator

the other hand, Figure 6.4 and 6.5 demonstrate that the set of mixed states which perform better than the classical strategies reduces for this representation.

6.9 CONCLUSIONS

In last two decades, the connection between the apparently unrelated fields of Bayesian games and quantum nonlocality has been studied in great details to understand the foundations of quantum mechanics and quantum information. In general, Bayesian games in the settings of a CHSH game whose foundation makes use of quantum correlations with the purpose of defeating classical players, rely on maximally entangled states. In this work, general two-qubit pure states and mixed states are used to analyse the game proposed by Pappa *et al.* It was found that all pure two-qubit entangled Bell states, set of Werner and Horodecki class states offer benefit when used as resources in comparison to classical strategies. Precisely, the players have higher advantage when they shared any general two-qubit maximally or non-maximally entangled pure Bell state over a Werner or a Horodecki class states.

The analysis for a fully conflicting interest Bayesian game as opposed to the game designed

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{1}{2} + \frac{\alpha}{4}, \frac{3}{2} + \frac{\alpha}{4}$	$\frac{\alpha}{4}, \frac{\alpha}{4}$
$y_A = 1$	$-\frac{\alpha}{4}, -\frac{\alpha}{4}$	$\frac{3}{2} - \frac{\alpha}{4}, \frac{1}{2} - \frac{\alpha}{4}$

(a) $(x_A = 0, x_B = 0)$ or $(x_A = 0, x_B = 1)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$\frac{1}{2}, \frac{3}{2}$	$0, 0$
$y_A = 1$	$0, 0$	$\frac{3}{2}, \frac{1}{2}$

(b) $(x_A = 1, x_B = 0)$

Alice \ Bob	$y_B = 0$	$y_B = 1$
$y_A = 0$	$0, 0$	$\frac{-1}{2}, \frac{5}{2}$
$y_A = 1$	$\frac{5}{2}, \frac{-1}{2}$	$-1, -1$

(c) $(x_A = 1, x_B = 1)$

Table 6.7 : Payoffs of Alice and Bob in a conflicting interest game setting involving the Chicken game as an example of an anti-coordination game where dependence of payoffs on type of players commensurate with the input-output relation in the tilted CHSH game

by Pappa *et al.* resulted in some interesting observations. The designed game is a replica of Battle of the Sexes game when type of at least one player is Type 0, otherwise the game enacts a Chicken or a Hawk-Dove game. Similar to the previous case, it is found that all pure states help attain higher payoff than the classical bound. Interestingly, Alice and Bob achieved higher total payoff when they shared a non-maximally entangled Bell state ($cos(40.188°)|00\rangle + sin(40.188°)|11\rangle$) with concurrence 0.986 rather than sharing a maximally entangled Bell state with unit concurrence. This anomaly can be attributed to an extra term in the total payoff when a Chicken game is involved, which leads to a less entangled state giving a higher payoff. Although mixed states are found to be useful as opposed to classical strategies for certain ranges of states parameters, the results suggested that classical strategies may be more useful than quantum strategies even in the range where mixed states violate the Bell-CHSH inequality. Therefore, mere violation of the Bell-CHSH inequality by a mixed state may not guarantee a team of quantum players a win over their classical opponents.

Inspired by the game set-up, a general Bayesian game representation of the tilted Bell-CHSH inequality is formulated. A similar phenomenon of higher payoff with less entangled states is observed after designing a Bayesian game based on the tilted Bell-CHSH inequality. Nevertheless, the analysis with the Bayesian game based on the standard Bell-CHSH inequality turned out to be more interesting due to the belief that maximally entangled pure Bell states are always more efficient than the non-maximally entangled pure states. Similar to the previous case, the combination of conflicting interest games led to interesting observations as opposed to the combinations of common interest games. Due to the uniqueness of tilted CHSH game, highly random non-maximally entangled states lead to higher payoff in the game, than maximally entangled Bell state. The same however is not true for all mixed states, e.g., Werner states are found to be independent of the tilt parameter and hence add no new interpretation in comparison to the Bayesian game based on the standard CHSH inequality. The set of mixed states represented in Eq. (6.23) are found to be slightly more useful resources for tilted game in comparison to Horodecki states. As the value of tilt parameter increases, classical strategies, however, lead to better efficiency in the game against the use of these mixed states. As a part of quantum strategy, it is specifically assumed that the players in the game perform one-parameter quantum measurements on their shared qubits.

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