

# **MODAL ANALYSIS AND NONLINEAR DYNAMICS OF MULTI-LINK FLEXIBLE MANIPULATOR WITH GENERIC PAYLOAD MOUNTED ON A MOVING BASE**

*A Thesis submitted by*  
**Pravesh Kumar**

*in partial fulfillment of the requirements for the award of the degree of*  
**Doctor of Philosophy**



॥ त्वं ज्ञानमयो विज्ञानमयोऽसि ॥

**Indian Institute of Technology Jodhpur**  
**Mechanical Engineering**  
*July 2020*





॥ त्वं ज्ञानमयो विज्ञानमयोऽसि ॥

Department of Mechanical Engineering  
Indian Institute of Technology Jodhpur  
India-342037

## Certificate

It is certified that the work contained in this thesis entitled “**Modal analysis and nonlinear dynamics of multi-link flexible manipulator with generic payload mounted on a moving base**” submitted by **Mr. Pravesh Kumar** to the Indian Institute of Technology Jodhpur for the award of the degree of Philosophy has been carried out under my supervision in the Department of Mechanical Engineering, Indian Institute of Technology Jodhpur. This work has not been submitted elsewhere for the award of any other degree or diploma.

The thesis, in my opinion, has reached the standard fulfilling the requirements for the award of the degree of Doctor of Philosophy in accordance with the regulations of the Institute.

(Dr. Barun Pratiher)

Associate Professor

Department of Mechanical Engineering

Indian Institute of Technology Jodhpur

India-342037

# Declaration

I hereby declare that the work presented in this thesis titled "*Modal Analysis and Nonlinear Dynamics of Multi-Link Flexible Manipulator with Generic Payload Mounted on a Moving Base*", submitted to Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of Dr. Barun Pratiher. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to any other Institute or University in India or abroad for the award of any degree or diploma.

*Pravesh Kumar*  
*P15ME003*

To my parents Late Shri G. Laxmanan Nair and Smt. Yashodha Devi, for instilling in me the belief that I could do this.

To my lovely wife Santoshi Goswami, whom I admire and love. I would like to thank you for your support, comprehension and love.

To my sweet brother Santosh Nair and grandmother who have been there to support me all through my life.

To my father who sacrificed a lot of his needs for my dreams.

In the present work, the detailed mathematical modeling, modal analysis, nonlinear characteristics and trajectory analysis and response of multi-link flexible manipulators of various configurations are addressed. The flexible and light weight manipulators have many advantages over the conventional rigid and heavy manipulators such as lower power consumption, higher speed, higher payload to manipulator weight capacity, smaller actuators for manipulator movement, larger work space, and lower inertia. In case of flexible manipulators, the control objective is to achieve the desired end-effector trajectory and vibration attenuation of the end-effector. The control design of such manipulator system necessitates the accurate dynamic modeling to achieve the desired performance requirements. The multi-link manipulator are preferred over the single-link manipulator in various applications involved in production and manufacturing industries, pick and drop tasks, space explorations, nuclear power plant, medical operations, and so on due to their large work space, better maneuverability and safer applications. From the available literature it is evident that the single-link manipulator has been extensively studied but in case of multi-link manipulator the researchers have confined their attention towards the control and vibration suppression of manipulator to achieve the desired motion. Also, the robotic manipulators engaged in various operations such as grinding, drilling, painting, spraying etc where the manipulator is subjected to external forces when the tool attached to the end-effector comes in contact with the working surface. It is apparent from the literature review that there have been trivial studies regarding the multi-link manipulator subjected to external excitations. Hence, in this work, an attempt has been made to study the flexible multi-link manipulator and to develop some control strategy to attenuate the unwanted vibrations.

In this thesis, a systematic dynamic modeling of flexible manipulator with single, two and multi-link using Hamilton's principle has been presented. The links have been modeled as Euler-Bernoulli beam element with payload at the distal end of the link. The cross-section of the links remain perpendicular to the bending axis and hence the rotary and shear deformation due to rotation of cross-section has been neglected. The payload is assumed to be either a point mass or an arbitrarily oriented sizeable mass having inertia and its centre of gravity is different than the point of attachment with the terminal link. The harmonic rotary and translatory motions have been imparted to the links of the manipulator is through the revolute and prismatic joints, respectively. The flexibilities have been induced in the revolute and prismatic joints by respectively, modeling them as torsional spring-inertia and linear spring-mass system. The coupled nonlinear governing equations and the boundary conditions are obtained by using extended Hamilton's principle with subsequent modal analysis to determine the eigenfrequencies and eigenspectrums of the system. The modal parameters thus developed are further used in the nonlinear analysis and trajectory analysis of the manipulator system. The influence of nondimensional system parameters such as payload mass, payload inertia, offset ratio, offset angle, beam mass density ratio, flexural rigidity ratio, actuator mass and joint frequency parameter on the system eigenfrequencies and eigenspectrums have been tabulated and graphically presented.

The nonlinear analysis is very imperative to determine the behaviors of a system under external and/or parametric excitation as the system exhibits inadmissible vibrations when the forcing frequency becomes equal or nearly equal to the system's natural frequency. Hence, it is the prime requirement for the design engineer to inhibit such vibrations by operating the system in safe zones or by manipulating the system parameters accordingly. Now, the governing equations of the links are discretized using the Galerkin's method along with the mode shapes of the system to obtain the nonlinear temporal differential governing equations of the motion of the links which are further nondimensionalized with respect to the system parameters. The governing temporal equations of motion of the links contain many nonlinear terms including the cubic nonlinearities due to axial stretching, nonlinear terms due to inertial coupling, damping terms, force and parametric excitation terms. The forcing and parametric excitation terms respectively arise due to the directly applied force at the end-effector and the

harmonic revolutes or cartesian motion of the manipulator. The closed form solution of the complex nonlinear temporal equations of motions is sought by using method of multiple scales as one of the perturbation techniques. The governing equations of motions are reduced to the autonomous set of first order equations in terms of the amplitude and the phase and finally the frequency response equations of the respective links are obtained for steady-state conditions. The effect of essential system parameters on the stability of steady-state solutions and bifurcation diagrams have been demonstrated through graphical illustrations. The system behavior under forced resonance, parametric resonance, internal resonance and combined resonance conditions have been studied using time response, phase portraits, FFTs and frequency response curves.

Further, to analyze the system parameters on the system responses such as angular tip positions, modal deflections and tip accelerations, numerical simulations have been carried out. Here, the joints of the manipulator are given a smooth sinusoidal torque as an input and system responses have been plotted. The significant influence of system parameters on the vibration characteristics provides a comparative analysis of performance and accuracy of manipulator. Finally, to control the trajectory tracking of flexible manipulator, a model based controller using stable inversion technique in conjunction with proportional-derivative controller is developed to examine the effect of system attributes on the control parameters.

Initially, the single-link and two-link manipulator with harmonically varying revolute or prismatic motion incorporating a generic payload with constraint pulsating axial force has been dynamically modeled with subsequent investigation of free vibration analysis. The influence of the payload and joint variables on the eigen-parameters has been investigated. The joints have been provided with harmonic motion causing the jump phenomenon and multiple steady-state solutions due to the primary and secondary resonance phenomena which may cause the catastrophic failure of the system. Further, due to the increased application of long reach manipulator in various industrial applications, the investigation of multi-link manipulator has been pursued. Five different models of two-link manipulator have been taken into considerations. Initially, the basic model of a two-link manipulator with the links connected with motors and payload designed as point mass is considered. In second model, the end-effector has been modeled as a generic payload with mass, inertia, offset and orientation and a harmonic axial force imitating its working environment is exerted. Further, in order to accomplish a realistic model of two-link manipulator, the prismatic and revolute motions are considered and the influence of their motions on the vibrational behavior of the system is demonstrated graphically. Forward and inverse dynamics with proportional-derivative controller of two-link manipulator with revolute pairs is explored to understand the consequence of parametric variation of system attributes on the system responses such as angular tip positions, modal deflections, control torques, and link accelerations. Finally, the mathematical modeling of a multi-link manipulator with bi-directional base motion which can incorporate any number of links and joints has been presented. The modal parameters along with the nonlinear behaviors and vibration characteristics have been thoroughly investigated. The results obtained in this thesis will alleviate the better design and manufacturing of flexible manipulators involved in various industrial applications and facilitate the efficient active/passive control or suppression of end point vibrations of manipulators engaged in precession operations. The analysis presented in the thesis can be extended to robotic manipulators with different configurations and joint dynamics involved in pick and drop operations, long reach manipulators where human interaction is not possible, manipulator involved in medical and nuclear industries

...

## Acknowledgements

This thesis incorporates the results of the research conducted within the **Department of Mechanical Engineering, Indian Institute of Technology Jodhpur**, under the supervision of **Dr. Barun Pratiher**. To him, I would like to express my sincere gratitude and appreciation for his invaluable guidance, significant support and technical assistance during my studies, research work and creation of this thesis. His brilliance, resourcefulness, and patience are greatly admired.

I would also like to thank **Dr. B. Ravindra, Dr. K. Hiremath** and **Dr. S. V. Shah** for their commitment to serve as the Members of my Doctoral Committee and for enhancing my knowledge by their comments and reviews at various stages of my Doctoral course. Moreover, I would also like to thank **Dr. B. Ravindra and Dr. K. Hiremath** for their excellent teaching and clear approach towards the courses which have assisted me during my research work.

I'm very much grateful to **Mr. H. P. Phadatare**, for giving me invaluable support during my numerical simulations of the thesis results. A special thanks to **Mr. Rohit Kumar, Mr. Aniket Monde, Mr. Shivam Chaturvedi** and **Mr. Ram Mohan** for helping me during my personal and financial crisis and making my stay pleasant and enjoyable.

Most importantly, I would like to thank my parents, **Late Shri G. Laxmanan Nair and Smt. Yashodha Devi**, and wife **Santoshi**, for their unconditional supports, love and affection. Their encouragement and never-ending kindness has made this journey possible. Also, I appreciate **Santoshi's** patience and the sacrifices that she made all this while. My sibling **Santosh Nair** has always been a source of untiring love and support in all walks of life.

**Pravesh Kumar**  
**IIT Jodhpur, 2020.**



# List of Figures

FIG. 1.1: (A) ROBOT PALLETIZER MACHINE AND (B) AXISIS ROBOT. ....	1
FIG. 1.2: (A) IROBOT 510 PACKBOT AND (B) HYDRAULIC ROBOT ARM – UNDERWATER.....	2
FIG. 3.1: SCHEMATIC DIAGRAM OF (A) ROTATING CARTESIAN MANIPULATOR WITH GENERIC PAYLOAD AND (B) FLEXIBLE ACTUATOR. ....	25
FIG. 3.2: VARIATION OF FIRST FOUR MODE SHAPES OF CARTESIAN MANIPULATOR WITH PAYLOAD MASS ( $\alpha_{mc}$ ) (A) MODE 1 (B) MODE 2 (C) MODE 3 (D) MODE 4. ....	33
FIG. 3.3: VARIATION OF MODE SHAPES OF CARTESIAN MANIPULATOR WITH (A) ACTUATOR MASS ( $\alpha_{ma}$ ) AND (B) PAYLOAD INERTIA ( $\alpha_i$ ). ....	34
FIG. 3.4: VARIATION OF MODE SHAPES OF CARTESIAN MANIPULATOR WITH (A) ACTUATOR FREQUENCY PARAMETER ( $\Omega_a$ ) AND (B) OFFSET RATIO $\alpha_c$ . ....	35
FIG. 3.5: INFLUENCE OF AXIAL MOTION ON THE MODE SHAPES OF CARTESIAN MANIPULATOR. ....	35
FIG. 3.6: FREQUENCY RESPONSE CHARACTERISTICS OF CARTESIAN MANIPULATOR (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE.....	36
FIG. 3.7: ANALYTICAL (A) NUMERICAL (B) TIME RESPONSE AND PHASE PORTRAIT , FFT (C) OF CRITICAL POINTS A, B, AND C IDENTIFIED IN FIG. 3.6.....	37
FIG. 3.8: EFFECT OF PAYLOAD MASS PARAMETER ( $\alpha_{mc}$ ) ON THE FREQUENCY RESPONSE CURVES (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	37
FIG. 3.9: EFFECT OF ACTUATOR MASS PARAMETER ( $\alpha_{ma}$ ) ON THE FREQUENCY RESPONSE CURVES (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	38
FIG. 3.10: EFFECT OF PAYLOAD INERTIA PARAMETER ( $\alpha_i$ ) ON THE FREQUENCY RESPONSE CURVES (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	38
FIG. 3.11: EFFECT OF ACTUATOR FREQUENCY PARAMETER ( $\Omega_a$ ) ON THE FREQUENCY RESPONSE CURVES (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	39
FIG. 3.12: EFFECT OF OFFSET RATIO ( $\alpha_c$ ) ON THE FREQUENCY RESPONSE CURVES (A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE.....	39
FIG. 3.13: EFFECT OF ROTATING FREQUENCY ( $\Omega$ ) ON THE FREQUENCY RESPONSE CURVES(A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	40
FIG. 3.14: EFFECT OF AMPLITUDE OF AXIAL FORCE AND BASE MOTION ON THE FREQUENCY RESPONSE CURVES(A) SUBHARMONIC RESONANCE (B) COMBINED PRIMARY-SUBHARMONIC RESONANCE. ....	40
FIG. 3.15: SCHEMATIC DIAGRAM OF (A) FLEXIBLE MANIPULATOR WITH (B) REVOLUTE HUB INCORPORATING A GENERIC PAYLOAD MOUNTED ON MOVING BASE. ....	41
FIG. 3.16: VARIATION OF EIGENFREQUENCIES WITH (A) PAYLOAD MASS AND (B) INERTIA PARAMETERS FOR $\alpha_{mh} = 1.0, \alpha_{ih} = 1.0, \dots$ .....	47
FIG. 3.17: VARIATION OF EIGENFREQUENCIES WITH (A) OFFSET RATIO AND (B) JOINT FREQUENCY PARAMETER FOR $\alpha_{mc} = 1.0, \alpha_i = 1.0, \alpha_{mh} = 1.0, \alpha_{ih} = 1.0, \gamma = 30^\circ$ . ....	48
FIG. 3.18: COMPARISON OF MODE SHAPES FOR FOUR DIFFERENT MODELS (A) MODE 1 (B) MODE 2. ....	48
FIG. 3.19: VARIATION OF MODE SHAPES WITH OFFSET MASS ( $\alpha_{mc}$ ) (A) MODE 1 (B) MODE 2. ....	49
FIG. 3.20: VARIATION OF MODE SHAPES WITH HUB MASS ( $\alpha_{mh}$ ) (A) MODE 1 (B) MODE 2.....	49
FIG. 3.21: VARIATION OF MODE SHAPES WITH PAYLOAD INERTIA ( $\alpha_i$ ) (A) MODE 1 (B) MODE 2. ....	49
FIG. 3.22: VARIATION OF MODE SHAPES WITH HUB FREQUENCY PARAMETER (A) $\Omega_h < 1$ (B) $\Omega_h = 1$ . ....	50
FIG. 3.23: VARIATION OF MODE SHAPES WITH OFFSET RATIO ( $\alpha_c$ ) (A) MODE 1 (B) MODE 2. ....	50
FIG. 3.24: VARIATION OF MANIPULATOR TIP DEFLECTIONS AND MODAL DISPLACEMENTS WITH (A) PAYLOAD MASS (B) PAYLOAD INERTIA (C) PAYLOAD OFFSET.....	51
FIG. 3.25: FREQUENCY RESPONSE CURVES FOR FLEXIBLE ROBOTIC MANIPULATOR FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE.....	52
FIG. 3.26: ANALYTICAL (A) NUMERICAL (B) TIME HISTORY, PHASE PORTRAIT, AND FFT (C) OF CRITICAL POINTS C, G, AND D IDENTIFIED IN FIG. 3.2 FOR PRIMARY RESONANCE CASE. ....	52

FIG. 3.27: VARIATION FREQUENCY RESPONSE CURVES WITH PAYLOAD MASS ( $\alpha_{mc}$ ) FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE.....	53
FIG. 3.28: VARIATION FREQUENCY RESPONSE CURVES WITH PAYLOAD INERTIA ( $\alpha_i$ ) FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE.....	53
FIG. 3.29: VARIATION FREQUENCY RESPONSE CURVES WITH ACTUATOR INERTIA ( $\alpha_h$ ) FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE.....	54
FIG. 3.30: VARIATION OF FREQUENCY RESPONSE CURVES WITH OFFSET RATIO ( $\alpha_c$ ) FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE.....	54
FIG. 3.31 : VARIATION OF FREQUENCY RESPONSE CURVES WITH FREQUENCY PARAMETER ( $\Omega_h$ ) FOR (A) PRIMARY AND (B) SUB-HARMONIC RESONANCE. ....	54
FIG. 4.1: SCHEMATIC DIAGRAM OF DEFLECTED PLANAR TWO-LINK FLEXIBLE MANIPULATOR.....	57
FIG. 4.4: MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR (A) MODE 1 (B) MODE 2 (C) MODE 3 (D) MODE 4. ....	66
FIG. 4.5: EFFECT OF SYSTEM ( $\alpha_{m1}, \alpha_{m2}$ ) MASS PARAMETERS ON THE MODE SHAPES OF THE TWO-LINK MANIPULATOR (A) MODE 1 (B) MODE 2. ....	66
FIG. 4.6: EFFECT OF FLEXURAL RIGIDITY RATIO ( $\chi$ ) ON THE MODE SHAPES OF THE TWO-LINK MANIPULATOR (A) MODE 1 (B) MODE 2. ....	67
FIG. 4.7: EFFECT OF BEAM MASS DENSITY RATIO ( $\alpha_M$ ) ON THE MODE SHAPES OF THE TWO-LINK MANIPULATOR (A) MODE 1 (B) MODE 2. ....	67
FIG. 4.8: REPRESENTATIVE (A) FREQUENCY RESPONSE CURVE AND (B) EFFECT OF PAYLOAD MASS PARAMETER ( $\alpha_{m2}$ ) FOR SECOND LINK. ....	68
FIG. 4.9: EFFECT OF (A) BEAM MASS DENSITY ( $\alpha_M$ ) AND (B) FLEXURAL RIGIDITY ( $\chi$ ) RATIO ON FREQUENCY RESPONSE CURVE OF SECOND LINK. ....	68
FIG. 4.10: EFFECT OF GEOMETRIC NONLINEARITY DUE TO (A) AXIAL STRETCHING ( $\alpha_1$ ) OF FIRST AND (B) SECOND ( $\alpha_3$ ) LINK ON FREQUENCY RESPONSE CURVE OF SECOND LINK.....	69
FIG. 4.11: EFFECT OF (A) DAMPING AND (B) AMPLITUDE OF FIRST LINK ON FREQUENCY RESPONSE CURVE OF SECOND LINK. ....	69
FIG. 4.13: EFFECT OF (A) PAYLOAD MASS ( $\alpha_{mc}$ ), (B) INERTIA ( $\alpha_i$ ), AND (C) OFFSET RATIO ( $\alpha_c$ ) ON THE EIGENFREQUENCIES OF FLEXIBLE TWO-LINK MANIPULATOR FOR ( $\alpha_{m1}$ ) =0, $\alpha_{m2}$ =1.0, $\chi$ =1.0, $\alpha_i$ =1.0, AND $\gamma$ =0.....	76
FIG. 4.14: VARIATION OF FIRST FOUR MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PAYLOAD MASS ( $\alpha_{mc}$ ) (A) MODE 1 (B) MODE 2 (C) MODE 3 (D) MODE4.....	77
FIG. 4.15: VARIATION OF MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PAYLOAD INERTIA PARAMETER ( $\alpha_i$ ) (A) MODE 1 (B) MODE 2. ....	77
FIG. 4.16: VARIATION OF MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH OFFSET RATIO ( $\alpha_c$ ) (A) MODE 1 (B) MODE 2. ....	78
FIG. 4.17: VARIATION OF MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH OFFSET ANGLE ( $\gamma$ ) (A) MODE 1 (B) MODE 2. ....	78
FIG. 4.18: FREQUENCY RESPONSE CHARACTERISTICS OF (A) FIRST AND (B) SECOND LINK OF TWO-LINK MANIPULATOR WITH GENERIC PAYLOAD. ....	79
FIG. 4.19: ANALYTICAL (A) NUMERICAL (B) TIME HISTORY, PHASE PORTRAIT, AND FFT (C) OF FIRST LINK IDENTIFIED IN FIG. 4.18 AT THE POINT A, B, AND C. ....	80
FIG. 4.20: COMPARISON OF (A) FOUR DIFFERENT PAYLOAD CONDITIONS AND (B) LINEAR AND NONLINEAR MODEL. ....	80
FIG. 4.21: VARIATION OF FREQUENCY RESPONSE CURVES OF (A) FIRST AND (B) SECOND LINK WITH PAYLOAD MASS ( $\alpha_{mc}$ ). ....	81
FIG. 4.22: VARIATION OF FREQUENCY RESPONSE CURVES OF (A) FIRST AND (B) SECOND LINK WITH PAYLOAD INERTIA ( $\alpha_i$ ) . ....	81
FIG. 4.23: VARIATION OF FREQUENCY RESPONSE CURVES OF (A) FIRST AND (B) SECOND LINK WITH PAYLOAD OFFSET ( $\alpha_c$ ).....	81

FIG. 4.24: VARIATION OF FREQUENCY RESPONSE CURVES OF (A) FIRST AND (B) SECOND LINK WITH BEAM MASS DENSITY RATIO ( $\alpha_M$ ).	82
FIG. 4.25: VARIATION OF FREQUENCY RESPONSE CURVES OF (A) FIRST AND (B) SECOND LINK WITH FLEXURAL RIGIDITY RATIO ( $\chi$ ).	82
FIG. 4.26: A PLANAR TWO-LINK FLEXIBLE ROBOTIC MANIPULATOR CONNECTED WITH FLEXIBLE REVOLUTE PAIR.	83
FIG. 4.27: VARIATION OF EIGENFREQUENCIES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH (A) PAYLOAD MASS PARAMETER ( $\alpha_{m2}$ ) AND (B) BEAM MASS DENSITY PARAMETER ( $\alpha_M$ ) FOR $\chi = 1.0$ , $\alpha_{lh1} = 1.0$ , $\alpha_{lh2} = 1.0$ , AND $\Omega_{h1,2} = 0.5$ .	93
FIG. 4.28: VARIATION OF EIGENFREQUENCIES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH (A) FLEXURAL RIGIDITY RATIO ( $\chi$ ) AND (B) JOINT FREQUENCY PARAMETERS ( $\Omega_{h1,2}$ ) FOR $\alpha_{m1} = 1.0$ , $\alpha_{m2} = 1.0$ , $\alpha_{lh1} = 1.0$ , $\alpha_{lh2} = 1.0$ , AND $\alpha_M = 1.0$ .	94
FIG. 4.29: VARIATION OF EIGENFREQUENCIES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH (A) AXIAL OFFSET LENGTH ( $\alpha_c$ ) AND (B) PAYLOAD INERTIA ( $\alpha_c$ ) FOR $\alpha_{mC} = 1.0$ , $\alpha_{m1} = 1.0$ , $\chi = 1.0$ , $\alpha_M = 1.0$ , $\alpha_{lh1} = 1.0$ , $\alpha_{lh2} = 1.0$ , $\Omega_{h1}, \Omega_{h2} = 0$ .	94
FIG. 4.31: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH VARYING PAYLOAD MASS PARAMETER ( $\alpha_{m1}, \alpha_{m2}$ ) (A) MODE 1 (B) MODE 2	95
FIG. 4.32: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH VARYING BEAM DENSITY PARAMETER ( $\alpha_M$ ) (A) MODE 1 (B) MODE 2	96
FIG. 4.33: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH VARYING FLEXURAL RIGIDITY RATIO ( $\chi$ ) (A) MODE 1 (B) MODE 2	96
FIG. 4.34: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH VARYING JOINT INERTIA PARAMETERS ( $\alpha_{lh1}, \alpha_{lh2}$ ) (A) MODE 1 (B) MODE 2	96
FIG. 4.35: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH VARYING JOINT FREQUENCY PARAMETERS ( $\Omega_{h1}, \Omega_{h2}$ ) (A) MODE 1 (B) MODE 2	97
FIG. 4.36: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH AXIAL OFFSET RATIO ( $\alpha_c$ ) (A) MODE 1 (B) MODE 2	97
FIG. 4.37: VARIATION OF MODE SHAPES OF TWO-LINK MANIPULATOR HAVING REVOLUTE PAIR WITH AXIAL OFFSET RATIO ( $\alpha_c$ ) (A) MODE 1 (B) MODE 2	98
FIG. 4.38: EFFECT OF NONLINEARITIES ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	99
FIG. 4.39: (A) NUMERICAL AND (B) ANALYTICAL TIME HISTORY AND PHASE PORTRAIT, (C) FFT AT CRITICAL POINTS A, B, AND C IDENTIFIED IN FIG. 4.38.	99
FIG. 4.40: EFFECT OF VARIATION OF PAYLOAD MASS PARAMETER ( $\alpha_{m2}$ ) ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	100
FIG. 4.41: EFFECT OF VARIATION OF BEAM DENSITY PARAMETER ( $\alpha_M$ ) ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	100
FIG. 4.42: EFFECT OF VARIATION OF FLEXURAL RIGIDITY RATIO ( $\chi$ ) ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	101
FIG. 4.44: EFFECT OF VARIATION OF SECOND FREQUENCY PARAMETER ( $\Omega_{h1,2}$ ) ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	102
FIG. 4.45: EFFECT OF DAMPING PARAMETERS ( $\xi_{1,2}$ ) ON FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK.	102
FIG. 4.46: INPUT TORQUE PROFILE TO THE JOINTS.	103
FIG. 4.47: EFFECT OF VARIATION OF PAYLOAD MASS ( $m_c$ ) ON THE ANGULAR POSITIONS, TIP DISPLACEMENTS AND TIP ACCELERATIONS OF (A) FIRST LINK (B) SECOND LINK.	104
FIG. 4.48: EFFECT OF VARIATION OF PAYLOAD INERTIA ( $I_c$ ) ON THE ANGULAR POSITIONS OF (A) FIRST AND (B) SECOND LINK.	105

FIG. 4.49: EFFECT OF VARIATION OF OFFSET LENGTH ( $c$ ) ON THE ANGULAR POSITIONS OF (A) FIRST AND (B) SECOND LINK. ....	105
FIG. 4.50: EFFECT OF VARIATION OF LENGTH OF FIRST LINK ( $L_1$ ) ON THE ANGULAR POSITIONS OF (A) FIRST AND (B) SECOND LINK. ....	105
FIG. 4.51: EFFECT OF VARIATION OF LENGTH OF SECOND LINK ( $L_2$ ) ON THE ANGULAR POSITIONS OF (A) FIRST AND (B) SECOND LINK. ....	106
FIG. 4.52: EFFECT OF JOINT MASS ( $m_1$ ) ON THE ANGULAR POSITIONS OF (A) FIRST AND (B) SECOND LINK. .	106
FIG. 4.53: EFFECT OF JOINT INERTIA ( $I_{h1,2}$ ) ON THE ANGULAR POSITIONS AND TIP ACCELERATIONS OF (A) FIRST LINK AND (B) SECOND LINK. ....	107
FIG. 4.54: EFFECT OF MATERIAL OF THE LINKS ON THE ANGULAR POSITIONS AND TIP ACCELERATIONS OF (A) FIRST AND (B) SECOND LINK. ....	107
FIG. 4.56: EFFECT OF VARIATION OF SECOND JOINT MASS ON THE INPUT TORQUES OF (A) FIRST LINK AND (B) SECOND LINK. ....	109
FIG. 4.57: VARIATION OF JOINT INERTIA ( $I_{h1,2}$ ) ON THE INPUT TORQUES, AND TIP DISPLACEMENTS OF (A) FIRST AND (B) SECOND LINK. ....	110
FIG. 4.58: VARIATION OF FIRST LINK LENGTH ( $L_1$ ) ON INPUT TORQUES, AND TIP DISPLACEMENTS OF (A) FIRST AND (B) SECOND LINK. ....	110
FIG. 4.59: VARIATION OF MATERIAL ON THE INPUT TORQUES, AND TIP DISPLACEMENTS OF (A) FIRST LINK AND (B) SECOND LINK. ....	111
FIG. 4.61: INFLUENCE OF (A) ACTUATOR MASS ( $\alpha_{ma}$ ) AND (B) FREQUENCY PARAMETER ( $\Omega_a$ ) ON EIGENFREQUENCIES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PRISMATIC AND REVOLUTE JOINTS FOR $\alpha_{m1,2}=1.0$ , $\alpha_{lh}=1.0$ , $\Omega_a=0.5$ , AND ( $\alpha_M$ )=1.0. ....	118
FIG. 4.63: INFLUENCE OF ACTUATOR MASS PARAMETER ( $\alpha_{ma}$ ) ON MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PRISMATIC AND REVOLUTE JOINTS (A) MODE 1 (B) MODE 2. ....	119
FIG. 4.64: INFLUENCE OF FLEXURAL RIGIDITY RATIO ( $\chi$ ) ON MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PRISMATIC AND REVOLUTE JOINTS (A) MODE 1 (B) MODE 2. ....	119
FIG. 4.66: INFLUENCE OF JOINT FREQUENCY PARAMETER ( $\Omega_h$ ) ON MODE SHAPES OF FLEXIBLE TWO-LINK MANIPULATOR WITH PRISMATIC AND REVOLUTE JOINTS (A) MODE 1 (B) MODE 2. ....	120
FIG. 4.67: FREQUENCY RESPONSE CURVE OF THE (A) FIRST AND (B) SECOND LINK OF FLEXIBLE TWO-LINK MANIPULATOR WITH PRISMATIC AND REVOLUTE JOINT FOR PRIMARY RESONANCE CASE. ....	121
FIG. 4.68: (A) NUMERICAL AND (B) ANALYTICAL TIME HISTORY, PHASE PORTRAIT, AND (C) FFT AT CRITICAL POINTS A, B, AND F IDENTIFIED IN FIG. 4.67. ....	122
FIG. 4.70: VARIATION OF FREQUENCY RESPONSE CURVE OF SECOND LINK WITH (A) ACTUATOR MASS PARAMETER ( $\alpha_{ma}$ ) AND (B) BEAM MASS DENSITY RATIO ( $\alpha_M$ ). ....	122
FIG. 4.71: VARIATION OF FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK WITH FLEXURAL RIGIDITY RATIO ( $\chi$ ). ....	123
FIG. 4.72: VARIATION OF FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK JOINT INERTIA PARAMETER ( $\alpha_{lh}$ ). ....	123
FIG. 4.73: VARIATION OF FREQUENCY RESPONSE CURVE OF (A) FIRST AND (B) SECOND LINK WITH ACTUATOR FREQUENCY PARAMETER ( $\Omega_a$ ) AND JOINT FREQUENCY PARAMETER ( $\Omega_h$ ). ....	124
FIG. 4.74: COMPARISON OF FREQUENCY RESPONSE CURVE OF SECOND LINK FOR (A) PRIMARY AND INTERNAL RESONANCE AND (B) EFFECT OF AMPLITUDE OF FIRST LINK ( $a_1$ ). ....	125
FIG. 4.75: VARIATION OF FREQUENCY RESPONSE CURVE OF SECOND LINK WITH (A) PAYLOAD MASS PARAMETER ( $\alpha_{m2}$ ) AND (B) JOINT INERTIA PARAMETER ( $\alpha_{lh}$ ) FOR INTERNAL RESONANCE CASE. ....	125
FIG. 5.2: INFLUENCE OF PAYLOAD MASS PARAMETER ( $\alpha_{m3}$ ) ON MODE SHAPES WITH $\alpha_{m1,2}=1.0$ , $\alpha_{M1,2}=1.0$ , $\alpha_{L1,2}=1.0$ , $\chi_{1,2}=1.0$ , $\Omega_{h1,2,3}=0.5$ , , AND $\alpha_{lh1,2,3}=1.0$ (A) MODE 1 (B) MODE 2. ....	138
FIG. 5.3: INFLUENCE OF FIRST AND SECOND BEAM MASS DENSITY PARAMETER ( $\alpha_{M1,2}$ ) ON MODE SHAPES WITH $\alpha_{m1,2,3}=1.0$ , $\chi_{1,2}=1.0$ , $\alpha_{L1,2}=1.0$ , $\Omega_{h1,2,3}=0.5$ , , AND $\alpha_{lh1,2,3}=1.0$ (A) MODE 1 (B) MODE 2. ....	139

FIG. 5.6: INFLUENCE OF HUB JOINT FREQUENCY PARAMETER ( $\Omega_{1,3}$ ) ON MODE SHAPES WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\alpha_{I_{h_{1,2,3}}} = 1.0$ , $\chi_{1,2} = 1.0$ , AND $\Omega_{h_2} = 1.0$ (A) MODE 1 (B) MODE 2.....	141
FIG. 5.7: INFLUENCE OF PAYLOAD MASS ( $m_3$ ) ON THE ANGULAR POSITIONS AND TIP ACCELERATIONS OF THE (A) FIRST LINK (B) SECOND LINK (C) THIRD LINK.....	142
FIG. 5.8: INFLUENCE THIRD JOINT MASS ( $m_2$ ) ON THE ANGULAR POSITIONS OF THE (A) FIRST LINK (B) SECOND LINK (C) THIRD LINK. ....	142
FIG. 5.9: INFLUENCE JOINT INERTIAS ( $I_{h_{1,2,3}}$ ) ON THE ANGULAR POSITIONS AND TIP ACCELERATIONS OF THE (A) FIRST LINK (B) SECOND LINK (C) THIRD LINK.....	143
FIG. 5.10: INFLUENCE LENGTH OF LINKS ( $L_2$ ) ON THE ANGULAR POSITIONS AND MODAL DISPLACEMENTS OF THE THE (A) FIRST LINK (B) SECOND LINK (C) THIRD LINK.....	143
FIG. 5.11: FREQUENCY RESPONSE CURVES OF (A) THE SECOND FOR INTERNAL RESONANCE AND (B) THE FIRST LINK FOR PRIMARY RESONANCE. ....	144
FIG. 5.12: (A) NUMERICAL AND (B) ANALYTICAL TIME HISTORIES, PHASE PORTRAIT AND (C) FFT OF CRITICAL POINTS G, H, AND I IDENTIFIED IN FIG. 5.11. ....	144
FIG. 5.13: INFLUENCE OF INITIAL EXCITATION ( $a_1$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	145
FIG. 5.14: INFLUENCE OF PAYLOAD MASS ( $\alpha_{m_3}$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	145
FIG. 5.15: INFLUENCE OF SECOND LINK FLEXURAL RIGIDITY RATIO ( $\chi_1$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_2 = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	146
FIG. 5.16: INFLUENCE OF THIRD LINK BEAM MASS DENSITY PARAMETER ( $\alpha_{M_2}$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_1} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	146
FIG. 5.17: INFLUENCE OF HUB AND JOINT INERTIA PARAMETER ( $\alpha_{I_{h_{1,2,3}}}$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , AND $\Omega_{h_{1,2,3}} = 0.5$ . ....	147
FIG. 5.18: INFLUENCE OF FIRST HUB JOINT FREQUENCY PARAMETER ( $\Omega_{h_1}$ ) ON FREQUENCY RESPONSE CURVES OF (A) SECOND AND (B) THIRD LINK FOR INTERNAL RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_{1,2}} = 1.0$ , $\Omega_{h_{2,3}} = 0.5$ . AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	147
FIG. 5.19: INFLUENCE OF PAYLOAD MASS ( $\alpha_{m_3}$ ) ON FREQUENCY RESPONSE CURVES OF (A) FIRST, (B) SECOND AND (C) THIRD LINK FOR SIMPLE RESONANCE WITH $\alpha_{m_{1,2}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_1} = 1.1$ , $\alpha_{L_2} = 1.2$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	148
FIG. 5.20: INFLUENCE OF SECOND LINK FLEXURAL RIGIDITY RATIO ( $\chi_1$ ) ON FREQUENCY RESPONSE CURVES OF (A) FIRST, (B) SECOND AND (C) THIRD LINK FOR SIMPLE RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_2 = 1.0$ , $\alpha_{L_1} = 1.1$ , $\alpha_{L_2} = 1.2$ , $\Omega_{h_{1,2,3}} = 0.5$ , $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	148
FIG. 5.21: INFLUENCE OF SECOND LINK BEAM MASS DENSITY PARAMETER ( $\alpha_{M_1}$ ) ON FREQUENCY RESPONSE CURVES OF (A) FIRST, (B) SECOND AND (C) THIRD LINK FOR SIMPLE RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_2} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_1} = 1.1$ , $\alpha_{L_2} = 1.2$ , $\Omega_{h_{1,2,3}} = 0.5$ , AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	148
FIG. 5.22: INFLUENCE OF HUB AND JOINT INERTIA PARAMETER ( $\alpha_{I_{h_{1,2,3}}}$ ) ON FREQUENCY RESPONSE CURVES OF (A) FIRST, (B) SECOND AND (C) THIRD LINK FOR SIMPLE RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_1} = 1.1$ , $\alpha_{L_2} = 1.2$ , AND $\Omega_{h_{1,2,3}} = 0.5$ . ....	149
FIG. 5.24: INFLUENCE OF THIRD HUB JOINT FREQUENCY PARAMETER ( $\Omega_{h_3}$ ) ON FREQUENCY RESPONSE CURVES OF (A) FIRST, (B) SECOND AND (C) THIRD LINK FOR SIMPLE RESONANCE WITH $\alpha_{m_{1,2,3}} = 1.0$ , $\alpha_{M_{1,2}} = 1.0$ , $\chi_{1,2} = 1.0$ , $\alpha_{L_1} = 1.1$ , $\alpha_{L_2} = 1.2$ , $\Omega_{h_{1,2}} = 0.5$ AND $\alpha_{I_{h_{1,2,3}}} = 1.0$ . ....	149

## List of Tables

TABLE 3-1: VARIATION OF EIGENFREQUENCIES WITH SYSTEM PARAMETERS.....	32
TABLE 4-1: VARIATION OF EIGENFREQUENCY PARAMETER WITH TIP MASS PARAMETER ( $\alpha_{m2}$ ).....	64
TABLE 4-2: VARIATION OF EIGENFREQUENCY PARAMETER WITH SYSTEM MASS PARAMETER ( $\alpha_{m1}, \alpha_{m2}$ ).....	64
TABLE 4-3: VARIATION OF EIGENFREQUENCY PARAMETER WITH FLEXURAL RIGIDITY RATIO ( $\chi$ ).....	65
TABLE 4-4: VARIATION OF EIGENFREQUENCY PARAMETER WITH BEAM MASS DENSITY PARAMETER ( $\alpha_M$ )...	65
TABLE 4-5: VARIATION OF EIGENFREQUENCY PARAMETER WITH LENGTH PARAMETER ( $\alpha_L$ ).....	65
TABLE 4-6: VARIATION OF EIGENFREQUENCIES ( $\bar{\delta}_i$ ) OF TWO-LINK MANIPULATOR WITH SYSTEM PARAMETERS.....	91

## List of Symbols

Symbol	Description
$\alpha_m$	Mass parameter.
$\alpha_{mC}$	Payload mass parameter.
$\alpha_{ma}$	Actuator mass parameter.
$\alpha_M$	Beam mass density ratio.
$\alpha_L$	Link length parameter.
$\alpha_{Ih}$	Joint inertia parameter.
$\chi$	Flexural rigidity ratio.
$\alpha_c$	Offset ratio.
$\alpha_{Ic}$	Payload inertia parameter.
$\alpha_{ah}$	Actuator inertia parameter.
$\gamma$	Offset angle.
$E$	Young's modulus of material of link.
$I$	Moment of inertia of link.
$I_C$	Mass moment of inertia of generic payload.
$k_a$	Stiffness of the Cartesian joint.
$k_\theta$	Stiffness of the revolute joint.
$\rho$	Mass density of the material of link.
$A$	Area of cross-section of the link.
$m_C$	Mass of generic payload.
$m_a$	Mass of actuator.
$x$	Abscissa of general point on the undeformed link.
$y$	Ordinate of general point on the undeformed link.
$(X, Y)$	Inertial co-ordinate system.
$g(t)$	Horizontal base motion.
$\eta(t)$	Vertical base motion.
$\delta$	Eigenfrequency of the manipulator.
$\omega_m$	Natural frequency of manipulator.
$n$	$n^{\text{th}}$ mode of vibration.
$\theta$	Angular motion of joint.
$u$	Axial deformation of link.
$w$	Transverse deformation of the link.
$I_h$	Mass moment of inertia of the revolute joint.
$I_a$	Mass moment of inertia of actuator.
$\Omega_a$	Frequency parameter of actuator.
$\Omega_h$	Frequency parameter of the revolute joint.

## List of Abbreviations

<i>Abbreviation</i>	<i>Full form</i>
<i>cc</i>	<i>Complex Conjugate</i>
<i>PD</i>	<i>Proportional Derivative</i>
<i>PID</i>	<i>Proportional Integral Derivative</i>
<i>MMS</i>	<i>Method of Multiple Scales</i>
<i>FFT</i>	<i>Fast Fourier Transform</i>
<i>IED</i>	<i>Improvised Explosive Device</i>
<i>FFT</i>	<i>Fast Fourier Transform</i>



