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Multidimensional Shift Spaces and Infinite Matrices

In this chapter, we address the problem of characterizing the elements of a multidimensional shift of finite type X using a square matrix. In the process, we characterize the elements of the multidimensional shift space using an infinite square matrix. We prove that a multidimensional shift of finite type is nonempty if the characterizing matrix is of positive dimension. Further, we prove that any submatrix of the characterizing matrix generates a proper subshift and hence the characterizing matrix is minimal in this sense. We prove that the elements of a shift of finite type can equivalently be characterized by limits of periodic configurations arising from cubes allowed for the shift space X. We introduce the concept of complementary set for a multidimensional shift space. Recall that, for a matrix M indexed by the set $\{\mathfrak{i} : \mathfrak{i} \in \mathfrak{I}\}$, a nonempty subset K of the index set $\mathfrak{I}$ is said to be complementary if for each $\mathfrak{i} \in K$, there exists $\mathfrak{j}, k \in K$ such that $\mathfrak{j}$ is u-related to $\mathfrak{i}$ and k is d-related to $\mathfrak{i}$. Generalizing the notions, for any two infinite strips P, Q of height l, we say that P is u-related (d-related) to Q for the shift space X if $\begin{pmatrix} P \\ Q \end{pmatrix}$ ($\begin{pmatrix} Q \\ P \end{pmatrix}$) is allowed for X. Further, a collection of infinite strips $\mathscr{A}$ (of height l) is complementary if for any $P \in \mathscr{A}$ there exists $Q, R \in \mathscr{A}$ such that Q is u-related to P and R is d-related to P. We prove that a shift space is nonempty if the set of indices of the characterizing matrix is a complementary set. We also characterize the periodic points of the shift space using complementary sets.

2.1 A BASIC OBSERVATION

Proposition 2.1.1 *X is a d-dimensional shift of finite type $\Longrightarrow$ there exists a set $\mathscr{C}$ of d-dimensional cubes such that $X = X_{\mathscr{C}}$.*

Proof. Let X be a shift of finite type and let $\mathscr{F}$ be the minimal forbidden set of patterns for the shift space X. It may be noted that $\mathscr{F}$ contains finitely many patterns defined over finite subsets of $\mathbb{Z}^d$. For any pattern p in $\mathscr{F}$, let l_p^i be the length of the pattern p in the i-th direction. Let $l_p = \max\{l_P^i : i = 1, 2, \ldots, d\}$ denote the width of the pattern p and let $l = \max\{l_p : p \in \mathscr{F}\}$. Let $\mathbb{C}_l$ be the collection of d-dimensional cubes of length l and let $\mathbb{E}_{\mathscr{F}}$ denote the set of extensions of patterns in $\mathscr{F}$. Let $\mathscr{C} = \mathbb{C}_l \cap \mathbb{E}_{\mathscr{F}}$. It may be observed that if p is a pattern with width l, forbidding a pattern p for X is equivalent to forbidding all extensions q of p in $\mathscr{C}_l$. Thus, each pattern in the forbidden set of width l can be replaced by an equivalent forbidden set of cubes of length l and $\mathscr{C}$ is an equivalent forbidden set for the shift space X. Consequently, $X = X_{\mathscr{C}}$ and the proof is complete. □

Remark 2.1.1 *The above result proves that every d-dimensional shift of finite type can equivalently be generated by a set of cubes of fixed finite length. The proof constructs an equivalent forbidden set by considering all the cubes which are extension of the set of patterns in $\mathscr{F}$. Such a consideration leads to an equivalent forbidden set for the multidimensional shift space X, which in general is not minimal. However, the cardinality of the new set can be reduced by considering only those cubes which are not proper extensions of patterns in $\mathscr{F}$ (but are of same size l). Such a construction reduces the cardinality of the forbidden set considerably and hence reduces the complexity of the* *system. It may be noted that the forbidden set obtained on reduction is still not minimal. However, the d-dimensional cubes generating the elements of X are of same size and can be used for generating the shift space X. We say that a shift of finite type X is generated by cubes of length l if there exists a set of cubes $\mathscr{C}$ of length l such that $X = X_{\mathscr{C}}$.*

2.2 CHARACTERIZATION OF TWO DIMENSIONAL SHIFT OF FINITE TYPE

Proposition 2.2.1 *Every 2-dimensional shift of finite type X can be characterized by an infinite square matrix.*

Proof. Let X be a 2-dimensional shift of finite type and let $\mathscr{F}$ be the equivalent set of forbidden cubes (of fixed length, say l) for the space X. Let $\mathscr{A}$ be the generating set of cubes (of length l) for the space X. It may be noted that as cubes of length l form a generating set for the shift space X, to verify whether any $x \in A^{\mathbb{Z}^d}$ belongs to X, it is sufficient to examine strips of height l in x.

$$\text{Let } \mathscr{A}^2 = \{\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} : S_1, S_2 \in \mathscr{A}, \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \text{ is allowed in } X\}.$$

By construction, $\mathscr{A}^2$ is a finite set of $2l \times l$ allowed rectangles, say $\{a_1, a_2, \ldots, a_k\}$, generating the shift space X.

Define a $k \times k$ matrix M as

$$M_{ij} = \begin{cases} 0, & (a_i a_j) \text{ is forbidden in } X; \\ 1 & (a_i a_j) \text{ is allowed in } X; \end{cases}$$

Let $\Sigma_M = \{(x_n) : M_{x_i x_{i+1}} = 1, \ \forall i\}$ be the shift of finite type generated by M. Note that for any (x_n) in Σ_M, (a_{x_n}) is a valid infinite strip of height $2l$ for the shift space X. Consequently, elements of Σ_M precisely generate the infinite strips (of height $2l$) allowed for the shift space X. It may be noted that any element in Σ_M is an element of the form $\begin{pmatrix} P \\ Q \end{pmatrix}$, where P and Q are allowed infinite strips of height l.

Generate an infinite matrix $\mathfrak{M}$, indexed by allowed infinite strips of height l, using the following algorithm:

1. Pick any $\begin{pmatrix} P \\ Q \end{pmatrix} \in \Sigma_M$ and index first two rows and columns of the matrix by P and Q. Set $\mathfrak{m}_{QP} = 1$.

2. For each $\begin{pmatrix} P \\ Q \end{pmatrix} \in \Sigma_M$, if the rows and columns indexed P and Q exist, set $\mathfrak{m}_{QP} = 1$. Else, label next row and/or column as P and/or Q (whichever required) and set $\mathfrak{m}_{QP} = 1$.

3. In the infinite matrix generated in step 2, set $\mathfrak{m}_{QP} = 0$, if $\mathfrak{m}_{QP}$ has so far not been assigned a value.

4. In the infinite matrix obtained, if there exists an index P such that the P-th row or column is zero, delete the P-th row and column from the matrix generated.

The above algorithm generates an infinite 0-1 matrix where $\mathfrak{m}_{QP} = 1$ if and only if $\begin{pmatrix} P \\ Q \end{pmatrix}$ is allowed in X, where P and Q are allowed infinite strips (of height l) in X. Let $\Sigma_{\mathfrak{M}}$ be the sequence space associated with the matrix $\mathfrak{M}$. Consequently, any sequence in $\Sigma_{\mathfrak{M}}$ gives a vertical arrangement of infinite allowed strips (of height l) such that the arrangement is allowed in X and hence generates an element in X. Conversely, any element in X is a sequential (vertical) arrangement of infinite strips of height l and hence is generated by a sequence in $\Sigma_{\mathfrak{M}}$. Consequently, $X = \Sigma_{\mathfrak{M}}$ and the proof is complete. □

Remark 2.2.1 *The above result characterizes elements of the shift space X by an infinite square matrix $\mathfrak{M}$. It may be noted that if the row/column corresponding to an index A is zero, the proposed algorithm deletes both (the row and the column) corresponding to the index A from the generating matrix $\mathfrak{M}$. Such a criteria reduces the size of the matrix $\mathfrak{M}$ and results in a matrix of dimension (size) 0, if the shift space is empty. Further, the characterization of the space may yield a matrix of infinite (uncountable) dimension. Consequently, it is undecidable whether a shift of finite type generated by a set of cubes $\mathscr{A}$ is nonempty (as observed in [Berger, 1966]). It may be noted that although the algorithm does not guarantee a positive dimensional matrix if the shift space X is nonempty the matrix generated is definitely of positive dimension and characterizes the elements in X. Further, as each row/column of the matrix generated has at least one nonzero entry, each block indexing the matrix can be extended to an element of X. Consequently, any submatrix of the matrix $\mathfrak{M}$ cannot generate the shift space X and hence the matrix $\mathfrak{M}$ is minimal in this sense. In light of the remark stated, we get the following corollary.*

Corollary 2.2.1 *A 2-dimensional shift of finite type is nonempty if and only if the characterizing matrix $\mathfrak{M}$ is of positive dimension (of nonzero size). Further, any proper submatrix of the matrix $\mathfrak{M}$ generates a proper subshift and hence the matrix $\mathfrak{M}$ is minimal.*

Remark 2.2.2 *For a shift space X, with generating set of cubes of height l, let $\mathscr{L}$ denote set of all allowed infinite strips of height l. Thus, the algorithm generates u-related (d-related) infinite strips for the shift space X which in turn generates an arbitrary element of X. As any element of the shift space is a sequential arrangement of u-related (d-related) infinite strips, the characterization of the elements of the space X by a matrix $\mathfrak{M}$ is equivalent to finding all the u-related (d-related) pairs of infinite strips for the space X. As any infinite strip of height l (say P) can be extended to an element of X only if there exist infinite strips Q, R of height l such that Q is u-related to P and R is d-related to P, only members of complementary family can form the building blocks for an element of X. As a result, we get the following corollary.*

Corollary 2.2.2 *Let X be a two dimensional shift space generated by cubes of length l and let $\mathfrak{B}$ be the infinite strips of height l allowed in X. Then, the shift space X is nonempty if and only if there exists nonempty set of indices $\mathfrak{B}_0 \subseteq \mathfrak{B}$ such that $\mathfrak{B}_0$ is complementary.*

Example 2.2.1 *Let Σ_2 be the two dimensional full shift over two symbols $\{0,1\}$ and let X be the shift of finite type generated by the forbidden set $\mathscr{F}_1 = \left\{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \ 11 \right\}$. Then, X is also generated by the frobidden set*

$$\mathscr{F}_2 = \left\{ \begin{smallmatrix} 0\,1 \\ 0\,1 \end{smallmatrix}, \begin{smallmatrix} 0\,0 \\ 1\,1 \end{smallmatrix}, \begin{smallmatrix} 1\,1 \\ 0\,0 \end{smallmatrix}, \begin{smallmatrix} 1\,0 \\ 1\,0 \end{smallmatrix}, \begin{smallmatrix} 0\,1 \\ 1\,1 \end{smallmatrix}, \begin{smallmatrix} 1\,0 \\ 1\,1 \end{smallmatrix}, \begin{smallmatrix} 1\,1 \\ 0\,1 \end{smallmatrix}, \begin{smallmatrix} 1\,1 \\ 1\,0 \end{smallmatrix}, \begin{smallmatrix} 1\,1 \\ 1\,1 \end{smallmatrix} \right\}$$

Thus, any element of X is a sequential (two dimensional) arrangement of blocks in $\mathscr{A}_2$ where

$$\mathscr{A}_2 = \left\{ \begin{smallmatrix} 0\,0 \\ 0\,0 \end{smallmatrix}, \begin{smallmatrix} 0\,0 \\ 0\,1 \end{smallmatrix}, \begin{smallmatrix} 0\,0 \\ 1\,0 \end{smallmatrix}, \begin{smallmatrix} 0\,1 \\ 0\,0 \end{smallmatrix}, \begin{smallmatrix} 1\,0 \\ 0\,0 \end{smallmatrix}, \begin{smallmatrix} 0\,1 \\ 1\,0 \end{smallmatrix}, \begin{smallmatrix} 1\,0 \\ 0\,1 \end{smallmatrix} \right\}$$

Further, note that the set of all infinite strips of height 2 avoiding adjacent placement of 1's is a complementary set and generates the elements of X (via sequential vertical arrangement of such *strips avoiding adjacent placement of 1's). Also, as any element of X is a vertical arrangement of such strips, the collection of strips under consideration is a complementary set characterizing the elements of X. Finally, if M is a matrix indexed by such strips then the matrix indicating the vertical compatibility of the strips characterizes the elements of X (via the proposed algorithm).*

Remark 2.2.3 *It may be noted that although the above algorithm characterizes the elements of the shift space using (possibly) a matrix of infinite dimension, the same can be achieved by approximating each point of X by a sequence of periodic points (which may not lie in the shift space X). To illustrate, let $\mathscr{A}$ is the collection of generating cubes (of size l) of X and let $\mathscr{A}^r$ be the collection of all allowed cubes of $rl \times rl$ obtained by $r \times r$ arrangement of elements of $\mathscr{A}$. Let X_r denote the all periodic configurations arising from the collection $\mathscr{A}^r$ and let $L = \{(x_n) : x_n \in X_n\}$. Then elements of X are precisely the limit points of the sequences in L. Consequently, any element of the shift space can be obtained by approximation through periodic points (which may not lie in X themselves). Hence we get the following result.*

Proposition 2.2.2 *Any point in a 2-dimensional shift of finite type can be approximated by a sequence of periodic points.*

Proof. Let $\mathscr{A}$ denote the collection of generating cubes (of size l) of X and $\mathscr{A}^r$ be the collection of all allowed cubes of $rl \times rl$ obtained by $r \times r$ arrangement of elements of $\mathscr{A}$. Note that as all central blocks of an element in X are allowed, any element is a limit of periodic configurations (generated by its central blocks). Also, if x is a limit of periodic configurations arising from the collection $\mathscr{A}^r$, then any central block of x is allowed and hence x is an element of the shift space X (proof follows from the fact that any element belongs to X if and only if all central blocks of x are allowed in X). Consequently, if X_n denotes the collection of the periodic configurations arising from $\mathscr{A}^r$ and $L = \{(x_n) : x_n \in X_n\}$ then, elements of X are precisely the limit points of sequences in L and hence every point in X can be approximated by a sequence of periodic points. □

Remark 2.2.4 *The above discussions provide an alternate view of the criteria established for the nonemptiness of the space X. The result verifies the nonemptiness for a given shift space using complementary sets. It may be noted that the set of indices for the matrix $\mathfrak{M}$ form a complementary set and consequently generates the shift space under consideration. Further, as the matrix $\mathfrak{M}$ characterizes the elements of the shift space X, any superset of set of indices of $\mathfrak{M}$ cannot be complementary. Consequently, the proof generates the maximal complementary set for the shift space X. Although the matrix generated characterizes the elements of the shift space X, one does not require the matrix $\mathfrak{M}$ for establishing the nonemptiness for the shift space. The set of indices of the matrix may be observed at each iteration and existence of a complementary subfamily can be used to establish the nonemptiness of the space X. However, as the algorithm does not provide any optimal technique for picking the block $\begin{pmatrix} P \\ Q \end{pmatrix}$ at each iteration, such a consideration does not reduce the time complexity of the problem. However, algorithms for optimal selection of the infinite blocks $\begin{pmatrix} P \\ Q \end{pmatrix}$ may be proposed which in turn may reduce the time complexity of the algorithm. As the above algorithm can be extended for a general d dimensional shift of finite type, similar results are true for a general d-dimensional shift of finite type. We include the proof of the result below.*

2.3 CHARACTERIZATION OF MULTIDIMENSIONAL SHIFT OF FINITE TYPE

Proposition 2.3.1 *If X is a d-dimensional shift of finite type, then the elements of X can be determined by an infinite square matrix.*

Proof. Let X be a d-dimensional shift of finite type and let $\mathscr{F}$ be the equivalent set of forbidden cubes (of fixed length, say l) for the space X. Let $\mathscr{A}$ be the generating set of cuboids of size $\underbrace{2l \times 2l \times \ldots 2l}_{d-1 \ times} \times l$ for the space X.

By construction, $\mathscr{A}$ is a finite set of allowed rectangles, say $\{a_1, a_2, \ldots, a_k\}$. Define a $k \times k$ matrix $\mathfrak{M}^0$ as

$$\mathfrak{M}^0_{ij} = \begin{cases} 0, & (a_i a_j) \text{ is forbidden in } X; \\ 1 & (a_i a_j) \text{ is allowed in } X; \end{cases}$$

where $(a_i a_j)$ denotes adjacent placement of a_j with a_i in the positive d-th direction.

Then, the sequence space corresponding to the matrix $\mathfrak{M}^0$, $\Sigma_{\mathfrak{M}^0} = \{(x_n) : \mathfrak{M}^0_{x_i x_{i+1}} = 1, \quad \forall i\}$ generates all allowed one directional (in d-th direction) infinite strips in X.

It may be noted that any element in $\Sigma_{\mathfrak{M}^0}$ is element of the form $\begin{pmatrix} P \\ Q \end{pmatrix}_0$, where P and Q are allowed infinite strips (in direction d) of dimension $\underbrace{2l \times 2l \times \ldots 2l}_{d-2 \ times} \times l \times \infty$ and $\begin{pmatrix} P \\ Q \end{pmatrix}_0$ denotes adjacent placement of Q with P in the negative $d-1$-th direction.

Generate an infinite matrix $\mathfrak{M}^1$, indexed by allowed infinite strips of dimension $\underbrace{2l \times 2l \times \ldots 2l}_{d-2 \ times} \times l \times \infty$, using the following algorithm:

1. Pick any $\begin{pmatrix} P \\ Q \end{pmatrix}_0 \in \Sigma_{\mathfrak{M}^0}$ and index first two rows and columns of the matrix by P and Q. Set $\mathfrak{m}_{QP} = 1$.

2. For each $\begin{pmatrix} P \\ Q \end{pmatrix}_0 \in \Sigma_{\mathfrak{M}^0}$, if the rows and columns indexed P and Q exist, set $\mathfrak{m}_{QP} = 1$. Else, label next row and/or column as P and/or Q (whichever required) and set $\mathfrak{m}_{QP} = 1$.

3. In the infinite matrix generated in step 2, set $\mathfrak{m}_{QP} = 0$, if $\mathfrak{m}_{QP}$ has so far not been assigned a value.

4. In the infinite matrix obtained, if there exists an index P such that the P-th row or column is zero, delete the P-th row and column from the matrix.

The above algorithm generates an infinite 0-1 matrix where $\mathfrak{m}_{QP} = 1$ if and only if $\begin{pmatrix} P \\ Q \end{pmatrix}_0$ is allowed in X, where P and Q are of dimension $\underbrace{2l \times 2l \times \ldots 2l}_{d-2 \ times} \times l \times \infty$. Let $\Sigma_{\mathfrak{M}^1}$ denote the sequence space corresponding to the matrix generated above. It can be seen that the space $\Sigma_{\mathfrak{M}^1}$ precisely is the collection of allowed bi-infinite strips (in direction d and $d-1$). Further, as any element in $\Sigma_{\mathfrak{M}^1}$ is of the form $\begin{pmatrix} P \\ Q \end{pmatrix}_1$, where P and Q are allowed infinite strips (in direction d and $d-1$) of dimension $\underbrace{2l \times 2l \times \ldots 2l}_{d-3 \ times} \times l \times \infty \times \infty$ and $\begin{pmatrix} P \\ Q \end{pmatrix}_1$ denotes adjacent placement of Q with P in the negative $d-2$-th direction, a repeated application of the algorithm generates a matrix $\mathfrak{M}^2$ which extends the infinite patterns in $\Sigma_{\mathfrak{M}^1}$ along the direction $d-3$ to generate the space $\Sigma_{\mathfrak{M}^2}$. Consequently, repeated application of the above algorithm extends the allowed patterns infinitely in all the d directions (one direction at each step) to obtain a point in X. Further, as any point in X can be visualized as such an extension of allowed cubes in the d directions, the matrix obtained (at the final step) characterizes the elements of the space X. □

Remark 2.3.1 *The above result generalizes the result obtained for a two dimensional shift of finite type to a general d dimensional shift of finite type. The characterization is obtained by repeated application of the 2-dimensional case, extending the allowed blocks in each of the d directions. In the process, at each step, we obtain an infinite matrix characterizing the extension of an allowed block in the i-th direction. Although the rows and columns of the characterizing matrix $\mathfrak{M}$ are indexed by infinite blocks allowed in X, their existence is guaranteed as they are procured from the allowed blocks obtained in the previous step. It may be noted that extension in any of the directions (at i-th step) does not guarantee an extension to the element of X. In particular, a block extendable in a direction i (or in a few directions $i_1, i_2, \ldots, i_r$) need not necessarily extend to an element in X. In particular, if the shift space is empty, the positive dimension of the matrix at i-th step does not guarantee a matrix of positive dimension at the final step. Consequently, once again, the shift space is nonempty if and only if the matrix generated (at the final step) is of positive dimension. Thus, we obtain the following corollary.*

Corollary 2.3.1 *A multidimensional shift of finite type is nonempty if and only if the characterizing matrix $\mathfrak{M}$ is of positive dimension. Further, any proper submatrix of $\mathfrak{M}$ generates a proper subshift, and hence the matrix $\mathfrak{M}$ is minimal.*

Remark 2.3.2 *It may be noted that the matrix characterizing the elements of the multidimensional shift space is once again (possibly) infinite. However, such a construction helps in better visualization of the problem and can help in better understanding of the subsystems of the shift space under consideration. It may be noted that the elements of the shift space can be obtained as sequential limits of the periodic points generated using allowed cubes of finite size (which may not lie in the shift space itself). Consequently, the points of the multidimensional shift space can be obtained by approximations through periodic points (which may not lie in the shift space X). Note that as the matrix $\mathfrak{M}$ characterizes the elements of the shift space X, any superset of the set of indices of $\mathfrak{M}$ is not complementary and the matrix $\mathfrak{M}$ generated is once again minimal. Thus we get the following result.*

Proposition 2.3.2 *Any point in a d-dimensional shift of finite type can be approximated by a sequence of periodic points.*

Proof. Let $\mathscr{A}$ denote the collection of generating cubes (of size l) of X and $\mathscr{A}^r$ be the collection of all allowed cubes of side rl. It may be noted that any element of $\mathscr{A}^r$ is an $\underbrace{r \times r \times \ldots \times r}_{d \ \ times}$ arrangement of elements of $\mathscr{A}$. Let X_r denote the collection of all periodic configurations generated by elements of $\mathscr{A}^r$. As all central blocks of an element in X are allowed, any element of X is a limit of periodic configurations (generated by its central blocks spread along all the d directions). Also, if x is a limit of periodic configurations spread across the collection $\mathscr{A}^r$ (a sequence (x_n) with $x_n \in X_n$), then any central block of x is allowed and hence x is an element of the shift space X (proof follows from the fact that any element belongs to X if and only if all central blocks of x are allowed in X). Consequently, any element of X can be approximated by a sequence of periodic points and the proof is complete. □

Remark 2.3.3 *The above proof characterizes the points of the shift space as limits of periodic points generated by the allowed cubes for the shift space. Note that although the periodic points generated* *are periodic in all the d-directions (with the same period), the construction of periodic points can be further simplified by constructing them as adjacent tiling of a single element (of $\mathscr{A}^r$) throughout the $\mathbb{Z}^d$ domain. As the arguments given in the proof hold good in this setting too, elements of the shift space can be realized as limits of periodic points constructed in this manner (note that as periodicity in one direction need not imply periodicity in the other, periodic points, in general, have infinite orbits in the multidimensional shift space). Once again, the construction of elements of the shift space can be captured through the notion of complementary sets. As any element of the shift space can be visualized as an alignment of elements of a complementary set, the shift space is nonempty if and only if the exists a subset $\mathfrak{B}_0$ of indices (of the matrix obtained at the final step) which forms a complementary set. The result is an analogous extension of the result obtained for the two dimensional case and hence characterize the elements of the shift space X. Hence we get the following corollary.*

Corollary 2.3.2 *Let X be a multidimensional shift space and let $\mathfrak{B}$ be the set of infinite strips of height l allowed in X. Then, the shift space X is nonempty if and only if there exists $\mathfrak{B}_0 \subseteq \mathfrak{B}$ such that $\mathfrak{B}_0$ is complementary.*

We now discuss the periodicity for a given multidimensional shift space.

Proposition 2.3.3 *Let X be a multidimensional shift space and let $\mathfrak{B}$ be the set of infinite strips of height l allowed in X. If there exists a finite complementary set $\mathfrak{B}_0 \subset \mathfrak{B}$, then the set of periodic points is nonempty.*

Proof. Let $\mathfrak{B}$ be the set of infinite strips of height l allowed in X and let $\mathfrak{B}_0 \subset \mathfrak{B}$ be a finite complementary set. By definition, elements of $\mathfrak{B}_0$ form indices (not all) for the matrix $\mathfrak{M}$. Let $\mathfrak{N}$ be the submatrix of $\mathfrak{M}$ indexed by elements of $\mathfrak{B}_0$. As the set $\mathfrak{B}_0$ is complementary, the shift generated by $\mathfrak{B}_0$ (say $\Sigma_{\mathfrak{B}_0}$) is nonempty. Further, as shift defined by a finite dimensional matrix contains periodic points, there exists periodic points for $\Sigma_{\mathfrak{B}_0}$ (and hence for the shift space X). □

Remark 2.3.4 *The above result establishes a sufficient condition for the existence of periodic points in a multidimensional shift space. However, the condition derived is sufficient in nature and the shift space may exhibit periodic points without exhibiting the derived condition. Note that the periodicity of a point in a direction d_k ensures (and is equivalent to) the existence of a finite complementary set in the direction d_k. Consequently, a point in the shift space is periodic in all the d directions if and only if there exists a finite complementary set for the shift space under consideration. Thus we get the following corollary.*

Corollary 2.3.3 *A shift space X contains a point periodic in all the directions if and only if there exists a finite set of finite patterns complementary for the shift space X.*

Proof. The proof follows from discussions in Remark 2.3.4. □

Example 2.3.1 ***(The Chessboard Problem)****: Let $n \geq 2$ and $A = \{0, 1, 2, \cdots, n-1\}$ denote the set of distinct colours. Let X be the collection of all two dimensional configurations in which adjacent lattice points have different colours. It may be noted that for $n = 2$, the matrix characterizing the elements of X is a 2×2 identity matrix (and hence the shift space X is finite). For $n \geq 3$, any horizontal arrangement of $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ and $\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}$ is a valid infinite strip (of height 2) for the shift space X. As any two such strips are vertically compatible, the characterizing matrix is of infinite (uncountable) dimension. Further, for a three (or higher) dimensional Chessboard problem, as any configuration is a sequential arrangement of two dimensional configurations, the characterizing matrix for the three (or higher) dimensional Chessboard problem is of infinite dimension. However, as finite blocks (horizontally and vertically compatible with itself) can be obtained in finite iterations, periodic points* *with finite orbits can be obtained in finite time (and hence the nonemptiness problem can be answered in finite time). Further, as a nontrivial set of complementary strips (of nonperiodic nature) can be obtained in finite iterations, existence of points with infinite orbits can be verified in finite time (and hence the nonemptiness problem can be answered in finite time without utilizing the periodic points). It is worth mentioning that for a general shift of finite type, while the matrix characterizing the shift space might be of infinite order, the structure of the matrix (at k-th iteration) can be used to generate complementary sets and hence the nonemptiness problem (problem of existence of periodic points) can be tackled in finite time. As the verification depends on the structure of the matrix being constructed, the nonemptiness problem (the problem of existence of periodic points) is still undecidable (as observed in [Berger, 1966]).*