3

Multidimensional Shift Spaces and Finite Matrices

In this chapter, we address the problem of characterizing the elements of a two dimensional shift of finite type using finite matrices. In the process, we characterize the elements of the shift of finite type using a sequence of finite matrices. We provide an algorithmic approach to generate finite matrices characterizing arbitrarily large blocks allowed for the shift space X. Consequently, we generate the elements of the shift space as the limits of the arbitrarily large allowed blocks and hence characterize the elements of the shift space using the proposed sequential approach. We extend the results obtained for a general d dimensional shift of finite type.

3.1 CHARACTERIZATION OF TWO DIMENSIONAL SHIFT OF FINITE TYPE

Proposition 3.1.1 *For every 2-dimensional shift of finite type, there exists a sequence of finite matrices characterizing the elements of X.*

Proof. Let X be the shift space generated by forbidding given set of patterns. In this proof, we provide an algorithm for generating finite matrices (of increasing size) characterizing the existence of arbitrary large squares (rectangles) allowed for the shift space X. The sequence of matrices in turn characterizes the elements of the shift space X and thus addresses the nonemptiness problem for the shift space generated by a given set of forbidden blocks. We now describe the proposed algorithm below:

Step 1: Computation of Generating Matrices

Let X be a 2-dimensional shift of finite type and let $\mathscr{P}$ be the finite set of forbidden patterns characterizing the shift space X. By Proposition 2.1.1, there exists a set of forbidden squares $\mathscr{S}$ (all of same size, say l) generating the shift space X. Let $B_1, B_2, \ldots, B_k$ be the the collection of all squares of size l. Let V^0 be the $k \times k$ matrix (indexed by $B_1, B_2, \ldots, B_k$) indicating vertical compatibility of the squares B_i. For notational convenience, let the index B_i be denoted by i, $\forall i = 1, 2, \ldots, k$. Then,

$$V^0_{ij} = \begin{cases} 1, & \begin{pmatrix} B_i \\ B_j \end{pmatrix} \text{ is allowed in } X \\ 0 & \text{otherwise} \end{cases}$$

Let H^0 be a matrix indexed by the set of rectangles of size $2l \times l$ indicating horizontal compatibility of rectangles of size $2l \times l$. As any rectangle of size $2l \times l$ is of the form $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$, H^0 is a $k^2 \times k^2$ matrix indexed by rectangles of the form $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i \\ B_j \end{pmatrix}$ be denoted by (ij), $\forall i,j \in \{1,2,\ldots,k\}$. Then,

$$H^0_{(ij)(rs)} = \begin{cases} 1, & \begin{pmatrix} B_i & B_r \\ B_j & B_s \end{pmatrix} \text{ is allowed in } X \\ 0 & \text{otherwise} \end{cases}$$

It may be noted that while entries of the matrix V^0 indicate the vertical compatibility of squares of length l, entries of the matrix H^0 characterize the existence of squares of side $2l$. In the next step, we use the matrices computed to verify the validity of rectangles (squares) of size $4l \times 2l$ $(4l \times 4l)$.

Step 2: One Step Extension: Computing V^1 and H^1

In the first step, we computed the matrices V^0 and H^0 characterizing existence of rectangles and squares of size $2l \times l$ and $2l \times 2l$ respectively. To add clarity to the structure of the matrix H^0, let the index set of H^0 follow the dictionary order, i.e., let the index set of H^0 be ordered as $\{(11),(12)\ldots,(1k),(21),(22),\ldots,(2k),\ldots,(k1),(k2),\ldots,(kk)\}$. Consequently, H^0 can be viewed as a block matrix of the form

$$H^0 = \begin{pmatrix} R^0_{11} & R^0_{12} & \cdots & R^0_{1k} \\ R^0_{21} & R^0_{22} & \cdots & R^0_{2k} \\ \vdots & \vdots & & \vdots \\ R^0_{k1} & R^0_{k2} & \cdots & R^0_{kk} \end{pmatrix}$$

where R^0_{ij} is a $k \times k$ matrix whose entries characterize the squares of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$. More precisely, (r,s)-th entry of R^0_{ij} is 1 if and only if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is allowed. Equivalently, the entries of the matrices R^0_{ij} characterize all squares of size $2l$ whose top half (which is rectangle of size $l \times 2l$) is $(B_i \;\; B_j)$.

Let V^1 be $k^4 \times k^4$ matrix indexed by squares of size $2l$ indicating vertical compatibility (of the squares of size $2l$). As any square of size $2l$ is of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$, V^1 is equivalently indexed by the squares of the form $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ be denoted by $(ijrs)$ $\forall 1 \leq i,j,r,s \leq k$. Consequently,

$$V^1_{(ijrs)(uvwz)} = \begin{cases} 1, & \begin{pmatrix} B_i & B_j \\ B_r & B_s \\ B_u & B_v \\ B_w & B_z \end{pmatrix} \text{ is allowed in } X; \\ 0 & \text{otherwise} \;; \end{cases}$$

For computational purposes, let the index set of V^1 be ordered using order $O_{\mathscr{H}}$. In particular, the index set of V^1 follows the following order:

$$\begin{array}{l}
(1111),(1112),\ldots,(111k),(1121),\ldots,(112k),\ldots,(11k1),\ldots,(11kk)\\
(1211),(1212),\ldots,(121k),(1221),\ldots,(122k),\ldots,(12k1),\ldots,(12kk)\\
\vdots\\
(1k11),(1k12),\ldots,(1k1k),(1k21),\ldots,(1k2k),\ldots,(1kk1),\ldots,(1kkk)\\
(2111),(2112),\ldots,(211k),(2121),\ldots,(212k),\ldots,(21k1),\ldots,(21kk)\\
(2211),(2212),\ldots,(221k),(2221),\ldots,(222k),\ldots,(22k1),\ldots,(22kk)\\
\vdots\\
(2k11),(2k12),\ldots,(2k1k),(2k21),\ldots,(2k2k),\ldots,(2kk1),\ldots,(2kkk)\\
\vdots\\
(k111),(k112),\ldots,(k11k),(k121),\ldots,(k12k),\ldots,(k1k1),\ldots,(k1kk)\\
(k211),(k212),\ldots,(k21k),(k221),\ldots,(k22k),\ldots,(k2k1),\ldots,(k2kk)\\
\vdots\\
(kk11),(kk12),\ldots,(kk1k),(kk21),\ldots,(kk2k),\ldots,(kkk1),\ldots,(kkkk).
\end{array}$$

It may be noted that for any index $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ for the matrix V^1, the corresponding row can be determined as follows:

1. If $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is forbidden, then the corresponding row is zero row.

2. If $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is allowed then the corresponding row is $R^0_{rs} \bigotimes H^0$.

Finally let the index set of V^1 be ordered using the order $O_{\mathscr{V}}$.

Note that if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is not allowed then it cannot be vertically aligned with another square to obtain a allowed pattern and hence the corresponding row is the zero row. Further, as the size of the squares B_k is determined by maximum possible length or breadth among patterns from $\mathscr{S}$, any pattern from $\mathscr{S}$ cannot be spread beyond a square of size $2l$. Thus, validity of any given pattern can be verified by examining the validity of all the squares of size $2l$ in the given pattern. Consequently, $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ is vertically compatible with $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ if and only if $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$, $\begin{pmatrix} B_r & B_s \\ B_u & B_v \end{pmatrix}$ and $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ are allowed. As $(R^0_{rs})_{uv}$ and $(R^0_{uv})_{wz}$ are 1 if and only if $\begin{pmatrix} B_r & B_s \\ B_u & B_v \end{pmatrix}$ and $\begin{pmatrix} B_u & B_v \\ B_w & B_z \end{pmatrix}$ are allowed, their product characterizes the validity of the block $\begin{pmatrix} B_i & B_j \\ B_r & B_s \\ B_u & B_v \\ B_w & B_z \end{pmatrix}$ under the validity of $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ and hence $(R^0_{rs}) \bigotimes H^0$ is the row corresponding to $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$ (under the validity of $\begin{pmatrix} B_i & B_j \\ B_r & B_s \end{pmatrix}$). Thus, the constructed matrix indeed characterizes the vertical compatibility of the squares of size $2l$ (and hence characterizes all rectangles of size $4l \times 2l$ allowed for the shift space X).

Let us investigate the structure of V^1 in detail. Note that the index set of V^1 (now ordered using the order $O_{\mathscr{V}}$) can be viewed as the block matrix of the form

$$V^1 = \begin{pmatrix} S^0_{11} & S^0_{12} & \cdots & S^0_{1k^2} \\ S^0_{21} & S^0_{22} & \cdots & S^0_{2k^2} \\ \vdots & \vdots & & \vdots \\ S^0_{k^2 1} & S^0_{k^2 2} & \cdots & S^0_{k^2 k^2} \end{pmatrix}$$

where each S^0_{ij} is a $k^2 \times k^2$ matrix. Note that for any $i \in \{1,2,\ldots,k^2\}$, there exists unique pair (p_i,q_i), $(p_i \in \{0,2,\ldots,k-1\},\quad q_i \in \{1,2,\ldots,k\})$ such that $i = kp_i + q_i$. Identifying i with $(p_i+1 \;\; q_i)$ $\forall i = 1,2,\ldots,k^2$, the matrix V^1 can be represented as

$$V^1 = \begin{pmatrix}
S^0_{(11)(11)} & S^0_{(11)(12)} & \cdots & S^0_{(11)(1k)} & S^0_{(11)(21)} & S^0_{(11)(22)} & \cdots & S^0_{(11)(2k)} & \cdots & S^0_{(11)(k1)} & S^0_{(11)(k2)} & \cdots & S^0_{(11)(kk)} \\
S^0_{(12)(11)} & S^0_{(12)(12)} & \cdots & S^0_{(12)(1k)} & S^0_{(12)(21)} & S^0_{(12)(22)} & \cdots & S^0_{(12)(2k)} & \cdots & S^0_{(12)(k1)} & S^0_{(12)(k2)} & \cdots & S^0_{(12)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(1k)(11)} & S^0_{(1k)(12)} & \cdots & S^0_{(1k)(1k)} & S^0_{(1k)(21)} & S^0_{(1k)(22)} & \cdots & S^0_{(1k)(2k)} & \cdots & S^0_{(1k)(k1)} & S^0_{(1k)(k2)} & \cdots & S^0_{(1k)(kk)} \\
S^0_{(21)(11)} & S^0_{(21)(12)} & \cdots & S^0_{(21)(1k)} & S^0_{(21)(21)} & S^0_{(21)(22)} & \cdots & S^0_{(21)(2k)} & \cdots & S^0_{(21)(k1)} & S^0_{(21)(k2)} & \cdots & S^0_{(21)(kk)} \\
S^0_{(22)(11)} & S^0_{(22)(12)} & \cdots & S^0_{(22)(1k)} & S^0_{(22)(21)} & S^0_{(22)(22)} & \cdots & S^0_{(22)(2k)} & \cdots & S^0_{(22)(k1)} & S^0_{(22)(k2)} & \cdots & S^0_{(22)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(2k)(11)} & S^0_{(2k)(12)} & \cdots & S^0_{(2k)(1k)} & S^0_{(2k)(21)} & S^0_{(2k)(22)} & \cdots & S^0_{(2k)(2k)} & \cdots & S^0_{(2k)(k1)} & S^0_{(2k)(k2)} & \cdots & S^0_{(2k)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(k1)(11)} & S^0_{(k1)(12)} & \cdots & S^0_{(k1)(1k)} & S^0_{(k1)(21)} & S^0_{(k1)(22)} & \cdots & S^0_{(k1)(2k)} & \cdots & S^0_{(k1)(k1)} & S^0_{(k1)(k2)} & \cdots & S^0_{(k1)(kk)} \\
S^0_{(k2)(11)} & S^0_{(k2)(12)} & \cdots & S^0_{(k2)(1k)} & S^0_{(k2)(21)} & S^0_{(k2)(22)} & \cdots & S^0_{(k2)(2k)} & \cdots & S^0_{(k2)(k1)} & S^0_{(k2)(k2)} & \cdots & S^0_{(k2)(kk)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
S^0_{(kk)(11)} & S^0_{(kk)(12)} & \cdots & S^0_{(kk)(1k)} & S^0_{(kk)(21)} & S^0_{(kk)(22)} & \cdots & S^0_{(kk)(2k)} & \cdots & S^0_{(kk)(k1)} & S^0_{(kk)(k2)} & \cdots & S^0_{(kk)(kk)}
\end{pmatrix}$$

To understand the generated submatrices better, let the k^4 rows (columns) be divided into k^2 groups of k^2 rows (columns) each. Note that if $i = kp_i + q_i$ $(i \in \{1,2,\ldots,k^2\})$, then the rows (columns) of the i-th group are indexed by squares of the form $\begin{pmatrix} B_{p_i+1} & * \\ B_{q_i} & * \end{pmatrix}$. As S^0_{ij} verifies the vertical compatibility of i-th group with the j-th group, entries of any $S^0_{(pq)(rs)}$ characterize all rectangles of size $4l \times 2l$ whose left half (which is a rectangle of size $4l \times l$) is $\begin{pmatrix} B_p \\ B_q \\ B_r \\ B_s \end{pmatrix}$. It is worth mentioning that for $i = kp_i + q_i, \quad j = kp_j + q_j$, S^0_{ij} is same as $S^0_{(p_i+1 \;\; q_i)(p_j+1 \;\; q_j)}$ and the expression above is another way of representing the same matrix.

Let H^1 be the matrix indexed by the set of rectangles of size $4l \times 2l$ indicating horizontal compatibility of the indices. As any rectangle of size $4l \times 2l$ is of the form $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$, H^1

is a $k^8 \times k^8$ matrix equivalently indexed by rectangles of the form $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$. For notational convenience, let the index $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ be denoted by $(ijrsuvwz)$.

For computational purposes, let the index set of H^1 be ordered using order $O_{\mathscr{V}}$. It may be noted that for any index $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ for the matrix H^1, the corresponding row can be determined as follows:

1. If $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is forbidden, then the corresponding row is zero row.

2. If $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is allowed then the corresponding row is $S^0_{(uv)(wz)} \bigotimes V^1$.

Finally let the index set of H^1 be ordered using the order $O_{\mathscr{H}}$.

Note that if a $4l \times 2l$ block is not allowed then it cannot be horizontally aligned with another $4l \times 2l$ block to obtain a allowed pattern and hence the corresponding row is the zero row. For any given pattern $\mathscr{P}$, as establishing the validity of all squares of size $2l$ (or rectangles or squares of greater size) embedded in $\mathscr{P}$ is sufficient to establish the validity of $\mathscr{P}$ for X, $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ is horizontally compatible with $\begin{pmatrix} B_{i'} & B_{u'} \\ B_{j'} & B_{v'} \\ B_{r'} & B_{w'} \\ B_{s'} & B_{z'} \end{pmatrix}$ if and only if $\begin{pmatrix} B_u & B_{i'} \\ B_v & B_{j'} \\ B_w & B_{r'} \\ B_z & B_{s'} \end{pmatrix}$ and $\begin{pmatrix} B_{i'} & B_{u'} \\ B_{j'} & B_{v'} \\ B_{r'} & B_{w'} \\ B_{s'} & B_{z'} \end{pmatrix}$ are allowed in X (under allowedness of $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$). Finally, as entries of any submatrix $S^0_{(i'j')(r's')}$ characterizes all allowed rectangles of size $4l \times 2l$ with fixed half $\begin{pmatrix} B_{i'} \\ B_{j'} \\ B_{r'} \\ B_{s'} \end{pmatrix}$, $S^0_{(uv)(wz)} \bigotimes V^1$ indeed is the row (of H^1) corresponding to $\begin{pmatrix} B_i & B_u \\ B_j & B_v \\ B_r & B_w \\ B_s & B_z \end{pmatrix}$ and the computation of H^1 is complete.

Step 3: Computation of V^{n+1} and H^{n+1}

Let us assume that the matrices $V^0, H^0, V^1, H^1, \ldots, V^n, H^n$ have been computed. Then, the matrix H^n can be visualized as

$$H^n = \begin{pmatrix} R^n_{11} & R^n_{12} & \cdots & R^n_{1k^{4^n}} \\ R^n_{21} & R^n_{22} & \cdots & R^n_{2k^{4^n}} \\ \vdots & \vdots & & \vdots \\ R^n_{k^{4^n}1} & R^n_{k^{4^n}2} & \cdots & R^n_{k^{4^n}k^{4^n}} \end{pmatrix}$$

where each R^n_{ij} is a $k^{4^n} \times k^{4^n}$ matrix. As H^n establishes the horizontal compatibility of rectangles of size $2^{n+1}l \times 2^n l$, entries of R^n_{ij} establishes the validity of squares of size $2^{n+1}l \times 2^{n+1}l$ (which are indices of V^{n+1}) with a fixed top half. Consequently, $R_{ij} \bigotimes H^n$ determines the rows of V^{n+1} (under allowedness of the index), when the index set of V^{n+1} is ordered using order $O_{\mathscr{H}}$. Further, note that the matrix V^{n+1} (ordered using order $O_{\mathscr{V}}$ after computation) can be visualized as:

$$V^{n+1} = \begin{pmatrix} S^{n+1}_{11} & S^{n+1}_{12} & \cdots & S^{n+1}_{1k^{2.4^n}} \\ S^{n+1}_{21} & S^{n+1}_{22} & \cdots & S^{n+1}_{2k^{2.4^n}} \\ \vdots & \vdots & & \vdots \\ S^{n+1}_{k^{2.4^n}1} & S^{n+1}_{k^{2.4^n}2} & \cdots & S^{n+1}_{k^{2.4^n}k^{2.4^n}} \end{pmatrix}$$

where each S^{n+1}_{ij} is a $k^{2.4^n} \times k^{2.4^n}$ matrix. As V^{n+1} establishes the vertical compatibility of squares of size $2^{n+1}l$, entries of S^{n+1}_{ij} establishes the validity of rectangles of size $2^{n+2}l \times 2^{n+1}l$ (which is an index of H^n) with a fixed left half. Consequently, $S^{n+1}_{ij} \bigotimes V^{n+1}$ determines the rows of H^{n+1} (when the index set of H^{n+1} is ordered using order $O_{\mathscr{V}}$) and thus computation of V^{n+1} and H^{n+1} is complete.

Finally, as entries of V^{n+1} (H^{n+1}) characterize all allowed rectangles (squares) of size $2^{n+2}l \times 2^{n+1}l$ ($2^{n+2}l \times 2^{n+2}l$ respectively), the shift space X is nonempty if and only if each V^i and H^i are nonzero for any $i \in \mathbb{N}$. Further, as any element of X can be viewed as a limit of finite blocks arising from V^i(or H^i), elements of the shift space are precisely the limits of the blocks arising from V^i(or H^i). □

Remark 3.1.1 *The above result provides an iterative procedure to generate arbitrarily large blocks for a shift space generated by finitely many forbidden blocks of finite size. Note that although the algorithm generates rectangles (squares) of size $2^{n+1}l \times 2^n l$ ($2^{n+1}l \times 2^{n+1}l$ respectively) using all squares (rectangles) of size $2^n l \times 2^n l$ ($2^{n+1}l \times 2^n l$ respectively), the matrices V^i (H^i) can be indexed by a smaller collection of all allowed squares (rectangles) to generate the same set and hence the order of the matrices V^i (and H^i) can be reduced. Further, as any compatible vertical (horizontal) alignment of indices of V^i (H^i) is an index for H^i (V^{i+1}), the number of 1's in V^i (H^i) characterizes the order of H^i (V^{i+1}) (while working with matrices V^i (H^i) of reduced size). Note that while the matrices V^i (and H^i) are computed with the indices being ordered using $O_{\mathscr{H}}$ (and $O_{\mathscr{V}}$ respectively), the indices are re-ordered after computation using $O_{\mathscr{V}}$ (and $O_{\mathscr{H}}$ respectively). Such an arrangement is useful as such an ordering of the index set helps represent the generated matrix in the form of a block matrix where each block characterizes rectangles (squares) of larger size with a fixed left (top) half which is useful to compute the rows of the next matrix H^n (V^{n+1}) (via the product*

$S^n_{ij}\bigotimes V^n$ and $(R^n_{ij}\bigotimes H^n)$ *respectively). Note that while the ability to construct arbitrarily large blocks for the space* X *guarantees nonemptiness of the shift space* X*, any element of the shift space is a limit of arbitrarily large blocks allowed for* X*. Consequently, the shift space* X *is nonempty if and only if the matrices* V^i *and* H^i *are nonzero at each iteration. It is worth mentioning that* V^i (H^i) *being nonzero at* i*-th iteration does not imply their non-trivialness in subsequent iterations.*

Remark 3.1.2 *It may be noted that the algorithm verifies horizontal (vertical) compatibility of rectangles (squares) using the matrices constructed to generate allowed blocks of arbitrarily large size. In the process, the algorithm verifies the compatibility of the blocks in each of the possible directions. Taking a note of the compatibility (between any two blocks) in each of the directions, one may construct a complementary set for the shift space in finite iterations (if exists). One may use the block structure of the matrices constructed to facilitate the construction of the complementary set. While the existence of a complementary set in finite iterations guarantees a point with finite orbit, such a construction in the limiting case verifies the nonemptiness of the given shift space. However, such a construction depends on the sequence of the matrices constructed (which in turn may be sparse) and hence the nonemptiness problem is still undecidable (as observed in [Berger, 1966]) for a general given shift of finite type.*

Example 3.1.1 *Let* Σ_2 *be the two dimensional full shift over two symbols* $\{0,1\}$ *and let* X *be the shift of finite type generated by the forbidden set* $\mathscr{F}_1=\left\{\begin{smallmatrix}1\\1\end{smallmatrix},\ 11\right\}$*. As already observed,* X *is also generated by the forbidden set*

$$\mathscr{F}_2=\left\{\begin{smallmatrix}0&1\\0&1\end{smallmatrix},\begin{smallmatrix}0&0\\1&1\end{smallmatrix},\begin{smallmatrix}1&1\\0&0\end{smallmatrix},\begin{smallmatrix}1&0\\1&0\end{smallmatrix},\begin{smallmatrix}0&1\\1&1\end{smallmatrix},\begin{smallmatrix}1&0\\1&1\end{smallmatrix},\begin{smallmatrix}1&1\\0&1\end{smallmatrix},\begin{smallmatrix}1&1\\1&0\end{smallmatrix},\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right\}$$

and any element of X *is a sequential (two dimensional) arrangement of blocks in* $\mathscr{A}_2$ *where*

$$\mathscr{A}_2=\left\{\begin{smallmatrix}0&0\\0&0\end{smallmatrix},\begin{smallmatrix}0&0\\0&1\end{smallmatrix},\begin{smallmatrix}0&0\\1&0\end{smallmatrix},\begin{smallmatrix}0&1\\0&0\end{smallmatrix},\begin{smallmatrix}1&0\\0&0\end{smallmatrix},\begin{smallmatrix}0&1\\1&0\end{smallmatrix},\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right\}$$

Note that for the shift space constructed, while V^0 *is a square matrix of size* 7*,* H^0 *is a square matrix of size* 41*. Further, it mat be observed that the sizes of* V^i (H^i) *in subsequent iterations increase exponentially, the complementary set generated in this case is infinite. Further, as finite complementary sets are generated at each of the iterations, the shift space possesses infinitely many periodic points with finite orbits. Note that as any sequential arrangement of* $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$ *and* $\begin{smallmatrix}1&0\\0&0\end{smallmatrix}$ *avoids adjacent placement of* 1*'s and hence is an allowed infinite strip for* X*. Consequently, for any such strip* H*,* $\{H\}$ *is a complementary set and hence sequential vertical arrangement of* H *generates a periodic point for* X *(and hence addresses the nonemptiness problem in finite iterations). It may be noted that, as infinite strips (of fixed height) can be generated (from* V^i (H^i)*) at any iteration, points with infinite orbits can be generated (by verifying the vertical compatibility of the strips generated). Consequently, the nonemptiness problem for the generated shift space can be addressed in finite iterations (without using the periodic points available) in this case.*

Remark 3.1.3 *Note that while* V^i (H^i) *inductively generate rectangles (squares) of a large order, detection of periodic points with finite orbits can be concluded after finite iterations. Note that if the shift space exhibits a periodic point with finite orbit, there exists a finite block (of appropriate size) which is a complementary set of size one (as observed in the previous example). Further, periodicity in any of the directions yields a finite complementary set (in that direction) which can be used to generate periodic points with infinite orbits (with reduced complexity). Finally, as the matrices are generated inductively using the matrices at previous iterations, the structure of the matrix can be used to tackle the nonemptiness problem and problem of existence of periodic points in a more efficient manner.*

Remark 3.1.4 *The above algorithm inductively generates the matrices V^i (H^i) using the matrices generated in the previous steps. As observed the size of V^i (H^i) can be reduced by indexing the matrix using allowed squares (rectangles). It is worth noting that the growth (or decline) in the order of the matrices V^i (H^i) is exponential. While exponential growth in the order of V^i (H^i) ensures faster convergence to an element of X (and hence implying a faster computation of a complementary set guaranteeing nonemptiness for the shift space X), exponential reduction in the size of the matrices generated may address to the complexity of the nonemptiness problem for the given shift space. In case the matrices generated are sparse, the matrices in subsequent iterations are also sparse matrices of large order. Such a phenomenon once again indicates the higher complexity of the nonemptiness problem which may turn out to be undecidable (as observed in [Berger, 1966]).*

Remark 3.1.5 *It may be noted that for a d-dimensional shift space, a similar algorithm yields a sequence of matrices $M_1^1, M_1^2, \ldots M_1^d, M_2^1, M_2^2, \ldots M_2^d \ldots, M_n^1, M_n^2, \ldots M_n^d \ldots$ (where M_n^i characterises the possible extensions in the i-th direction at n-th iteration) such that the d-dimensional shift space is nonempty if and only if M_n^i $(i = 1, 2, \ldots, d)$ are nonzero at each iteration. Consequently, a similar process extended in d mutually orthogonal directions yeilds the criteria for the verification of nonemptiness problem for a d dimensional shift space. We now derive the stated criteria below.*

3.2 CHARACTERIZATION OF MULTIDIMENSIONAL SHIFT OF FINITE TYPE

Proposition 3.2.1 *For every d-dimensional shift of finite type, there exists a sequence of finite matrices characterizing the elements of X.*

Proof. The proof is a natural extension of the 2 dimensional case and it follows directly by first extending the allowed d-dimensional cubes in each of the d directions and iteratively generating cubes (cuboids) of arbitrarily large size. For the sake of clarity, we provide an outline of the proof below.

Step 1: Computing Generating Matrices

Let X be a d-dimensional shift space generated by a finite set of forbidden patterns. By Proposition 2.1.1, there exists a set of forbidden cubes $\mathscr{S}$ (all of same size, say l) generating the shift space X. Let $e_1, e_2, \ldots, e_d$ be the set of d mutually orthogonal directions (along the directions of standard basis vectors of $\mathbb{R}^d$) and let $B_1, B_2, \ldots, B_k$ be the the collection of all allowed cubes of size l. Let M_1^1 be the $k \times k$ matrix (indexed by $B_1, B_2, \ldots, B_k$) indicating the compatibility of the cubes B_i in e_1 direction. Consequently, entries of M_1^1 characterizes the validity of all blocks of size $2l \times \underbrace{l \times l \times \ldots \times l}_{d-1 \ \ times}$ for the space X. Similarly, if M_1^2 be the matrix indexed by all allowed blocks of size $2l \times \underbrace{l \times l \times \ldots \times l}_{d-1 \ \ times}$ indicating the compatibility of the indices in e_2 direction then, entries of M_1^2 characterizes the validity of all $2l \times 2l \times \underbrace{l \times l \times \ldots \times l}_{d-2 \ \ times}$ for the space X. Inductively, for $i = 3, 4, \ldots, d$, if M_1^i is indexed by all allowed blocks of size $\underbrace{2l \times 2l \times \ldots \times 2l}_{i-1 \ \ times} \times \underbrace{l \times l \times \ldots \times l}_{d-i+1 \ \ times}$ indicating the the compatibility of the indices in i-th direction, entries of M_1^i characterize the validity of all blocks of size $\underbrace{2l \times 2l \times \ldots \times 2l}_{i \ \ times} \times \underbrace{l \times l \times \ldots \times l}_{d-i \ \ times}$. In particular, M_1^d is a matrix indexed by all allowed blocks of size $\underbrace{2l \times \ldots 2l}_{d-1 \quad times} \times l$ indicating the compatibility of the indices in the direction e^d and hence characterizes all allowed cubes of size $2l$.

Step 2: Application of Induction and Generating Cubes (Cuboids) of Arbitrarily Large Size

Let us assume that $M_1^1, M_1^2, \ldots M_1^d, M_2^1, M_2^2, \ldots M_2^d \ldots, M_n^1, M_n^2, \ldots M_n^d$ have been computed. Note that as M_n^i is indexed by all allowed blocks of size $\underbrace{2^n l \times 2^n l \times \ldots \times 2^n l}_{i-1 \quad times} \times \underbrace{2^{n-1} l \times 2^{n-1} l \times \ldots \times 2^{n-1} l}_{d-i+1 \quad times}$, entries of M_n^i characterize all allowed blocks of size $\underbrace{2^n l \times 2^n l \times \ldots \times 2^n l}_{i \quad times} \times \underbrace{2^{n-1} l \times 2^{n-1} l \times \ldots \times 2^{n-1} l}_{d-i \quad times}$.

Further, in order for the proof of Proposition 3.1.1 to be extendable to the d-dimensional case, while the indices of M_n^i need to be ordered using $(i-1)$-th direction (mod d) while computing M_n^i, they need to be ordered using $(i+1)$-th direction (mod d) after computation, to facilitate the computation of the next matrix. Finally, note that as the sequence M_n^i generates arbitrarily large cubes (and cuboids), the process yields an element of the shift if each M_n^i is nonempty. Further, as any element of sequence can be viewed as a limit of arbitrarily large cubes (or cuboids), the elements of the shift space are characterized by the sequence generated. □

Remark 3.2.1 *The above corollary extends the algorithm given in Proposition 3.1.1 for generating elements of d-dimensional shift space. The algorithm generates cubes (cuboids) of arbitrarily large size by iteratively extending the allowed blocks in each of the d directions. Note that as the compatibility of two blocks in one of the directions cannot indicate its extension in any other direction, the matrices $M_1^1, M_1^2, \ldots, M_1^d$ are independent of each other (cannot be determined from each other). Further, as the size of the matrices $B_1, B_2, \ldots, B_k$ is the maximum possible length (in any of the directions) of the generating set of forbidden patterns $\mathscr{P}$, to examine the validity of any given pattern in $\mathbb{R}^d$, it is sufficient to examine the validity of all cubes of size $2l$ (or cubes/cuboids of a larger size). Finally, as any element of X can be visualized as a limit of finite cubes (cuboids) and generation of arbitrarily large cubes (cuboids) yields an element of X, the sequence generated characterizes the elements of X. To realize the convergence mathematically, one may visualize the allowed cube as an element which is obtained by placing the cube (centre of the cube) at the origin and assigning 0 at all the other places. Consequently, a sequence of arbitrarily large blocks yields a Cauchy sequence in the full shift without forbidden blocks in the central cube (which is of arbitrarily large size) and hence converges to an element of X.*

Remark 3.2.2 *It may be noted that, the shift space X is nonempty if and only if the matrices generated at each of the iterations is nonempty. This follows from the fact that for any element of X, its central blocks are allowed blocks (of arbitrarily large size) which in turn will be generated by M_i^j. Although, the matrices M_i^j are generated using matrices obtained in the previous iteration, non-trivialness of these matrices at one iteration need not imply non-trivialness at subsequent iterations. As M_i^d generates cubes of higher order, the structure of the matrix can be utilize to generate a complementary set. In case the shift space possesses a periodic point with finite orbit, such a complementary set is guaranteed to be obtained in finite iterations. Similar to the two dimensional case such a construction depends on the matrices constructed and the nonemptiness problem is still undecidable (as observed in [Berger, 1966]). It may be noted that, one may work with matrices M_i^j of reduced size to reduce the complexity of the proposed algorithm.*

Example 3.2.1 ***(The Chessboard Problem)**: Let $n \geq 2$ and $A = \{0, 1, 2, \cdots, n-1\}$ denote the set of distinct colours. Let X be the collection of all three dimensional configurations in which adjacent lattice points have different colours. It may be noted that for $n = 2$, each M_i^j is a 2×2 identity* *matrix. Consequently, each index of M_i^j is a complementary set (of size one) which generates a point with finite orbit and hence the shift space contains precisely two points in this case. For the case $n \geq 3$, as any three dimensional configuration can be visualized as a sequential arrangement of two dimensional configurations, the orders of matrices M_i^j increase exponentially (generating a nonperiodic point in the limiting case). However, as the matrices generated contain* 1 *on the diagonal, periodic points with finite orbits are generated in finite iterations. Further, as the sequence of finite matrices generates a complementary set in the limiting case, all the points with infinite orbits are generated as limits of sequences spread across elements generated by indices of (M_i^j). Such an observation helps characterizing the periodic points for the shift space X and can be used to generate any general point for the shift space X.*