4
On Periodicity In Multidimensional Shift Spaces

In this chapter, we investigate the structural and existential problems of periodic points with finite orbits in a multidimensional shift space. We prove that in a d dimensional shift space, a periodic point has a finite orbit if and only if the lattice of periods is d dimensional. We prove that in a multidimensional shift space, any periodic point with infinite orbit can be represented as an arithmetic progressive arrangement of shifts of lower dimensional infinite strips. In the process, we relate the dimension of the lattice of periods with the dimension of generating lower dimensional infinite strip. We also derive necessary and sufficient conditions for a periodic point with finite orbit to belong to a given multidimensional shift space X. In particular, we prove that any element of the full shift belongs to X if and only if it is a limit of a sequence of elements from $\mathscr{B}_n^X$. We extend our result to a general point in the multidimensional full shift.

4.1 PERIODIC POINTS WITH FINITE ORBITS IN MULTIDIMENSIONAL SHIFT SPACES

Proposition 4.1.1 *Let x be a periodic point in a two dimensional shift space. Then, the orbit of x is infinite if and only if the lattice of periods is one dimensional.*

Proof. Let x be a periodic point with infinite orbit in a two dimensional shift space and let Γ_x be the lattice of periods for x. If Γ_x is two dimensional, there exists $(p,q),(r,s) \in \Gamma_x$ such that $(p,q) \notin (r,s)\mathbb{Z}$. Also, as integral combination of periods of x is a period of x, there exists $k_1,k_2 \in \mathbb{N}$ such that $(k_1+1,0)$ and $(0,k_2+1)$ are periods for x. Thus, x is an infinite repetitive arrangement of a rectangle of size $k_1 \times k_2$ (placed at origin in x). Consequently, the orbit of x is finite (which is a contradiction) and hence Γ_x is one dimensional.

Conversely, if Γ_x is one dimensional, there exists $(m,n) \in \mathbb{N}^2$ such that $\Gamma_x = (m,n)\mathbb{Z}$. Thus x is not periodic under $\sigma_{(m,n+1)}$ and hence the orbit of x is infinite. □

Remark 4.1.1 *The above result relates the orbit of the periodic point with the dimension of the lattice of periods in a two dimensional shift space. The result establishes that if the dimension of the lattice of periods coincides with the dimension of the multidimensional shift space, then the periodic point has a finite orbit. The result holds in the other direction and hence the result provides a necessary and sufficient condition for a point to have a finite orbit. Note that if x is a point in d dimensional shift space such that the lattice of periods has dimension d, then, as integral combination of periods of x is a period of x, an appropriate combination of the basis of Γ_x yields the periodicity in each of the d directions and hence the above result can be extended for a d dimensional shift space. We now provide a proof for the observation below.*

Proposition 4.1.2 *Let x be a periodic point in a d dimensional shift space. Then, the orbit of x is finite if and only if the lattice of periods is d dimensional.*

Proof. Let x be a point with finite orbit. As nonperiodicity in a direction d yields an infinite orbit in the direction d, x is periodic in each of the directions in the set $\{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$ and hence the lattice of periods is d dimensional.

Conversely, let x be a periodic point with infinite orbit in a d dimensional shift space and let Γ_x be the lattice of periods for x. Note that if Γ_x is d dimensional, there exists a set $S = \{(p_1^1,p_2^1,\ldots,p_d^1),(p_1^2,p_2^2,\ldots,p_d^2),\ldots,(p_1^d,p_2^d,\ldots,p_d^d)\} \subset \Gamma_x$ such that S is rationally independent. Consequently, S (by taking integral combinations of elements of S) can be transformed into a set of the form $S^* = \{(n_1+1,0,0,\ldots,0),(0,n_2+1,0,0,\ldots,0),\ldots,(0,0,\ldots,0,n_d+1)\}$. As integral combination of periods of x is a period of x, x is periodic in each of the d directions. Hence, x is an infinite repetitive arrangement of a rectangle of size $n_1 \times n_2 \times \ldots \times n_d$ and the orbit of x is finite. □

Example 4.1.1 *Let Σ_2 be the two dimensional full shift over two symbols $\{0,1\}$ and let X be the collection of all configurations for which $x(m,n)+x(m+1,n)+x(m,n+1) \equiv 0(mod\,2) \quad \forall m,n \in \mathbb{Z}$. Then, the collection X is a shift of finite type. Note that the square $\begin{smallmatrix}1&0&1\\0&1&1\\1&1&0\end{smallmatrix}$ is a valid block for the shift space X. Further, as horizontal (vertical) arrangement of the block with itself is valid for the shift space, the block generates a periodic point (by repetitive arrangement in both directions) for shift space X. It is worth mentioning that the point generated is periodic with periods $(1,1)$ and $(3,0)$ and thus the lattice of periods is two dimensional. As any periodic point with finite orbit is necessarily periodic in both the directions, the lattice of periods of any such point is of full dimension. Conversely, as any periodic point having the lattice of periods of dimension two is periodic in both the directions, any such point is a repetitive arrangement of a finite block (and hence has a finite orbit). As the arguments given are independent of the dimension of the shift space X, the example can be extended to any d dimensional shift space to validate the results above.*

4.2 PERIODIC POINTS WITH INFINITE ORBITS IN MULTIDIMENSIONAL SHIFT SPACES

Proposition 4.2.1 *If x is a periodic point with infinite orbit in a two dimensional shift space then, x is an arrangement of shifts of a one dimensional infinite strip.*

Proof. Let x be a periodic point with infinite orbit. By Proposition 4.1.1, the lattice of periods Γ_x for x is one dimensional. Thus there exists $(m,n) \in \mathbb{N}^2$ such that $\Gamma_x = (m,n)\mathbb{Z}$. Note that if $m = 0 \ \ (n = 0)$, then periodicity of x ensures that x is a repetitive vertical (horizontal) arrangement of an infinite strip. Further for $m,n > 0$, if x is visualised as an infinite rectangular arrangement of rectangles of size $(m-1)\times(n-1)$, say $M_{ij} \ \ (i,j \in \mathbb{Z})$, then, periodicity under shift by (m,n) ensures $M_{ij} = M_{(i+1)(j+1)}$ for all $i,j \in \mathbb{Z}$. Thus, if x is visualised as vertical superposition of infinite strips of height $n-1$, say $S_j \quad (j \in \mathbb{Z})$, then $\sigma_{(m,0)}(S_j) = S_{j+1}$. Consequently, x is a vertical arrangement of shifts of an infinite strip and the proof is complete. □

Remark 4.2.1 *The above proof establishes the representation of a periodic point in a two dimensional shift space (with infinite orbit) using an infinite strip of fixed height. Note that while every periodic point with infinite orbit is proved to have such a representation, the converse is not true in general. The proof follows from the fact that for any transitive point x of one dimensional full shift, the ordered arrangement of $\{\sigma^{2^k}(x) : k \in \mathbb{Z}\}$ is an arrangement of shifts of an infinite strip but is not periodic in the two dimensional shift space. Further, it may be noted that the arrangements of shifts of an infinite strip yields a periodic point if and only if the order of the shifts forms an arithmetic progression. We now give the detailed proofs of our claims below.*

Example 4.2.1 *Let X be one dimensional full shift and let x be a point transitive for (X,σ). Let $\mathbb{Z}\times\mathbb{Z}$ be visualized as copies of $\mathbb{Z}$ placed at different heights and let x^* be a point in the two dimensional full shift obtained by placing $\sigma^{2^r}(x)$ at height r. As shifting x by (m,n) yields a point where the*

$\sigma^{2^{n+r}+m}(x)$ is placed at height r, the point x^* is not periodic in the two dimensional shift space and hence arrangement of shifts of an infinite strip need not be periodic in the multidimensional shift space.

Corollary 4.2.1 *For any point x with infinite orbit in a two dimensional shift space, x is periodic if and only if there exists $k \in \mathbb{N}$ such that x can be visualised as an arithmetic progressive arrangement of shifts of an infinite strip of height (width) k.*

Proof. The proof of forward part follows from directly from discussions in Proposition 4.2.1. Conversely, note that if x can be visualised as an arithmetic progressive arrangement of shifts of an infinite strip of height k with common difference r, then x is periodic with period (r,k) and hence the proof is complete. □

Remark 4.2.2 *The above result characterizes the representation of a periodic point with an infinite orbit in a two dimensional shift space. For $m,n > 0$, note that while the proof visualizes x as a vertical arrangement of shifts of an infinite horizontal strip, the point can alternatively be visualized as horizontal arrangement of shifts of infinite vertical strips. Also, the horizontally (vertically) periodic points can be visualized as a repetitive arrangements of vertical (horizontal) infinite strips. Consequently, the variations of the point in one direction are uniquely determined by the other (direction) and hence the point can be visualized as a one dimensional point. Further, the arguments do hold in higher dimensions and an analogous result holds for a d dimensional shift space. For the sake of completion, we establish the result below.*

Proposition 4.2.2 *For any point x with infinite orbit in a d dimensional shift space, x is periodic with r dimensional lattice of periods if and only if x can be visualised as an arithmetic progressive arrangement of shifts of an $d-r$ dimensional infinite strip.*

Proof. Let x be a periodic point with an r dimensional lattice of periods. Let $S = \{(p_1^1, p_2^1, \cdots, p_d^1), (p_1^2, p_2^2, \cdots, p_d^2), \ldots, (p_1^r, p_2^r, \cdots, p_d^r)\}$ be the basis for the lattice of periods. As S is rationally independent, taking appropriate integral combinations, S can be transformed into a set S^* such that r columns (say $i_1, i_2, \ldots, i_r$) of S^* are vectors in r of the directions of standard basis. As integral combinations of periods of x is a period of x, x is periodic in these r directions (say with period m_j in direction i_j) and the variations of the point in these directions is uniquely determined by other $d-r$ directions. Consequently, if y is an $d-r$ dimensional infinite strip obtained by restricting x in the remaining $d-r$ directions, then x can be visualized as an arrangement of the shifts of the $d-r$ dimensional strip in the other r directions (where the shift in direction i_j is by m_j) and the proof of the forward part is complete.

Conversely, if x can be visualised as an arithmetic progressive arrangement of shifts of an $d-r$ dimensional infinite strip, with common difference m_i in i-th of the remaining r directions, then x is periodic in these r directions (under shifts by m_i in i-th of these r directions) and the proof of converse is complete. □

Example 4.2.2 ***(The Chessboard Problem)**: Let $n \geq 2$ and $A = \{0, 1, 2, \cdots, n-1\}$ denote the set of distinct colours. Let X be the collection of all two dimensional configurations in which adjacent lattice points have different colours. It may be noted that for $n = 2$, X contains precisely two points (a periodic point of period two). In this case both the points are periodic with periods $(2,0)$ and $(0,2)$ and hence the lattice of period is two dimensional. A similar conclusion holds for $n \geq 3$ when discussed for periodic points with finite orbits and hence any point with finite orbit has a lattice of periods of full dimension. Let $u = \left(\begin{smallmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{smallmatrix}\right)^{\infty}$ and let $v = \left(\begin{smallmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{smallmatrix}\right)^{\infty}$. Let (sequence (u_n)) be defined as $u_i = u$ for $n(n+1)+1 \leq |i| \leq (n+1)^2$ and $u_i = v$ for $(n+1)^2+1 \leq |i| \leq (n+1)(n+2)$. Then, the vertical arrangement of u_i's generates a point which is nonperiodic in the vertical direction. Also* *as the generating block is a repetition of finite block (in horizontal direction), the generated point is a periodic point with infinite orbit. It may be noted that the lattice of periods in this case is one dimensional (generated by* $(4,0)$*). A similar argument holds for any periodic point with infinite orbit and hence any periodic point with infinite orbit has a one dimensional lattice of periods. Similar arguments establish a periodic point with* r *dimensional lattice of periods to be arithmetic progressive arrangement of a* $d-r$ *dimensional infinite strips and hence validates the above results.*

We now give some results for validating the presence of a given periodic point (with finite orbit) in a given multidimensional shift space X.

Proposition 4.2.3 *For any point* x *with finite orbit in the* d *dimensional full shift,* x *belongs to a shift space* X *if and only if there exists* $a \in \mathbb{Z}^d$ *such that* $x \in \bigcap_{n=1}^{\infty} \mathscr{B}^X_{na}$.

Proof. Let x be a point with finite orbit in a d dimensional shift space. If x has a finite orbit, x can be visualized as a repetitive arrangement (in each of the d directions) of a finite d dimensional cuboid (say C). Let a_i be the length of cuboid in the i-th direction. Then for $a = (a_1, a_2, \ldots, a_d)$, as repetitive arrangement of C in each of the d directions is allowed in X, $x \in \mathscr{B}^X_{na}$ for all $n \in \mathbb{N}$ and the proof for forward part is complete.

Conversely, let $x \in \bigcap_{n=1}^{\infty} \mathscr{B}^X_{na}$ and let C be the cuboid of size a in x (placed at origin). Then, as $x \in \mathscr{B}^X_{na}$ for all $n \in \mathbb{N}$, cubes (of arbitrarily large finite size) obtained by repetitive arrangement of C are allowed in X. Consequently, $x \in X$ and the proof is complete. □

Proposition 4.2.4 *For any point* x *with finite orbit in the* d *dimensional full shift,* x *belongs to a shift of finite type* X *if and only if there exists* $a \in \mathbb{Z}^d$ *and* $M \in \mathbb{N}$ *such that* $x \in \bigcap_{n=1}^{2M} \mathscr{B}^X_{na}$.

Proof. Let x be a point with finite orbit in a d dimensional shift space. Let the shift space X be determined by a finite set (say $\mathscr{F}$) of forbidden cubes of size M. As orbit of x is finite, x can be visualized as a repetitive arrangement (in each of the d directions) of a finite d dimensional cuboid (say C). Let a_i be the length of cuboid in the i-th direction. Then for $a = (a_1, a_2, \ldots, a_d)$, as repetitive arrangement of C in each of the d directions is allowed in X, $x \in \mathscr{B}^X_{na}$ for all $n = 1, 2, \ldots, 2M$. Thus, $x \in \bigcap_{n=1}^{2M} \mathscr{B}^X_{na}$ and the proof for forward part is complete.

Conversely, let $x \in \bigcap_{n=1}^{2M} \mathscr{B}^X_{na}$ and let C be the cuboid of size a in x (placed at origin). Then, as $x \in \mathscr{B}^X_{na}$ for all $n = 1, 2, \ldots, 2M$, repetitive $2M \times 2M$ placement of the cube C is allowed in X. Further, as any forbidden block is spread across a cube of size less than $2M$, validity of $\underbrace{2M \times 2M \times \ldots \times 2M}_{d \ \ times}$ arrangement of C ensures compatibility of C with itself in all the d directions and hence infinite repetition of C is allowed in each of the d directions. Thus $x \in X$ and the proof is complete. □

Remark 4.2.3 *The above results provide a necessary and sufficient criteria for a point with finite orbit to belong to a shift space* X*. Note that in general, while the validity of arbitrarily large sizes needs to be validated to ensure the presence of* x *in the shift space* X*, the same can be verified in finite steps if* X *is a shift of finite type. The proof follows from the fact that if a cube of size* $2M$ *obtained by a repetitive arrangement is allowed in* X *then the pattern can be repeated infinitely to obtain the point in the shift space* X*. Also, as each point of* $\mathscr{B}^X_{na}$ *has a finite orbit, such a criteria cannot be used to validate the presence of periodic points with infinite orbit. However, any point of*

X can be approximated by arbitrarily large cubes allowed in X and hence can be visualized as a limit of elements from $\mathscr{B}_n^X$. More precisely, any element x of the full shift belongs to the shift space X if and only if there exists a sequence (x_n) $(x_n \in \mathscr{B}_n^X)$ such that $x_n \to x$ (in the product topology). Finally, note that if the set of periodic points with finite orbits are dense in X then the shift space can be determined completely in the countable number of steps. Thus we get the following corollaries.

Corollary 4.2.2 *For any shift space X, a point x of the full shift belongs to X if and only if there exists a sequence (x_n), $(x_n \in \mathscr{B}_n^X)$ such that $x_n \to x$.*

Proof. Let X be a shift space and let $x \in X$. As $x \in X$, its restriction to a cube of size n (placed at origin) is allowed in X and hence repetitive arrangement of the cube (say x_n) belongs to $\mathscr{B}_n^X$. Finally as x coincides with x_n in cubes of size n, $x_n \to x$ and the proof of forward part is complete.

Conversely, let x be a point of the full shift and let (x_n) be a sequence $(x_n \in \mathscr{B}_n^X)$ such that $x_n \to x$. As $x_n \to x$, for each $r \in \mathbb{N}$ there exists $n_r \in \mathbb{N}$ such that $d(x_n, x) < \frac{1}{2^r}$ $\forall n \geq n_r$ and hence central block (of size r) of x and x_{n_r} coincide. As x_{n_r} is a repetitive arrangement of an allowed block (of size n_r) in X, central block of x (of size r) is allowed in X. As $r \in \mathbb{N}$ is arbitrary, arbitrarily large central blocks of x are allowed in X. Hence $x \in X$ and the proof is complete. □

Corollary 4.2.3 *For any shift space X, If periodic points with finite orbits are dense in X, then the shift space X can be determined in countable time.*

Proof. Let X be a shift space such that periodic points with finite orbits are dense in X. As the presence of each point with finite orbit can be validated in countable time, the periodic points with finite orbits can be determined in countable time. Further, as a set of such points is dense in X, the shift space X can be determined in countable time and the proof is complete. □