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Introduction

Dynamical systems is an important branch of Mathematics that studies the long-term behavior of a system whose states evolve over time. The time-evolutionary process generated either by linear or nonlinear equations gives the dynamical systems. In the early 1600s, Johannes Kepler and Galileo Galilei used dynamical systems in astronomy investigations. In the mid 1600s, Isaac Newton invented calculus, differential equations, the laws of motion and universal gravitation, and combined them to Kepler's laws of planetary motion. Further, in late 1800s, Henri Poincaré introduced qualitative rather than quantitative approach, for analyzing the behavior of a system and investigated the three body problem in celestial mechanics with the help of theory of dynamical systems. In 1900s, A. M. Lyapunov, G. D. Birkhoff, A. A. Andronov, V. I. Arnold, J. Moser and others flourished the qualitative approach for analyzing the dynamical evolution of a system. Later, E. Lorenz studied a simplified three variable model of convection in the atmosphere. This field continues to develop rapidly in many directions, and its implication continue to grow.

One dimensional dynamical systems have an extensive history. The general theory of one dimensional dynamics is very well developed. One dimensional dynamical system models some real time phenomena. The central theme of the one dimensional theory is the geometric rigidity of the attractors. Renormalization is the key technique to study the rigidity of the attractors. The historical review on renormalization is vast. The aim of this introduction is just to give an idea of different directions and developments of the theory.

Renormalization is a technique to analyze maps having the property that the first return map to a small part of the phase space resembles the original map itself. M. Feigenbaum [Feigenbaum, 1978], [Feigenbaum, 1979] and P. Coullet and C. Tresser [Coullet and Tresser, 1978] introduced period doubling renormalization operator to describe the asymptotic small scale geometry of the attractor of one parameter family of one dimensional maps near the accumulation of period doubling. Renormalization is the main tool to investigate the geometrical properties of the attractors at smaller scales. If the microscopic geometry of an attractor is not changed, this phenomenon is called rigidity. The geometric rigidity of the attractors is the center of attention in one dimensional theory. The smoothness of the maps under consideration plays a crucial role for rigidity. M. Feigenbaum, P. Coullet and C. Tresser disclosed some interesting results on rigidity in one dimensional dynamics, i.e., the microscopic geometry of the invariant Cantor set of generic smooth maps at the phase transition from simple to chaotic being independent of the considered one parameter family of unimodal maps. In particular, M. Feigenbaum considered a family of quadratic maps such as Logistic family f_μ for analyzing the phase transition, where,

$$f_\mu(x) = \mu x(1-x), \quad x \in [0,1], \quad \mu > 0.$$

By computer experiments, the parameter values μ_n where 2^n cycle first occurred in this family was obtained. Then, it was observed that the ratio of distance between two consecutive period doubling bifurcations approaches to a constant, i.e.,

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = 4.669... = \delta.$$

This number δ is a universal constant and known as Feigenbaum constant (or Feigenbaum delta).

To explain this phenomenon, a nonlinear operator, called as period doubling renormalization operator, defined on the space of unimodal maps was introduced. Initially, Coullet and Tresser conjectured that the period doubling renormalization operator defined on the space of smooth enough generic unimodal maps has a unique hyperbolic fixed point with a one dimensional unstable direction and the universal constant δ is the eigenvalue of the derivative of the renormalization operator at the fixed point.

O. Lanford [Lanford III, 1982] confirmed this conjecture through a computer assisted proof. Also, Eckmann and Wittwer [Eckmann and Wittwer, 1987] proved this conjecture using rigorous computer estimates. Without essential help from the computer, M. Campanino and H. Epstein [Campanino and Epstein, 1981], Campanino et al. [Campanino et al., 1982] and Epstein [Epstein, 1986] proved the existence of the renormalization fixed point (but neither uniqueness nor hyperbolicity). Later on, the initial conjecture was generalized as:

Renormalization conjecture: *The limit set of the renormalization operator in the space of maps of bounded combinatorial type is a hyperbolic Cantor set where the operator acts as the full shift in a finite number of symbols.*

From the last four decades, a lot of mathematical theory have been developed for the renormalization theory in low-dimensional dynamics. Several milestones have been achieved by a number of mathematicians. Especially, D. Sullivan [Sullivan, 1992] showed the convergence of renormalizations. Moreover, all limits of renormalization are quadratic-like maps with a definite modulus. J. Hu [Hu, 1995] proved that the real polynomial map having the periodic points of all power of 2 is infinitely renormalizable. Further, McMullen [McMullen, 1996] proved the exponential convergence towards the limit set of renormalization. Also, M. Martens [Martens, 1998] developed an approach to prove the existence of periodic points of the renormalization operator defined on smooth unimodal maps with arbitrary combinatorial type. M. Lyubich considered the renormalization with bounded combinatorics in [Lyubich, 1999]. A. Davie [Davie, 1996] showed the hyperbolicity of a unique analytic renormalization fixed point in the space of $C^{2+\alpha}$ ($\alpha < 1$) unimodal maps. Moreover, he proved that the renormalization fixed point has codimension one stable manifold which coincides with infinitely renormalizable maps, and a one dimensional unstable manifold which consists of analytic maps.

Using the results of Lyubich [1999]; E. de Faria, W. de Melo and A. Pinto [de Faria et al., 2006] extended to more general types of renormalization in the space C^r, provided $r \geq 2+\alpha$ with α close to one. Later, Chandramouli, Martens, de Melo, Tresser [Chandramouli et al., 2009], proved that the period doubling renormalization converges to the analytic generic fixed point proving it to be globally unique in a class $C^{2+|\cdot|}$ which is bigger than $C^{2+\alpha}$ (for any positive $\alpha \leq 1$). It was observed that the fixed point of renormalization operator is not hyperbolic in the space of C^2 unimodal maps. Furthermore, they showed that the uniqueness is lost below C^2 space and other asymptotic behavior encountered. A. Avila and M. Lyubich [Avila and Lyubich, 2011] developed a new approach to convergence of renormalization for unimodal maps. Recently, O. Kozlovski and S. van Strien [Kozlovski and van Strien, 2020] proved the existence of a period doubling infinitely renormalizable asymmetric unimodal map with non universal scaling laws.

Importance of Low-Smooth Maps: Many of real time applications can be modeled by one dimensional system which are not enough smooth. In dissipative systems, the states are bundled in stable manifolds, and different states in a stable manifold have the same future. In general, the stable manifolds are not bundled smoothly. Like, the Lorenz flow is a three dimensional flow associated with a two dimensional stable manifold and one dimensional unstable manifold. These stable manifolds are bundled in a foliation which is non-smooth. The Lorenz flow can be studied

by an interval map with below C^2 smoothness.
Therefore, the thesis work focuses on renormalization with specified combinatorics, of unimodal maps and bimodal maps whose smoothness is below C^2.

Before proceeding further, we recall some basic definitions and concepts.
Let $I = [a,b]$ be a closed interval.
A point $c \in I$ is called *critical point* of a map $f : I \to I$ if $Df(c) = 0$. The critical point c is said to be *non-flat critical point of order* κ, where κ is a positive integer, if f is $C^{\kappa+1}$ in a neighborhood of c and $Df(c) = D^2 f(c) = \cdots = D^{\kappa-1} f(c) = 0$ and $D^{\kappa} f(c) \neq 0$.
A map $f : I \to I$, is a C^1 map with a unique critical point $c \in I$, is called *unimodal* map.
A function $f : I \to \mathbb{R}$ is said to be a *Lipschitz function* on I if there exists a constant $\lambda_f > 0$ such that

$$|f(x) - f(y)| \leq \lambda_f \, |x - y| \quad \forall\, x,\, y \,\in\, I.$$

A unimodal map $f : I \to I$ is a C^{1+Lip} mapping with the following properties:

- $f \in C^1$,
- Df is a Lipschitz function.

Let c be a critical point of a unimodal map f has a *quadratic tip* if there exists a sequence $\{y_n\}$ approaches to c and a constant $l > 0$ such that

$$\lim_{n\to\infty} \frac{f(y_n) - f(c)}{(y_n - c)^2} = -l.$$

A interval $J \subset I$ is called a *periodic interval of period n* if $f^n(J) = J$ for some positive integer $n \in \mathbb{N}$.
A unimodal map f is called *period n-renormalizable* if there exists a proper subinterval J of I and a positive integer $n \geq 2$ such that
(i) $f^i(J)$, $i = 0, 1, \ldots.., n-1$, have no pairwise interior intersection,
(ii) $f^n(J) \subset J$.
Then $f^n : J \to J$ is called a *pre-renormalization* of f.
A map $f : I \to I$ is *infinitely renormalizable* map if there exists an infinite sequence $\{I_m\}_{m=0}^{\infty}$ of nested intervals and an infinite sequence $\{k(m)\}_{m=0}^{\infty}$ of positive integers such that $f^{k(m)}|_{I_m} : I_m \to I_m$ are pre-renormalizations of f and the length of I_m tends to zero as $n \to \infty$.

Note that, for $n = 2, 3, 4, 5$, period $n-$renormalization is said to be period doubling, tripling, quadrupling or quintupling renormalization respectively.

Let U be the set of unimodal maps and $U_0 \subset U$ contains the set of period tripling renormalizable unimodal maps.
Let $f \in U_0$. Then, the period tripling renormalization operator

$$R : U_0 \to U$$

is defined by

$$Rf(x) = h^{-1} \circ f^3 \circ h(x),$$

where $h : I \to J$ is the orientation reversing affine homeomorphism. The map Rf is again a unimodal map. This is illustrated in Figure 1.1. The set of *period tripling infinitely renormalizable* maps is denoted by

$$W_0 = \bigcap_{m\geq 1} R^{-m}(U_0).$$

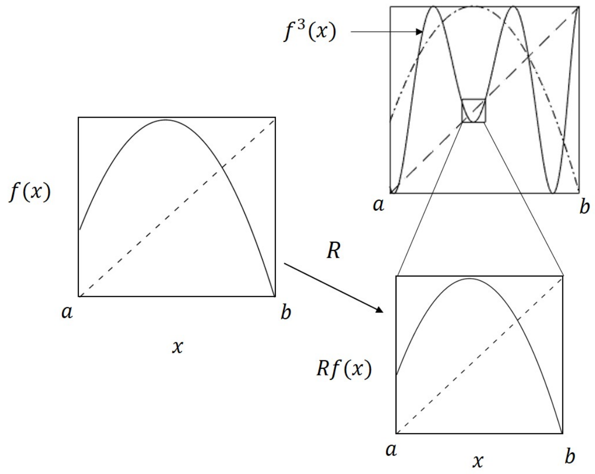

Figure 1.1 : Period tripling renormalization of a unimodal map f

The first part of the thesis deals with the renormalizations of unimodal maps whose smoothness are below C^2.

In chapter 2, we primarily consider the renormalization operator as a period tripling and quintupling renormalizations (i.e., odd period renormalizations) below C^2 class of unimodal maps.

In the case of period $n-$renormalization, where $n \in \{3,5\}$, for a proper scaling data s^* we construct a nested sequence of affine pieces whose end-points lie on the unimodal map and shrinking down to the critical point of the unimodal map. This allows us to prove the existence of the fixed point of renormalization operators acting on the space of piece-wise affine infinitely renormalizable maps. The class U_∞ consists of piece-wise affine period tripling infinitely renormalizable maps. This will lead to the following proposition,

Proposition 1.0.1. *There exists a map $f_{s^*} \in U_\infty$, where s^* is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

In particular, $U_\infty = \{f_{s^*}\}$.

Further, the renormalization fixed point f_{s^*} is extended to C^{1+Lip} unimodal map $\mathscr{F}_{s^*}$ with a quadratic tip. We have the following theorem,

Theorem 1.0.2. *There exists a period tripling infinitely renormalizable C^{1+Lip} unimodal map $\mathscr{F}_{s^*}$ with a quadratic tip such that*

$$R\mathscr{F}_{s^*} = \mathscr{F}_{s^*}.$$

The *topological entropy* of a system defined on a non-compact space is defined to be the Supremum of topological entropies contained in compact invariant subsets.
Using this statement, we investigate the topological entropy of renormalization operators defined on the space of C^{1+Lip} unimodal maps. Then, we have,

Theorem 1.0.3. *The period tripling renormalization operator R defined on the space of C^{1+Lip} unimodal maps has infinite topological entropy.*

Further, we consider $\varepsilon-$variation on the scaling data. This helps us show the existence of another fixed point of the renormalization operator. Then, we obtain the following theorem,

Theorem 1.0.4. *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} unimodal maps.*

Moreover, we prove the following result,

Theorem 1.0.5. *There exists a period tripling infinitely renormalizable C^{1+Lip} unimodal map k with quadratic tip such that $\{c_n\}_{n\geq 0}$, where c_n is the critical point of $R^n k$, is dense in a Cantor set.*

Furthermore, there are two other possible period 5 combinatorics for period quintupling renormalization, which are $(I_3^n \to I_5^n \to I_1^n \to I_2^n \to I_4^n \to I_3^n)$ and $(I_4^n \to I_5^n \to I_1^n \to I_2^n \to I_3^n \to I_4^n)$. Consequently, one can construct a renormalization fixed point and obtain the above results corresponding to each combinatorics.

These results described in chapter 2 of the thesis work are based on the following article. Rohit Kumar, V.V.M.S. Chandramouli, *Period Tripling and Quintupling Renormalizations Below C^2 Space*, Discrete and Continuous Dynamical Systems, doi:10.3934/dcds.2021091.

Additionally, in subsection 2.3, we also describe the period quadrupling renormalization by considering period quadruple combinatorics. Consequently, we observe that there are two main differences among period tripling, quadrupling and quintupling renormalizations as follows:
As n increases for period $n-$renormalization, the number of the possible combinatorics will increase and each of them will lead to such construction of renormalization fixed point.
Moreover, the geometry of respective invariant Cantor set become more complex.

In the context of circle diffeomorphisms, M. Herman [Herman, 1979] proved the rigidity result, using real variable techniques. J. C. Yoccoz [Yoccoz, 1984] proved the other fundamental rigidity results by using conformal surgery, where Herman's theorem holds in the real-analytic category. Further, K. M. Khanin and Y. G. Sinai [Khanin and Sinai, 1987] gave a proof of M. Herman's theorem which is based on the thermodynamic formalism and ergodic properties for the corresponding random variables. Later, M. Yampolsky [Yampolsky, 2001] proved the rigidity of circle map with a critical point. Furthermore, the rigidity theory for circle maps with break type singularities have been developed by K. M. Khanin, S. Kocić, A. Teplinsky, E. Mazzeo [Khanin and Kocić, 2013] [Khanin and Teplinsky, 2013] [Khanin, k. et al., 2017] [Khanin and Kocić, 2018], K. Cunha and D. Smania [Cunha and Smania, 2014], H. Akhadkulov et al. [Akhadkulov et al., 2017]. In the context of interval maps, the rigidity phenomena is understood for $C^{2+\alpha}$ $(\alpha > 0)$ smooth maps. Further, W. de Melo and A. Pinto [de Melo and Pinto, 1999] proved the rigidity of C^2 infinitely renormalizable unimodal maps with bounded combinatorial type. The measure-theoretical properties of real family of unimodal maps are studied by M. Lyubich, A. Avila, W. de Melo, H. Bruin, W. Shen, S. van Strien, C. S. Moreira, D. Smania and M. Todd. Later, M. Lyubich [Lyubich, 2002] proved that almost any real quadratic map has either an attracting cycle or an absolutely continuous invariant measure. Further, A. Avila, M. Lyubich and W. de Melo [Avila et al., 2003] extended these result for any non-trivial real analytic family of quasiquadratic maps. H. Bruin, W. Shen and S. van Strien [Bruin et al., 2006] showed that almost every unicritical polynomial with even critical order greater than or equal to 2, admits a physical measure, which is either supported on an attracting periodic orbit, or is absolutely continuous, or is supported on the postcritical set. Further, C. S. Moreira and D. Smania [Moreira and Smania, 2014] showed the rigidity of infinitely renormalizable Fibonacci unimodal maps with even critical order and having negative Schwarzian derivative. H. Bruin and M. Todd [Bruin and Todd, 2015] proved the existence of wild attractor for a countably piece-wise linear infinitely renormalizable Fibonacci unimodal map with infinite critical order.

In the context of two dimensional maps, A. de Carvalho, M. Lyubich and M. Martens [Carvalho et al., 2005] showed the non-rigidity of Cantor attractors of Hénon-like maps. Further, P. Hazard, M. Lyubich and M. Martens [Hazard et al., 2012] discussed the unbounded geometry of Cantor attractor of strongly dissipative infinitely renormalizable Hénon-like map with stationary combinatorics. In case of Lorenz maps, M. Martens and B. Winckler [Martens and Winckler, 2014], [Martens and Winckler, 2018] studied the hyperbolicity of Lorenz renormalization and also proved the non-existence of physical measures for Lorenz maps which are infinitely renormalizable.

With a relatively complete understanding of the period doubling renormalization of unimodal maps, recent research in dynamical systems has either focused on more complicated maps of the real line or other low dimensional maps. L. Jonker and D. Rand [Jonker and Rand, 1980], and S. van Strien [Strien, 1988] used renormalization as a natural vehicle to decompose the non-wandering set in a hierarchical manner, for unimodal maps. The multimodal maps are interesting as generalizations of unimodal maps, as well as for their applications. For example, in the case of bimodal maps, they are essential to understand the non-invertible circle maps which have been used extensively to model the transitions to chaos in two frequency systems [Mackay and Tresser, 1986]. R. S. Mackay and J. van Zeijts [MacKay and Zeijts, 1988] explained the period doubling renormalization of two parameter families of bimodal maps in the term of a horseshoe with a Cantor set of two dimensional unstable manifold. Also, they calculated the periodic points of renormalization up to period five. D. Veitch [Veitch, 1994] presented some work on topological renormalization of C^0 bimodal maps with zero and positive entropy. Further, D. Smania developed a combinatorial theory for certain kind of multimodal maps and proved that for the same combinatorial type the renormalizations of infinitely renormalizable smooth multimodal maps are exponentially close [Smania, 2001], [Smania, 2005]. Later, D. Smania [Smania, 2020] proved the hyperbolicity of renormalization for real analytic multimodal maps with bounded combinatorics.

As the first part of the thesis discusses the period tripling renormalization of unimodal maps. This motivates us to describe the period tripling renormalization in the context of symmetric bimodal maps of the interval with low smoothness. The second part of this thesis also contains interesting results on the non-rigidity of infinitely renormalizable symmetric bimodal maps.

An interval map f is *piece-wise monotone* if there exists a partition of I into finitely many subintervals on each of which the restriction of f is continuous and strictly monotonic.
A map f is called a *bimodal* map if three is the minimal number of such subintervals.

Definition 1.0.1. Let $f : I \to I$ be a C^1 map with two subsets J_l and J_r such that $\overset{\circ}{J_l} \cap \overset{\circ}{J_r} = \emptyset$. If $f|_{J_l}$ and $f|_{J_r}$ are unimodal maps which are concave up and concave down respectively, their *join*, denoted by $f|_{J_l} \oplus f|_{J_r}$, is a bimodal map whose graph is obtained by joining $\big(max(J_l), f(max(J_l))\big)$ and $\big(min(J_r), f(min(J_r))\big)$ by a C^{1+Lip} curve.

We define the renormalization of bimodal maps associated with period tripling combinatorics. Let $I = [0,1]$ be a closed interval.

Definition 1.0.2. A bimodal map $b : I \to I$, is a C^1 map having two critical points c_l and c_r, which is said to be *renormalizable* if there exists two disjoint intervals I_l containing c_l and I_r containing c_r such that

(i) $b^i(I_l) \cap b^j(I_l) = \emptyset$, for each $i \neq j$ and $i, j \in \{0,1,2\}$,
$b^i(I_r) \cap b^j(I_r) = \emptyset$, for each $i \neq j$ and $i, j \in \{0,1,2\}$,

(ii) $b^3(I_l) \subset I_l$ and $b^3(I_r) \subset I_r$,

(iii) The unimodal maps $\hat{b}_l : [0,\ b(0)] \to [0,\ b(0)]$ and $\hat{b}_r : [b(1), 1] \to [b(1), 1]$ are joined to generate a bimodal map $\hat{b}_l \oplus \hat{b}_r$. The unimodal maps $\hat{b}_l$ and $\hat{b}_r$ are defined as

$$\hat{b}_l(x) = h_1^{-1} b^3 h_1(x)$$

and

$$\hat{b}_r(x) = h_2^{-1} b^3 h_2(x),$$

where $h_1 : [0,\ b(0)] \to I_l$ and $h_2 : [b(1), 1] \to I_r$ are the affine orientation reversing homeomorphisms.

The renormalization of a symmetric bimodal map is illustrated in Figure 1.2.

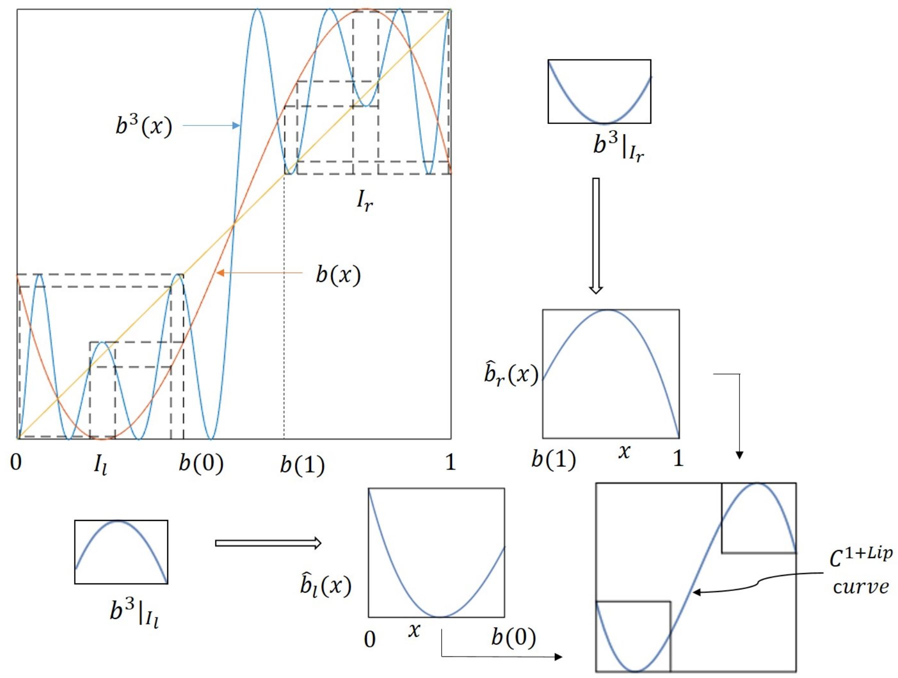

Figure 1.2 : Pairwise period tripling Renormalization of a bimodal map b

Furthermore, by reducing the periodicity of combinatorics, we define the pairwise period doubling renormalization of bimodal maps.

Definition 1.0.3. A bimodal map $b : I \to I$, is a C^1 map with two critical points c_l and c_r, is said to be *pairwise period doubling renormalizable* if there exists a pair of disjoint intervals (I_l, I_r), with $I_l \ni c_l$ and $I_r \ni c_r$, such that
(i) $I_l \cap b(I_l) = \emptyset,\quad I_r \cap b(I_r) = \emptyset,$
(ii) I_l and I_r are b^2-invariant,
(iii) The unimodal maps $\hat{b}_l : [0,\ b(0)] \to [0,\ b(0)]$ and $\hat{b}_r : [b(1), 1] \to [b(1), 1]$ are joined to generate a bimodal map $\hat{b}_l \oplus \hat{b}_r$. The unimodal maps $\hat{b}_l$ and $\hat{b}_r$ are defined as

$$\hat{b}_l(x) = h_l^{-1} b^2 h_l(x)$$

and

$$\hat{b}_r(x) = h_r^{-1} b^2 h_r(x),$$

where $h_l : [0,\ b(0)] \to I_l$ and $h_r : [b(1), 1] \to I_r$ are the affine orientation reversing homeomorphisms.

The pairwise period doubling renormalization of a symmetric bimodal map is illustrated in Figure 1.3.

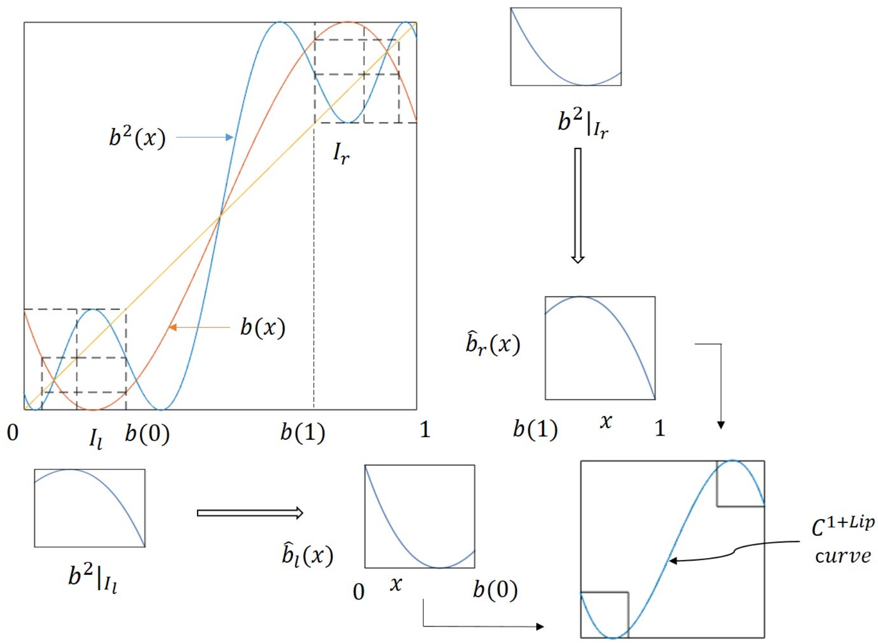

Figure 1.3 : Pairwise period doubling renormalization of a bimodal map b

The results based on renormalizations of symmetric bimodal maps by considering period doubling and tripling combinatorics are discussed in the second part of the thesis.
In Chapter 3, the renormalization operator R is a pair of period tripling renormalization operators R^l and R^r which are defined on piece-wise affine period tripling infinitely renormalizable maps corresponding to a proper scaling data s_l and s_r, respectively.
In Chapter 4, the renormalization operator R is a pairwise period doubling renormalization operators R^l and R^r which are defined on piece-wise affine period doubling infinitely renormalizable maps corresponding to a proper scaling data s_l and s_r, respectively.[1] Firstly, we show that there exists a sequence of affine pieces which are nested and shrinking down to the critical points of the bimodal map corresponding to a pair of proper scaling data $s^* = (s_l^*, s_r^*)$. This helps us to show the existence of a fixed point f_{s^*} of the renormalization operator defined on the space of piece-wise affine infinitely renormalizable maps, which is denoted by W, corresponding to a pair of proper scaling data s^*. This gives us the following result.

Theorem 1.0.6. *There exists a map $f_{s^*} \in W$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

In particular, $W = \{f_{s^*}\}$.

Afterwards, we explain the extension of the renormalization fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map g_{s^*}. Then, we have the following theorem,

Theorem 1.0.7. *There exists an infinitely renormalizable C^{1+Lip} symmetric bimodal map g_{s^*} such that*

$$Rg_{s^*} = g_{s^*}.$$

[1] In both chapters 3 and 4, we use the same notations R, R^l and R^r but the contextual meaning is different.

Also, we describe the topological entropy of renormalization defined on the space of C^{1+Lip} symmetric bimodal maps. Then we obtain the following theorem,

Theorem 1.0.8. *The renormalization operator R acting on the space of C^{1+Lip} symmetric bimodal maps has unbounded topological entropy.*

Furthermore, we discuss the existence of another fixed point of renormalization by considering the small perturbation on the scaling data. Then, we get the following result,

Theorem 1.0.9. *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} symmetric bimodal maps.*

Consequently, this result leads to the non-rigidity of the Cantor attractors of infinitely renormalizable symmetric bimodal maps, whose smoothness is below C^2.

The results described in chapter 3 of the thesis work are published in the following article. Rohit Kumar, V.V.M.S. Chandramouli, *Renormalization of Symmetric Bimodal Maps with Low Smoothness*, Journal of Statistical Physics, 183, 29 (2021). doi:10.1007/s10955-021-02764-8

Finally, in chapter 5, we present comprehensive conclusions concerning all the key findings of the constitutive chapters 2-4 of the thesis. Also, we highlight a grand overview of the thesis along with the future research scope.

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