3
Renormalization of symmetric bimodal maps with low smoothness

This chapter deals with the renormalization of symmetric bimodal maps with low smoothness associated with the period tripling combinatorics. Here, the renormalization operator acts as a pair of period tripling renormalization operators corresponding to left and right critical points of symmetric bimodal map. Firstly, for a given pair of proper scaling data, we construct a nested sequence of affine pieces whose end-points lie on the symmetric bimodal map and shrinking down to the critical points of the map. We show that there exists a sequence of affine pieces which are nested and contract to the critical points of the bimodal map corresponding to a pair of proper scaling data. This helps us to prove that the renormalization operator defined on the space of piece-wise affine infinitely renormalizable maps has a fixed point, denoted by f_{s^*}, corresponding to a pair of proper scaling data s^*. In the next section 3.2, we explain the extension of the renormalization fixed point f_{s^*} to a C^{1+Lip} symmetric bimodal map. In section 3.3, we describe the topological entropy of renormalization defined on the space of C^{1+Lip} symmetric bimodal maps. Furthermore, we prove the existence of another fixed point of renormalization by considering the small perturbation on the scaling data. Consequently, for two different perturbed scaling data we get two Cantor attractors of renormalization fixed points. This leads to the non-rigidity of the Cantor attractors of renormalizable symmetric bimodal maps with low smoothness.

We recall some basic definitions. Let $I = [0,1]$ be a closed interval.
A unimodal map $\mathfrak{u} : I \to I$, which is a C^1 map having a unique non-flat critical point c, is called *period tripling renormalizable map* if there exists a proper subinterval $J \subset I$ with $c \in J$ such that
(1) J, $\mathfrak{u}(J)$ and $\mathfrak{u}^2(J)$ are pairwise disjoint,
(2) $\mathfrak{u}^3(J) \subset J$.
Then $\mathfrak{u}^3 : J \to J$ is called a pre-renormalization of $\mathfrak{u}$.
Where, u^n denotes n fold composition of u with itself.

Let $\mathscr{U}$ be the collection of unimodal maps and $\mathscr{U}_\infty(\subset \mathscr{U})$ be the collection of period tripling infinitely renormalizable unimodal maps.

An interval map f is *piece-wise monotone* if there exists a partition of I into finitely many subintervals on each of which the restriction of f is continuous and strictly monotonic.
A map f is called a *bimodal* map if three is the minimal number of such subintervals.

Definition 3.0.1. Let $f : I \to I$ be a C^1 map with two subsets J_l and J_r such that $\overset{o}{J_l} \cap \overset{o}{J_r} = \emptyset$. If $f|_{J_l}$ and $f|_{J_r}$ are unimodal maps which are concave up and concave down respectively, their *join*, denoted by $f|_{J_l} \oplus f|_{J_r}$, is a bimodal map whose graph is obtained by joining $\big(max(J_l), f(max(J_l))\big)$ and $\big(min(J_r), f(min(J_r))\big)$ by a C^{1+Lip} curve.

Definition 3.0.2. A bimodal map $b : I \to I$, is a C^1 map having two critical points c_l and c_r, which is said to be *renormalizable* if there exists two disjoint intervals I_l containing c_l and I_r containing c_r such that

(i) $b^i(I_l) \cap b^j(I_l) = \emptyset$, for each $i \neq j$ and $i, j \in \{0,1,2\}$,
$b^i(I_r) \cap b^j(I_r) = \emptyset$, for each $i \neq j$ and $i, j \in \{0,1,2\}$,

(ii) $b^3(I_l) \subset I_l$ and $b^3(I_r) \subset I_r$,

(iii) The unimodal maps $\hat{b}_l : [0,\ b(0)] \to [0,\ b(0)]$ and $\hat{b}_r : [b(1), 1] \to [b(1), 1]$ are joined to generate a bimodal map $\hat{b}_l \oplus \hat{b}_r$. The unimodal maps $\hat{b}_l$ and $\hat{b}_r$ are defined as

$$\hat{b}_l(x) = h_1^{-1} b^3 h_1(x)$$

and

$$\hat{b}_r(x) = h_2^{-1} b^3 h_2(x)$$

where $h_1 : [0,\ b(0)] \to I_l$ and $h_2 : [b(1), 1] \to I_r$ are the affine orientation reversing homeomorphisms.

The renormalization of a bimodal map is illustrated in Figure 3.1.

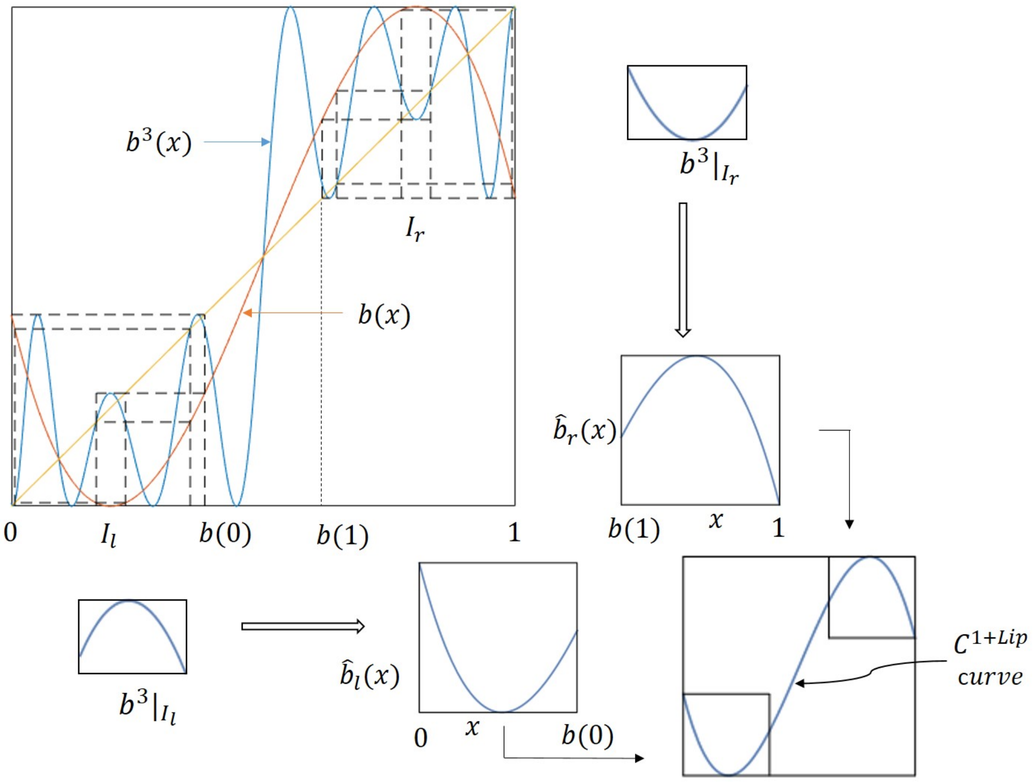


Figure 3.1 : Renormalization of a bimodal map

In the next section, we construct the renormalization operator defined on the space of piece-wise affine maps which are infinitely renormalizable maps.

3.1 PIECE-WISE AFFINE RENORMALIZABLE MAPS

A symmetric bimodal map $b : [0,1] \to [0,1]$ of the form $b(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, for $a_3 < 0$, is a C^1 map with the following conditions

- $b(0) = 1 - b(1)$,
- $b(\frac{1}{2}) = \frac{1}{2}$,

- let c_l and c_r be the two critical points of $b(x)$, then $b(c_l) = 0$ and $b(c_r) = 1$.

Let us consider a one parameter family of symmetric bimodal maps $\mathscr{B}_c : [0,1] \to [0,1]$ which are increasing on the interval between the critical points and decreasing elsewhere. then, we obtained a family of bimodal maps as

$$\begin{aligned} \mathscr{B}_c(x) &= \begin{cases} 1 - \frac{1-6c+9c^2-4c^3+6cx-6c^2x-3x^2+2x^3}{(1-2c)^3}, & \text{if } c \in \left[0, \frac{1}{4}\right] \\ 1 - \frac{4c^3-3c^2+6cx-6c^2x-3x^2+2x^3}{(2c-1)^3}, & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases} \\ &\equiv \begin{cases} b_c(x), & \text{if } c \in \left[0, \frac{1}{4}\right] \\ \tilde{b}_c(x), & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases} \end{aligned} \tag{3.1}$$

Note that the bimodal maps b_c and $\tilde{b}_c$ are identical maps.

Let us define an open set

$$T_3 = \left\{ (s_0, s_1, s_2) \in \mathbb{R}^3 \ : \ s_0, s_1, s_2 > 0, \ \sum_{i=0}^{2} s_i < 1 \right\}.$$

Each element (s_0, s_1, s_2) of T_3 is called a scaling tri-factor. A pair of scaling tri-factors $(s_{0,l}, s_{1,l}, s_{2,l})$ and $(s_{0,r}, s_{1,r}, s_{2,r})$ induces two sets of affine maps $(F_{0,l}, F_{1,l}, F_{2,l})$ and $(F_{0,r}, F_{1,r}, F_{2,r})$ respectively. For each $i = 0, 1, 2$,

$$F_{i,l} : I_L = [0, b_c(0)] \longrightarrow I_L$$

are defined as

$$\begin{aligned} F_{0,l}(t) &= b_c(0) - s_{0,l} \cdot t, \\ F_{1,l}(t) &= b_c^2(0) - s_{1,l} \cdot t, \\ F_{2,l}(t) &= s_{2,l} \cdot t \end{aligned}$$

and

$$F_{i,r} : I_R = [\tilde{b}_c(1), 1] \longrightarrow I_R$$

are defined as

$$\begin{aligned} F_{0,r}(t) &= \tilde{b}_c(1) + s_{0,r} \cdot (1 - t), \\ F_{1,r}(t) &= \tilde{b}_c^2(1) + s_{1,r} \cdot (1 - t), \\ F_{2,r}(t) &= 1 - s_{2,r} \cdot (1 - t). \end{aligned}$$

Note that ${I_L}^{\circ} \cap {I_R}^{\circ} = \phi$, for $c \in [0, \frac{3-\sqrt{3}}{6}]$.
The functions $s_l : \mathbb{N} \to T_3$ and $s_r : \mathbb{N} \to T_3$ are said to be a scaling data. We set scaling tri-factors $s_l(n) = (s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in T_3$ and $s_r(n) = (s_{0,r}(n), s_{1,r}(n), s_{2,r}(n)) \in T_3$,
so that $s_l(n)$ and $s_r(n)$ induce the triplets of affine maps $(F_{0,l}(n)(t), F_{1,l}(n)(t), F_{2,l}(n)(t))$ and $(F_{0,r}(n)(t), F_{1,r}(n)(t), F_{2,r}(n)(t))$ as described above.
For $i = 0, 1, 2$, let us define the intervals

$$I_{i,l}^n = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ \ldots\ldots \circ F_{1,l}(n-1) \circ F_{i,l}(n)([0, b_c(0)]).$$

Also,

$$I_{i,r}^n = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ \ldots\ldots \circ F_{1,r}(n-1) \circ F_{i,r}(n)([\tilde{b}_c(1), 1]).$$

Definition 3.1.1. A scaling data $s_j \equiv \{s_j(n)\}$, for $j = l, r$, is said to be proper if, for each $n \in \mathbb{N}$,

$$d(s_j(n), \partial T_3) \geq \varepsilon, \quad \text{for some} \quad \varepsilon > 0.$$

Where $d(s_j(n), \partial T_3)$ stands for the Euclidean distance between $s_j(n)$ and the closest boundary point of T_3.

A pair of proper scaling data $s_l : \mathbb{N} \to T_3$ and $s_r : \mathbb{N} \to T_3$, which is denoted by $s = (s_l, s_r)$, induce the sets $D_{s_l} = \bigcup_{n \geq 1} (I^n_{0,l} \cup I^n_{2,l})$ and $D_{s_r} = \bigcup_{n \geq 1} (I^n_{0,r} \cup I^n_{2,r})$, respectively. Consider a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0,1]$$

defined as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

where $f_{s_l}|_{I^n_{0,l}}$ and $f_{s_l}|_{I^n_{2,l}}$ are the affine extensions of $b_c|_{\partial I^n_{0,l}}$ and $b_c|_{\partial I^n_{2,l}}$ respectively. Similarly, $f_{s_r}|_{I^n_{0,r}}$ and $f_{s_r}|_{I^n_{2,r}}$ are the affine extensions of $b_c|_{\partial I^n_{0,r}}$ and $b_c|_{\partial I^n_{2,r}}$ respectively. These affine extensions are shown in Figure 3.2. The end points of the intervals at each level are labeled by

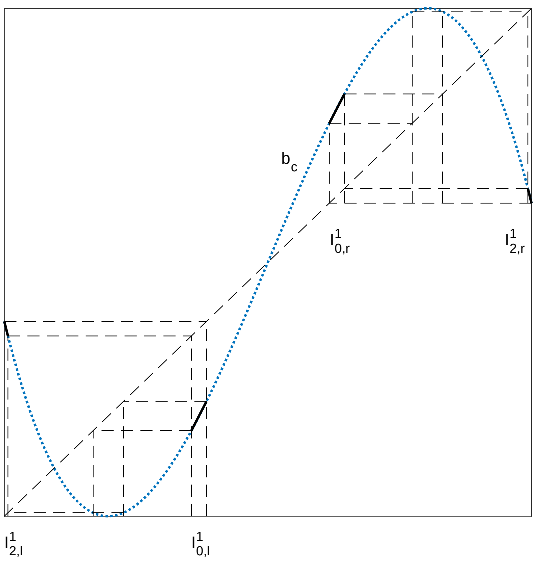


Figure 3.2 : Piece-wise affine extension

$$y_0 = 0,\ z_0 = b_c(0),\ I^0_{1,l} = I_L = [0, b_c(0)]$$

and for $n \geq 1$

$$\begin{aligned} x_n &= \partial I^n_{0,l} \backslash \partial I^{n-1}_{1,l} \\ y_{2n-1} &= max\{\partial I^{2n-1}_{1,l}\} \\ y_{2n} &= min\{\partial I^{2n}_{1,l}\} \\ z_{2n-1} &= min\{\partial I^{2n-1}_{1,l}\} \\ z_{2n} &= max\{\partial I^{2n}_{1,l}\} \\ w_n &= \partial I^n_{2,l} \backslash \partial I^{n-1}_{1,l}, \end{aligned}$$

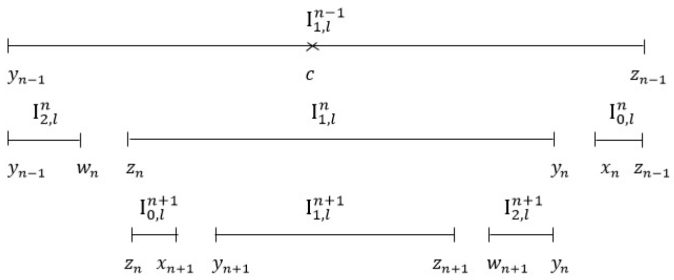


Figure 3.3 : Formation of interval $I_{1,l}^{n-1}$ into three sub-intervals $I_{2,l}^{n}$, $I_{1,l}^{n}$ and $I_{0,l}^{n}$.

These points are illustrated in Figure 3.3.

Also, the end points of the intervals at each level are labeled by

$$z'_0 = \tilde{b}_c(1),\ y'_0 = 1,\ I_{1,r}^0 = I_R = [\tilde{b}_c(1), 1]$$

and for $n \geq 1$

$$\begin{aligned} x'_n &= \partial I_{0,r}^n \backslash \partial I_{1,r}^{n-1} \\ y'_{2n-1} &= min\{\partial I_{1,r}^{2n-1}\} \\ y'_{2n} &= max\{\partial I_{1,r}^{2n}\} \\ z'_{2n-1} &= max\{\partial I_{1,r}^{2n-1}\} \\ z'_{2n} &= min\{\partial I_{1,r}^{2n}\} \\ w'_n &= \partial I_{2,r}^n \backslash \partial I_{1,r}^{n-1}. \end{aligned}$$

These points are illustrated in Figure 3.4.

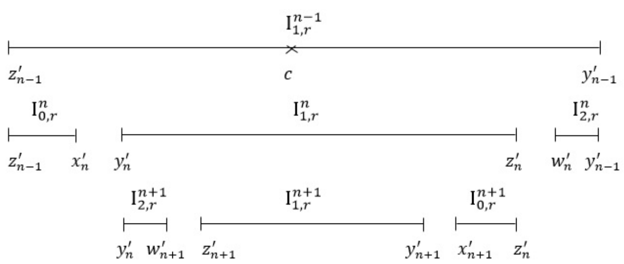


Figure 3.4 : Formation of interval $I_{1,r}^{n-1}$ into three sub-intervals $I_{0,r}^{n}$, $I_{1,r}^{n}$ and $I_{2,r}^{n}$.

Definition 3.1.2. For a given pair of proper scaling data $s_l, s_r : \mathbb{N} \to T_3$, a map f_s is said to be *infinitely renormalizable* if for $n \geq 1$,

1(i) $[0, f_{s_l}(y_n)]$ is the maximal domain containing 0 on which $f_{s_l}^{3^n-1}$ is defined affinely, $[f_{s_l}^2(y_n), f_{s_l}(0)]$ is the maximal domain containing $f_{s_l}(0)$ on which $f_{s_l}^{3^n-2}$ is defined affinely,

(ii) $[f_{s_r}(y'_n), 1]$ is the maximal domain containing 1 on which $f_{s_r}^{3^n-1}$ is defined affinely and $[f_{s_r}(1), f_{s_r}^2(y'_n)]$ is the maximal domain containing $f_{s_r}(1)$ on which $f_{s_r}^{3^n-2}$ is defined affinely,

2(i) $f_{s_l}^{3^n-1}([0, f_{s_l}(y_n)]) \qquad = I_{1,l}^n$,

(ii) $f_{s_l}^{3^n-2}([f_{s_l}^2(y_n), f_{s_l}(0)]) = I_{1,l}^n$,

(iii) $f_{s_r}^{3^n-1}([f_{s_r}(y_n'), 1]) \quad = I_{1,r}^n$,

(iv) $f_{s_r}^{3^n-2}([f_{s_r}(1), f_{s_r}^2(y_n')]) = I_{1,r}^n$.

Define $W = \{f_s : f_s$ is infinitely renormalizable map$\}$.
Further using definition 3.1.2, we write $W_l = \{f_{s_l} : f_{s_l}$ satisfies 1(i), 2(i) and 2(ii)$\}$
and $W_r = \{f_{s_r} : f_{s_r}$ satisfies 1(ii), 2(iii) and 2(iv)$\}$.
Note that W_l and W_r be the collection of the piece-wise affine period tripling infinitely renormalizable maps f_{s_l} on I_L and f_{s_r} on I_R, respectively.
The combinatorics for renormalization of f_{s_l} and f_{s_r} are shown in the following Figures 3.5a and 3.5b.

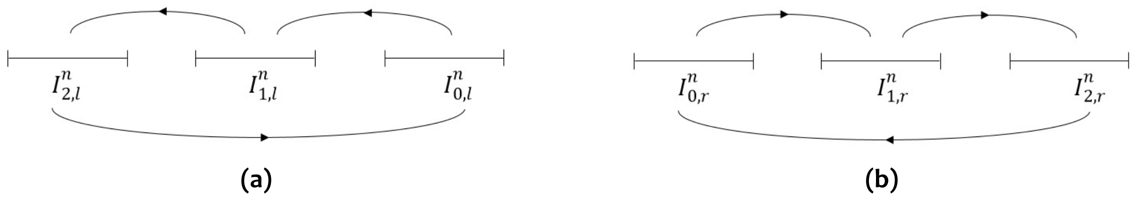


Figure 3.5 : The combinatorics: (a) corresponding to f_{s_l}, $(I_{1,l}^n \to I_{2,l}^n \to I_{0,l}^n \to I_{1,l}^n)$ and (b) corresponding to f_{s_r}, $(I_{1,r}^n \to I_{2,r}^n \to I_{0,r}^n \to I_{1,r}^n)$.

3.1.1 Renormalization on $I_L = [0, b_c(0)]$

Let $f_{s_l} \in W_l$ be given by the proper scaling data $s_l : \mathbb{N} \to T_3$ and define

$$\tilde{I}_{1,l}^n = [b_c^2(y_n), b_c(0)] = [f_{s_l}^2(y_n), f_{s_l}(0)],$$

and

$$\hat{I}_{1,l}^n = [0, b_c(y_n)] = [0, f_{s_l}(y_n)].$$

Let

$$h_{s_l,n} : [0, b_c(0)] \to I_{1,l}^n$$

be defined by

$$h_{s_l,n} = F_{1,l}(1) \circ F_{1,l}(2) \circ F_{1,l}(3) \circ \ldots\ldots \circ F_{1,l}(n)$$

Furthermore, let

$$\tilde{h}_{s_l,n} : [0, b_c(0)] \to \tilde{I}_{1,l}^n \text{ and } \hat{h}_{s_l,n} : [0, b_c(0)] \to \hat{I}_{1,l}^n$$

be the affine orientation preserving homeomorphisms. Then define

$$R_n^l f_{s_l} : h_{s_l,n}^{-1}(D_{s_l} \cap I_{1,l}^n) \to [0, b_c(0)]$$

by

$$R_n^l f_{s_l}(x) = \begin{cases} R_n^{l-} f_{s_l}(x), & \text{if } x \in h_{s_l,n}^{-1}(\underset{m \geq n+1}{\cup} I_{0,l}^m) \\ R_n^{l+} f_{s_l}(x), & \text{if } x \in h_{s_l,n}^{-1}(\underset{m \geq n+1}{\cup} I_{2,l}^m) \end{cases}$$

where,

$$R_n^{l-} f_{s_l} : h_{s_l,n}^{-1}(\underset{m \geq n+1}{\cup} I_{0,l}^m) \to [0, b_c(0)]$$

and

$$R_n^{l+}f_{s_l}: h_{s_l,n}^{-1}(\underset{m\geq n+1}{\cup} I_{2,l}^m) \to [0, b_c(0)]$$

are defined by

$$R_n^{l-}f_{s_l}(x) = \tilde{h}_{s_l,n}^{-1}\circ f_{s_l}^2 \circ h_{s_l,n}(x)$$
$$R_n^{l+}f_{s_l}(x) = \hat{h}_{s_l,n}^{-1}\circ f_{s_l} \circ h_{s_l,n}(x),$$

which are illustrated in Figure 3.6. Let $\sigma: T_3^{\mathbb{N}} \to T_3^{\mathbb{N}}$ be the shift map defined as

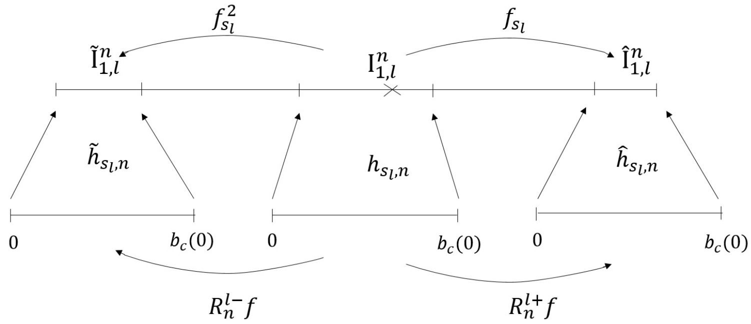


Figure 3.6 : Illustration of operators R_n^{l-} and R_n^{l+}

$$\sigma(s_l(1)s_l(2)s_l(3)s_l(4)....) = (s_l(2)s_l(3)s_l(4)....),$$

where $s_l(i) \in T_3$ for all $i \in \mathbb{N}$.

Note that the operator R_n^l normalize the affine pieces $f_{s_l}^2(\underset{m\geq n+1}{\cup} I_{0,l}^m)$ and $f_{s_l}(\underset{m\geq n+1}{\cup} I_{2,l}^m)$ to I_L with the help of affine homeomorphism $\tilde{h}_{s_l,n}^{-1}$ and $\hat{h}_{s_l,n}^{-1}$, respectively.
This implies, $R_n^l f_{s_l}$ is a piecewise affine map associated with the scaling data $(s_l(n+1)s_l(n+2)s_l(n+3)\ldots)$. Thus,

$$R_n^l f_{s_l} = f_{s_l(n+1)s_l(n+2)s_l(n+3)\ldots}.$$

The above explanation leads the following lemma.

Lemma 3.1.1. *Let $s_l: \mathbb{N} \to T_3$ be proper scaling data such that f_{s_l} is infinitely renormalizable. Then*

$$R_n^l f_{s_l} = f_{\sigma^n(s_l)}.$$

Let f_{s_l} be infinitely renormalization, then for $n \geq 0$, we have

$$f_{s_l}^{3^n}: D_{s_l} \cap I_{1,l}^n \to I_{1,l}^n$$

is well defined.
Define the renormalization $R^l: W_l \to W_l$ by

$$R^l f_{s_l} = h_{s_l,1}^{-1} \circ f_{s_l}^3 \circ h_{s_l,1}.$$

The maps $f_{s_l}^{3^n-2}: \tilde{I}_{1,l}^n \to I_{1,l}^n$ and $f_{s_l}^{3^n-1}: \hat{I}_{1,l}^n \to I_{1,l}^n$ are the affine homeomorphisms whenever $f_{s_l} \in W_l$.

One can observe that, for each $n \in \mathbb{N}$, $\underset{m\geq n+1}{\cup} I_{0,l}^m \subset I_{1,l}^n$ and $\underset{m\geq n+1}{\cup} I_{2,l}^m \subset I_{1,l}^n$.
By the definition of R_n^l, the operator R_n^l is just normalizing the affine pieces, which are contained in $I_{1,l}^n$, to I_L. Also, $I_{1,l}^n$ are the renormalization intervals corresponding to n^{th} renormalization operator $(R^l)^n$. Then, we have the following lemma,

Lemma 3.1.2. *We have $(R^l)^n f_{s_l} : D_{\sigma^n(s_l)} \to [0, b_c(0)]$ and $(R^l)^n f_{s_l} = R^l_n f_{s_l}$.*

Using Lemma 3.1.1 and Lemma 3.1.2, now we are in a position to state the following proposition:

Proposition 3.1.3. *There exists a map $f_{s_l^*} \in W_l$, where s_l^* is characterized by*

$$R^l f_{s_l^*} = f_{s_l^*}.$$

Proof. Consider $s_l : \mathbb{N} \to T_3$ be proper scaling data such that f_{s_l} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_l)}$. Then

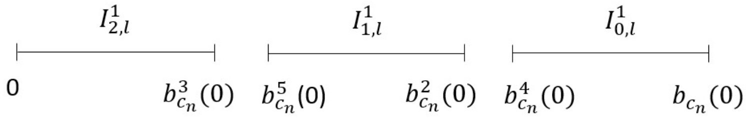


Figure 3.7 : Length of intervals

we have the following scaling ratios which are illustrated in Figure 3.7

$$s_{0,l}(n) = \frac{b_{c_n}(0) - b^4_{c_n}(0)}{b_{c_n}(0)} \tag{3.2}$$

$$s_{1,l}(n) = \frac{b^2_{c_n}(0) - b^5_{c_n}(0)}{b_{c_n}(0)} \tag{3.3}$$

$$s_{2,l}(n) = \frac{b^3_{c_n}(0)}{b_{c_n}(0)} \tag{3.4}$$

$$c_{n+1} = \frac{b^2_{c_n}(0) - c_n}{s_{1,l}(n)} \equiv \mathscr{R}(c_n). \tag{3.5}$$

Since $(s_{0,l}(n), s_{1,l}(n), s_{2,l}(n)) \in T_3$, this implies the following conditions

$$s_{0,l}(n),\ s_{1,l}(n),\ s_{2,l}(n) > 0 \tag{3.6}$$

$$s_{0,l}(n) + s_{1,l}(n) + s_{2,l}(n) < 1 \tag{3.7}$$

As the intervals $I^n_{i,l}$, for $i = 0, 1, 2$, are mutually disjoint, we denote the gap ratios as $g^n_{0,l}$ and $g^n_{1,l}$ which are in between $I^n_{0,l}$ & $I^n_{1,l}$ and $I^n_{1,l}$ & $I^n_{2,l}$ respectively. The gap ratios are defined as, for $n \in \mathbb{N}$,

$$g^n_{0,l} = \frac{b^4_{c_n}(0) - b^2_{c_n}(0)}{b_{c_n}(0)} \equiv G_{0,l}(c_n) > 0 \tag{3.8}$$

$$g^n_{1,l} = \frac{b^5_{c_n}(0) - b^3_{c_n}(0)}{b_{c_n}(0)} \equiv G_{1,l}(c_n) > 0 \tag{3.9}$$

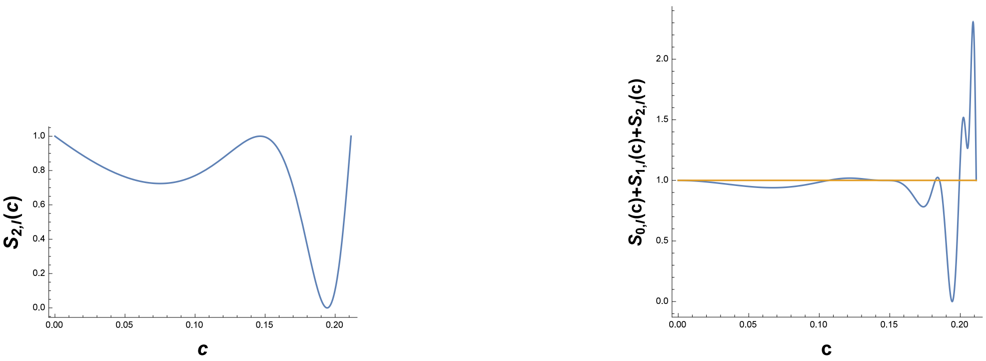
$$0 < c_n < \frac{3 - \sqrt{3}}{6} \tag{3.10}$$

We use Mathematica for solving the equations (3.2), (3.3) and (3.4), then we get the expressions for $s_{0,l}(n)$, $s_{1,l}(n)$ and $s_{2,l}(n)$.
Let $s_{i,l}(n) \equiv S_{i,l}(c_n)$ for $i = 0, 1, 2$. The graphs of $S_{i,l}(c)$ are shown in Figures 3.8a, 3.8b and 3.9a.
Note that the conditions (3.6), (3.8) and (3.9) give the condition (3.7)

$$0 < \sum_{i=0}^{2} s_{i,l}(n) < 1.$$

(a) (b)

Figure 3.8 : (a) and (b) shows the graph of $S_{0,l}(c)$, and $S_{1,l}(c)$ respectively.



(a) (b)

Figure 3.9 : (a) and (b) shows the graph of $S_{2,l}(c)$, and $(S_{0,l}+S_{1,l}+S_{2,l})(c)$ respectively.

The conditions (3.6) together with (3.8) to (3.10) define the feasible domain F_d^l is to be:

$$F_d^l = \left\{ c \in \left(0,\ \frac{3-\sqrt{3}}{6}\right)\ :\ S_{i,l}(c) > 0 \text{ for } i = 0,1,2, G_{0,l}(c) > 0, G_{1,l}(c) > 0 \right\}. \tag{3.11}$$

To compute the feasible domain F_d^l, we need to find subinterval(s) of $\left(0, \frac{3-\sqrt{3}}{6}\right)$ which satisfies the conditions of (3.11). By using Mathematica software, we employ the following command to obtain the feasible domain

$$\text{N[Reduce[}\left\{S_{0,l}(c) > 0, S_{1,l}(c) > 0, S_{2,l}(c) > 0, G_{0,l}(c) > 0, G_{1,l}(c) > 0, 0 < c < \frac{3-\sqrt{3}}{6}\right\}, c]].$$

This yields:

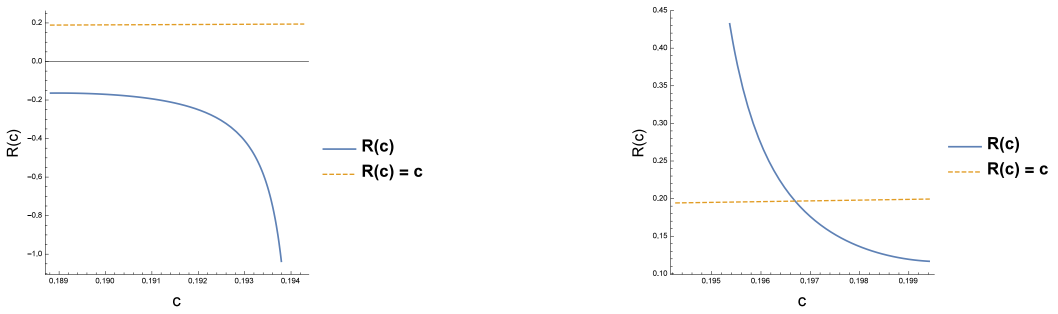
$$F_d^l = (0.188816...,\ 0.194271...) \cup (0.194271...,\ 0.199413...) \equiv F_{d_1}^l \cup F_{d_2}^l.$$

From the Eqn.(3.5), the graphs of $\mathscr{R}(c)$ are plotted in the sub-domains $F_{d_1}^l$ and $F_{d_2}^l$ of F_d^l which are shown in Figure 3.10. The map $\mathscr{R} : F_d^l \to \mathbb{R}$ is expanding in the neighborhood of fixed point c_l^* which is illustrated in Figure 3.10b. By Mathematica computations, we get an unstable fixed points $c_l^* = 0.196693...$ in F_d^l such that

$$\mathscr{R}(c_l^*) = c_l^*$$

corresponds to an infinitely renormalizable maps $f_{s_l^*}$. We observe that the map $f_{s_l{}^*}$ corresponding to c_l^* has the following property

$$\{c_l^*\} = \bigcap_{n\geq 1} I_{1,l}^n.$$



(a) $\mathscr{R}$ has no fixed point in $F^l_{d_1}$.

(b) $\mathscr{R}$ has only one fixed point in $F^l_{d_2}$.

Figure 3.10 : The graph of $\mathscr{R} : F^l_d \to \mathbb{R}$ and the diagonal $\mathscr{R}(c) = c$.

In other words, consider the scaling data $s_l{}^* : \mathbb{N} \to T_3$ with

$$\begin{aligned} s_l{}^*(n) &= (s^*_{0,l}(n), s^*_{1,l}(n), s^*_{2,l}(n)) \\ &= \left(\frac{b_{c^*_l}(0) - b^4_{c^*_l}(0)}{b_{c^*_l}(0)}, \frac{b^2_{c^*_l}(0) - b^5_{c^*_l}(0)}{b_{c^*_l}(0)}, \frac{b^3_{c^*_l}(0)}{b_{c^*_l}(0)} \right). \end{aligned}$$

Then $\sigma(s^*_l) = s^*_l$ and using Lemma 3.1.1 we have

$$R^l f_{s^*_l} = f_{s^*_l}.$$

□

3.1.2 Renormalization on $I_R = [\tilde{b}_c(1), 1]$

In subsection 3.1.1, the bimodal map $b_c(x)$ has two critical points $c \in I_L$ and $1 - c \in I_R$ and we define the piece-wise renormalization on I_L. In similar fashion, to define the renormalization on I_R with $c \in I_R$, from Equation 3.1, we consider

$$\tilde{b}_c(x) = 1 - \frac{4c^3 - 3c^2 + 6cx - 6c^2x - 3x^2 + 2x^3}{(2c-1)^3}$$

where $x \in [0,1]$ and $c \in [\frac{3}{4}, 1]$.

Note that $I_L{}^o \cap I_R{}^o = \phi$, for $c \in [\frac{3+\sqrt{3}}{6}, 1]$.

Let $f_{s_r} \in W_r$ be given by the proper scaling data $s_r : \mathbb{N} \to T_3$ and define

$$\tilde{I}^n_{1,r} = [\tilde{b}_c(1), \tilde{b}^2_c(y'_n)] = [f_{s_r}(1), f^2_{s_r}(y'_n)],$$

and

$$\hat{I}^n_{1,r} = [\tilde{b}_c(y'_n), 1] = [f_{s_r}(y'_n), 1].$$

Let

$$h_{s_r,n} : [\tilde{b}_c(1), 1] \to I^n_{1,r}$$

be defined by

$$h_{s_r,n} = F_{1,r}(1) \circ F_{1,r}(2) \circ F_{1,r}(3) \circ \ldots.. \circ F_{1,r}(n).$$

Furthermore, let

$$\tilde{h}_{s_r,n} : [\tilde{b}_c(1), 1] \to \tilde{I}^n_{1,r} \text{ and } \hat{h}_{s_r,n} : [\tilde{b}_c(1), 1] \to \hat{I}^n_{1,r}$$

be the affine orientation preserving homeomorphisms. Then define

$$R^r_n f_{s_r} : h^{-1}_{s_r,n}(D_{s_r} \cap I^n_{1,r}) \to [\tilde{b}_c(1), 1]$$

by

$$R^r_n f_{s_r}(x) = \begin{cases} R^{r-}_n f_{s_r}(x), & \text{if } x \in h^{-1}_{s_r,n}(\underset{m \geq n+1}{\cup} I^m_{0,r}) \\ R^{r+}_n f_{s_r}(x), & \text{if } x \in h^{-1}_{s_r,n}(\underset{m \geq n+1}{\cup} I^m_{2,r}) \end{cases}$$

where,

$$R^{r-}_n f_{s_r} : h^{-1}_{s_r,n}(\underset{m \geq n+1}{\cup} I^m_{0,r}) \to [\tilde{b}_c(1), 1]$$

and

$$R^{r+}_n f_{s_r} : h^{-1}_{s_r,n}(\underset{m \geq n+1}{\cup} I^n_{2,r}) \to [\tilde{b}_c(1), 1]$$

are defined by

$$R^{r-}_n f_{s_r}(x) = \tilde{h}^{-1}_{s_r,n} \circ f^2_{s_r} \circ h_{s_r,n}(x)$$
$$R^{r+}_n f_{s_r}(x) = \hat{h}^{-1}_{s_r,n} \circ f_{s_r} \circ h_{s_r,n}(x),$$

which are illustrated in Figure 3.11.

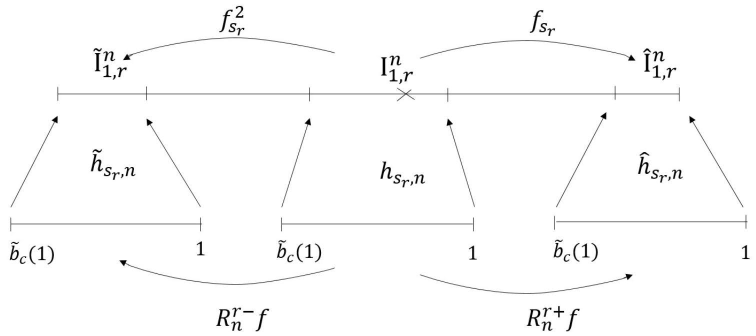


Figure 3.11 : Illustration of operators R^{r-}_n and R^{r+}_n

Let $\sigma : T_3^{\mathbb{N}} \to T_3^{\mathbb{N}}$ be the shift map which is defined as

$$\sigma(s_r(1)s_r(2)s_r(3)s_r(4)....) = (s_r(2)s_r(3)s_r(4)....),$$

where $s_r(i) \in T_3$ for all $i \in \mathbb{N}$.

Lemma 3.1.4. *Let $s_r : \mathbb{N} \to T_3$ be proper scaling data such that f_{s_r} is infinitely renormalizable. Then*

$$R^r_n f_{s_r} = f_{\sigma^n(s_r)}.$$

Let f_{s_r} be infinitely renormalization, then for $n \geq 0$, we have

$$f^{3^n}_{s_r} : D_{s_r} \cap I^n_{1,r} \to I^n_{1,r}$$

is well defined.
Define the renormalization $R^r : W_r \to W_r$ by

$$R^r f_{s_r} = h^{-1}_{s_r,1} \circ f^3_{s_r} \circ h_{s_r,1}.$$

The maps $f^{3^n-2}_{s_r} : \tilde{I}^n_{1,r} \to I^n_{1,r}$ and $f^{3^n-1}_{s_r} : \hat{I}^n_{1,r} \to I^n_{1,r}$ are the affine homeomorphisms whenever $f_{s_r} \in W_r$. Then we have:

Lemma 3.1.5. *We have* $(R^r)^n f_{s_r} : D_{\sigma^n(s_r)} \to [\tilde{b}_c(1), 1]$ *and* $(R^r)^n f_{s_r} = R^r_n f_{s_r}$.

From the above Lemma 3.1.4 and Lemma 3.1.5, consequently we get

Proposition 3.1.6. *There exists a map* $f_{s_r^*} \in W_r$*, where* s_r^* *is characterized by*

$$R^r f_{s_r^*} = f_{s_r^*}.$$

Proof. Consider $s_r : \mathbb{N} \to T_3$ be proper scaling data such that f_{s_r} is an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_r)}$. Then from Figure 3.12, we have the following scaling ratios

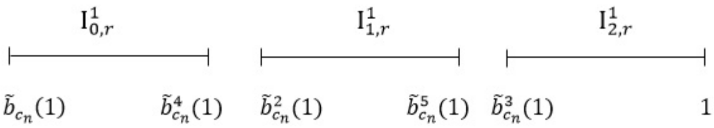


Figure 3.12 : Length of intervals

$$s_{0,r}(n) = \frac{\tilde{b}^4_{c_n}(1) - \tilde{b}_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{3.12}$$

$$s_{1,r}(n) = \frac{\tilde{b}^5_{c_n}(1) - \tilde{b}^2_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{3.13}$$

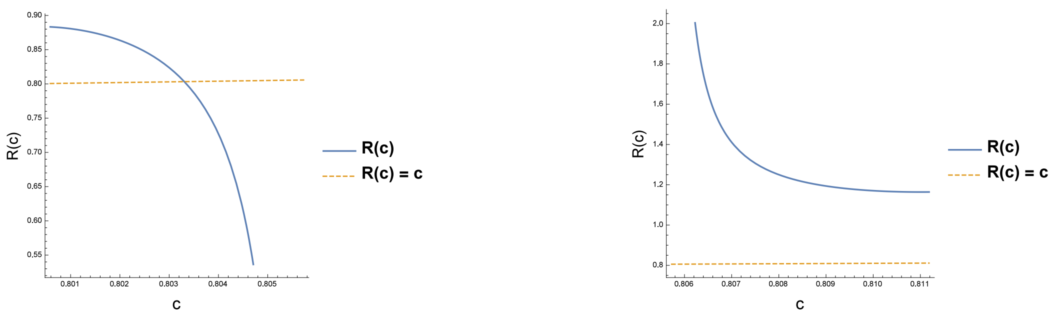
$$s_{2,r}(n) = \frac{1 - \tilde{b}^3_{c_n}(1)}{1 - \tilde{b}_{c_n}(1)} \tag{3.14}$$

$$c_{n+1} = 1 - \frac{c_n - \tilde{b}^2_{c_n}(1)}{s_{1,r}(n)} \equiv \mathscr{R}(c_n). \tag{3.15}$$

Use the same argument as given in subsection 3.1.1, one can compute feasible domain F^r_d. Finally, we get

$$F^r_d = (0.800587...,\ 0.805729...) \cup (0.805729...,\ 0.811184...) \equiv F^r_{d_1} \cup F^r_{d_2}.$$

From the Eqn.(3.15), the graphs of $\mathscr{R}(c)$ are plotted in the sub-domains $F^r_{d_1}$ and $F^r_{d_2}$ of F^r_d which are shown in Figure 3.13.



(a) $\mathscr{R}$ has only one fixed point in $F^r_{d_1}$.

(b) $\mathscr{R}$ has no fixed point in $F^r_{d_2}$.

Figure 3.13 : The graph of $\mathscr{R} : F^r_d \to \mathbb{R}$ and the diagonal $\mathscr{R}(c) = c$.

The map $\mathscr{R} : F_d^r \to \mathbb{R}$ is expanding in the neighborhood of fixed point c_r^* which is illustrated in Figure 3.13a. By Mathematica computations, we get an unstable fixed points $c_r^* = 0.803307...$ in F_d^r such that

$$\mathscr{R}(c_r^*) = c_r^*$$

corresponds to an infinitely renormalizable maps $f_{s_r^*}$. We observe that the map $f_{s_r{}^*}$ corresponding to c_r^* has the following property

$$\{c_r^*\} = \bigcap_{n \geq 1} I_{1,r}^n.$$

In other words, consider the scaling data $s_r{}^* : \mathbb{N} \to T_3$ with

$$\begin{aligned} s_r{}^*(n) &= (s_{0,r}^*(n), s_{1,r}^*(n), s_{2,r}^*(n)) \\ &= \left(\frac{\tilde{b}_{c_r^*}^4(1) - \tilde{b}_{c_r^*}(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{\tilde{b}_{c_r^*}^5(1) - \tilde{b}_{c_r^*}^2(1)}{1 - \tilde{b}_{c_r^*}(1)}, \frac{1 - \tilde{b}_{c_r^*}^3(1)}{1 - \tilde{b}_{c_r^*}(1)} \right). \end{aligned}$$

Then $\sigma(s_r^*) = s_r^*$ and using Lemma 3.1.4 we have

$$R^r f_{s_r^*} = f_{s_r^*}.$$

□

For a given pair of proper scaling data $s = (s_l, s_r)$, we defined a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0, 1]$$

as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

Then, the renormalization of f_s is defined as

$$R f_s(x) = \begin{cases} R^l f_{s_l}(x), & \text{if } x \in D_{s_l} \\ R^r f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

From proposition 3.1.3 and 3.1.6, we conclude that the period tripling infinitely renormalizable maps $f_{s_l^*}$ and $f_{s_r^*}$ are fixed points of R^l and R^r corresponding to the proper scaling data s_l^* and s_r^*, respectively. Then, for a given pair of scaling data $s^* = (s_l^*, s_r^*)$, we have

$$\begin{aligned} R f_{s^*}(x) &= \begin{cases} R^l f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ R^r f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= \begin{cases} f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= f_{s^*}(x) \end{aligned}$$

This will give us the following theorem,

Theorem 3.1.7. *There exists a map $f_{s^*} \in W$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$R f_{s^*} = f_{s^*}.$$

In particular, $W = \{f_{s^*}\}$.

Remark 3.1.1. *The constructed map f_{s^*} with a pair of proper scaling data $s^* = (s_l^*, s_r^*)$ holds the following conditions,*

(i) $s_{2,l}^* \leq (s_{1,l}^*)^2$

(ii) $s_{2,r}^* \leq (s_{1,r}^*)^2$

Note that for $i \in \{0,1,2\}$, the scaling ratios $s_{i,l}(n)$ are the expressions in the terms of c_n which are described in equations (3.2)-(3.4). Therefore, one can easily compute $s_{0,l}^*$, $s_{1,l}^*$ and $s_{2,l}^*$ by substituting $c_n = c_l^*$ in the respective expressions. Then,

$$s_{2,l}^* = s_{2,l}(n)\big|_{c_n=c_l^*} \leq \left(s_{1,l}(n)\big|_{c_n=c_l^*}\right)^2 = \left(s_{1,l}^*\right)^2.$$

Similarly,

$$s_{2,r}^* = s_{2,r}(n)\big|_{c_n=c_r^*} \leq \left(s_{1,r}(n)\big|_{c_n=c_r^*}\right)^2 = \left(s_{1,r}^*\right)^2.$$

Remark 3.1.2. *The invariant Cantor set of the map f_{s^*}, namely Λ_{s^*}, is next in complexity to the invariant doubling Cantor set, namely Λ_{σ^*}, of piece-wise affine period doubling infinitely renormalizable map f_{σ^*} [Chandramouli et al., 2009] in the following sense,*

(i) like the both Cantor sets Λ_{s^*} and Λ_{σ^*}, on each scale and everywhere the same scaling ratio s^* and σ^* are used respectively,

(ii) but unlike the doubling Cantor set Λ_{σ^*}, there are now a pair of three different ratios at each scale corresponding to s^*.

Furthermore, the geometry of the invariant Cantor set of f_{s^*} is different from the geometry of the invariant Cantor set of piece-wise affine period tripling renormalizable map because the Cantor set of f_{s^*} has $2-$copy of Cantor set of Kumar and Chandramouli [2021].

3.2 C^{1+Lip} EXTENSION OF f_{s^*}

In Section 3.1, we have constructed a piece-wise affine infinitely renormalizable map f_{s^*} corresponding to the pair of scaling data $s^* = (s_l^*,\ s_r^*)$. Let us define a pair of scaling functions

$$S_l : [0,\ b_{c_l^*}(0)]^2 \to [0,\ b_{c_l^*}(0)]^2$$

$$S_r : [\tilde{b}_{c_r^*}(1),\ 1]^2 \to [\tilde{b}_{c_r^*}(1),\ 1]^2$$

as

$$S_l \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{c_l^*}^2(0) - s_{1,l}^* \cdot x \\ s_{2,l}^* \cdot y \end{pmatrix}; \qquad S_r \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{b}_{c_r^*}^2(1) + s_{1,r}^* \cdot (1-x) \\ 1 - s_{2,r}^* \cdot (1-y) \end{pmatrix}.$$

Let G be the graph of g_{s^*} which is an extension of f_{s^*} where $f_{s^*} : D_{s_l^*} \cup D_{s_r^*} \to [0,\ 1]$. Let G_l^1 and G_l^2 are the graphs of $g_{s^*}|_{[y_1,\ z_0]}$ which is a C^{1+Lip} extension of f_{s^*} on $D_{s_l^*} \cap [y_1,\ z_0]$ and $g_{s^*}|_{[y_0,\ z_1]}$ which is a C^{1+Lip} extension of f_{s^*} on $D_{s_l^*} \cap [y_0,\ z_1]$ respectively. Also, G_r^1 and G_r^2 are the graphs of $g_{s^*}|_{[z_0',\ y_1']}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_r^*} \cap [z_0',\ y_1']$ and $g_{s^*}|_{[z_1',\ y_0']}$ which is an C^{1+Lip} extension of f_{s^*} on $D_{s_r^*} \cap [z_1',\ y_0']$ respectively which are shown in Figure 3.14. Also, note that G_r^1 and G_r^2 are the reflections of G_l^1 and G_l^2 across the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ respectively. Define

$$G_l = \cup_{n\geq 0} S_l^n (G_l^1 \cup G_l^2) \ \text{ and } \ G_r = \cup_{n\geq 0} S_r^n (G_r^1 \cup G_r^2).$$

Then, G_l is the graph of a unimodal map $g_{s_l^*}$ which extends $f_{s_l^*}$ and G_r is the graph of a unimodal map $g_{s_r^*}$ which extends $f_{s_r^*}$. Consequently, G is the graph of $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$. We claim that g_{s^*} is a C^{1+Lip} symmetric bimodal map. Let $B_l^0 = [0,\, b_{c_l^*}(0)] \times [0,\, b_{c_l^*}(0)]$ and $B_r^0 = [\tilde{b}_{c_r^*}(1),\, 1] \times [\tilde{b}_{c_r^*}(1),\, 1]$. For $n \in \mathbb{N}$, define

$$B_l^n = S_l^n(B_l^0) \qquad \text{and} \qquad B_r^n = S_r^n(B_r^0)$$

as

$$B_l^n = \begin{cases} [z_n,\, y_n] \times [0,\, \hat{y}_n], & \text{if } n \text{ is odd} \\ [y_n,\, z_n] \times [0,\, \hat{y}_n], & \text{if } n \text{ is even} \end{cases}$$

and

$$B_r^n = \begin{cases} [y'_n,\, z'_n] \times [\hat{y'}_n,\, 1], & \text{if } n \text{ is odd} \\ [z'_n,\, y'_n] \times [\hat{y'}_n,\, 1], & \text{if } n \text{ is even.} \end{cases}$$

Let p_l^n and p_r^n be the points on the graph of the bimodal map $b_{c_l^*}(x)$ and $b_{c_r^*}(x)$ respectively. For all $n \in \mathbb{N}$, p_l^n and p_r^n are defined as

$$p_l^n = \begin{cases} \begin{pmatrix} y_{\frac{n+1}{2}} \\ \hat{y}_{\frac{n+1}{2}} \end{pmatrix}, & \text{if } n \text{ is odd} \\ \begin{pmatrix} z_{\frac{n}{2}} \\ \hat{z}_{\frac{n}{2}} \end{pmatrix}, & \text{if } n \text{ is even} \end{cases}$$

$$p_r^n = \begin{cases} \begin{pmatrix} y'_{\frac{n+1}{2}} \\ \hat{y'}_{\frac{n+1}{2}} \end{pmatrix}, & \text{if } n \text{ is odd} \\ \begin{pmatrix} z'_{\frac{n}{2}} \\ \hat{z'}_{\frac{n}{2}} \end{pmatrix}, & \text{if } n \text{ is even} \end{cases}$$

where $\hat{y}_n = b_{c_l^*}(y_n)$, $\hat{z}_n = b_{c_l^*}(z_n)$, $\hat{y'}_n = \tilde{b}_{c_r^*}(y'_n)$ and $\hat{z'}_n = \tilde{b}_{c_r^*}(z'_n)$.

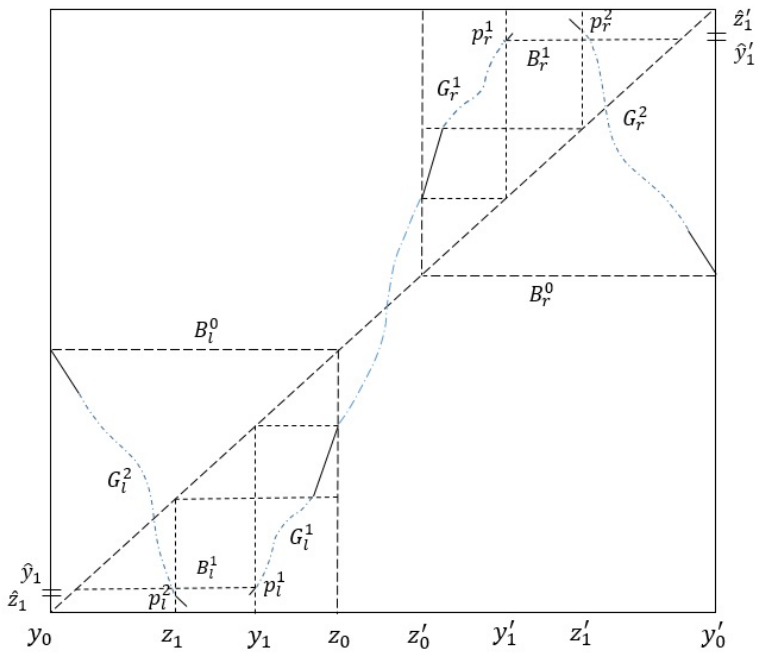


Figure 3.14 : Extension of f_{s^*}

Then the above construction will lead to following proposition,

Proposition 3.2.1. *G is the graph of g_{s^*} which is a C^1 extension of f_{s^*}.*

Proof. Since G_l^1 and G_l^2 are the graph of $f_{s_l^*}|_{[y_1,z_0]}$ and $f_{s_l^*}|_{[y_0,z_1]}$, respectively, and G_r^1 and G_r^2 are the graph of $f_{s_r^*}|_{[z'_0,y'_1]}$ and $f_{s_r^*}|_{[z'_1,y'_0]}$, respectively, we obtain $G_l^{2n+1} = S_l^n(G_l^1)$ and $G_l^{2n+2} = S_l^n(G_l^2)$ for each

$n \in \mathbb{N}$. Note that G_l^n is the graph of a C^1 function defined

$$\begin{aligned} &\text{on } [z_{\frac{n-1}{2}},\, y_{\frac{n+1}{2}}] && \text{if } n \in 4\mathbb{N}-1,\\ &\text{on } [z_{\frac{n}{2}},\, y_{\frac{n}{2}-1}] && \text{if } n \in 4\mathbb{N},\\ &\text{on } [y_{\frac{n+1}{2}},\, z_{\frac{n-1}{2}}] && \text{if } n \in 4\mathbb{N}+1,\\ \text{and } &\text{on } [y_{\frac{n}{2}-1},\, z_{\frac{n}{2}}] && \text{if } n \in 4\mathbb{N}+2. \end{aligned}$$

Also, we have $G_r^{2n+1} = S_r^n(G_r^1)$ and $G_r^{2n+2} = S_r^n(G_r^2)$ for each $n \in \mathbb{N}$. Note that G_r^n is the graph of a C^1 function defined

$$\begin{aligned} &\text{on } [y'_{\frac{n+1}{2}},\, z'_{\frac{n-1}{2}}] && \text{if } n \in 4\mathbb{N}-1,\\ &\text{on } [y'_{\frac{n}{2}-1},\, z'_{\frac{n}{2}}] && \text{if } n \in 4\mathbb{N},\\ &\text{on } [z'_{\frac{n-1}{2}},\, y'_{\frac{n+1}{2}}] && \text{if } n \in 4\mathbb{N}+1,\\ \text{and } &\text{on } [z'_{\frac{n}{2}},\, y'_{\frac{n}{2}-1}] && \text{if } n \in 4\mathbb{N}+2. \end{aligned}$$

To prove the proposition, we have to check continuous differentiability at the points p_l^n and p_r^n. Consider the neighborhoods $(y_1-\varepsilon, y_1+\varepsilon)$ around y_1 and $(z_1-\varepsilon, z_1+\varepsilon)$ around z_1, the slopes are given by an affine pieces of $f_{s_l^*}$ on the subintervals $(y_1-\varepsilon, y_1)$ and $(z_1, z_1+\varepsilon)$ and the slopes are given by the chosen C^1 extension on $(y_1, y_1+\varepsilon)$ and $(z_1-\varepsilon, z_1)$. This implies, G_l^1 and G_l^2 are C^1 at p_l^1 and p_l^2, respectively.
Let $\gamma_1 \subset G_l$ be the graph over the interval $(y_1-\varepsilon, y_1+\varepsilon)$ and $\gamma_2 \subset G_l$ be the graph over the interval $(z_1-\varepsilon, z_1+\varepsilon)$,
then the graph G_l locally around p_l^n is equal to $\begin{cases} S_l^{\frac{n-1}{2}}(\gamma_1) & \text{if } n \text{ is odd} \\ S_l^{\frac{n-2}{2}}(\gamma_2) & \text{if } n \text{ is even} \end{cases}$. This implies, for $n \in \mathbb{N}$, G_l^{2n-1} is C^1 at p_l^{2n-1} and G_l^{2n} is C^1 at p_l^{2n}.
Hence G_l is a graph of a C^1 function on $[0, b_{c_l^*}(0)] \setminus \{c_l^*\}$.
We note that the horizontal contraction of S_l is smaller than the vertical contraction. This implies that the slope of G_l^n tends to zero when n is large. Therefore, G_l is the graph of a C^1 function $g_{s_l^*}$ on $[0, b_{c_l^*}]$. In similar way, one can prove that G_r is the graph of a C^1 function $g_{s_r^*}$ on $[\tilde{b}_{c_r^*}, 1]$. Therefore, $G = G_l \oplus G_r$ is the graph of a C^1 bimodal map $g_{s^*} = g_{s_l^*} \oplus g_{s_r^*}$ which is a C^1 extension of f_{s^*}. □

Proposition 3.2.2. *Let g_{s^*} be the function whose graph is G then g_{s^*} is a C^{1+Lip} symmetric bimodal map.*

Proof. As the function g_{s^*} is a C^1 extension of f_{s^*}. We have to show that, for $i \in \{l,\ r\}$, G_i^n is the graph of a C^{1+Lip} function

$$g_{s_i^*}^n : Dom(G_i^n) \to [0,1]$$

with an uniform Lipschitz bound.
That is, for $n \geq 1$,

$$Lip((g_{s_i^*}^{n+1})') \leq Lip((g_{s_i^*}^n)')$$

let us assume that $g_{s_l^*}^n$ is C^{1+Lip} with Lipschitz constant λ_n for its derivatives. We show that $\lambda_{n+1} \leq \lambda_n$.
For given $\begin{pmatrix} u \\ v \end{pmatrix}$ on the graph of $g_{s_l^*}^n$, there is $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = S_l \begin{pmatrix} u \\ v \end{pmatrix}$ on the graph of $g_{s_l^*}^{n+1}$, this implies

$$g_{s_l^*}^{n+1}(\tilde{u}) = s_{2,l}^* \cdot g_{s_l^*}^n(u)$$

Since $u = \frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*}$, we have

$$g_{s_l^*}^{n+1}(\tilde{u}) = s_{2,l}^* \cdot g_{s_l^*}^n \left(\frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*} \right)$$

Differentiate both sides with respect to $\tilde{u}$, we get

$$\left(g_{s_l^*}^{n+1}\right)'(\tilde{u}) = -\frac{s_{2,l}^*}{s_{1,l}^*} \cdot \left(g_{s_l^*}^{n}\right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}}{s_{1,l}^*}\right).$$

Therefore,

$$\begin{aligned}\left|\left(g_{s_l^*}^{n+1}\right)'(\tilde{u}_1) - \left(g_{s_l^*}^{n+1}\right)'(\tilde{u}_2)\right| &= \left|\frac{s_{2,l}^*}{s_{1,l}^*}\right| \cdot \left|\left(g_{s_l^*}^{n}\right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}_1}{s_{1,l}^*}\right) - \left(g_{s_l^*}^{n}\right)' \left(\frac{b_{c_l^*}^2(0) - \tilde{u}_2}{s_{1,l}^*}\right)\right| \\ &\le \frac{s_{2,l}^*}{(s_{1,l}^*)^2} \cdot \lambda \left(g_{s_l^*}^{n}\right)' |\tilde{u}_1 - \tilde{u}_2|\end{aligned}$$

From remark 3.1.1, we have $(s_{1,l}^*)^2 \ge s_{2,l}^*$. Then,

$$\lambda(g_{s_l^*}^{n+1})' \le \lambda(g_{s_l^*}^{n})' \le \lambda(g_{s_l^*}^{1})'.$$

Similarly, one can show that

$$\lambda(g_{s_r^*}^{n+1})' \le \lambda(g_{s_r^*}^{n})' \le \lambda(g_{s_r^*}^{1})'.$$

Therefore, choose $\lambda = max\{\lambda(g_{s_l^*}^1)',\ \lambda(g_{s_r^*}^1)'\}$ is the uniform Lipschitz bound. This completes the proof. □

Note that for a given pair of proper scaling data $s^* = (s_l^*, s_r^*)$, the piece-wise affine map f_{s^*} is infinitely renormalizable and g_{s^*} is a C^{1+Lip} extension of f_{s^*}. This implies g_{s^*} is also renormalizable map. Further, we observe that Rg_{s^*} is an extension of Rf_{s^*}. Therefore Rg_{s^*} is renormalizable. Hence, g_{s^*} is infinitely renormalizable map which is not a C^2 map. Then we have the following theorem,

Theorem 3.2.3. *There exists an infinitely renormalizable C^{1+Lip} symmetric bimodal map g_{s^*} such that*

$$Rg_{s^*} = g_{s^*}.$$

3.3 TOPOLOGICAL ENTROPY OF RENORMALIZATION

In this section, we calculate the topological entropy of the renormalization operator defined on the space of C^{1+Lip} bimodal maps.
Let us consider three pairs of C^{1+Lip} maps $\phi_i : [0, z_1] \cup [y_1, b_{c_l^*}(0)] \to [0, b_{c_l^*}(0)]$ and $\psi_i : [\tilde{b}_{c_r^*}(1), y_1'] \cup [z_1', 1] \to [\tilde{b}_{c_r^*}(1), 1]$, for $i = 0, 1, 2$, which extend f_{s^*}. Because of symmetricity, $\psi_i(x) = 1 - \phi_i(1 - x)$. For a sequence $\alpha = \{\alpha_n\}_{n\ge1} \in \Sigma_3$,
where $\Sigma_3 = \{\{x_n\}_{n\ge1} : x_n \in \{0,\ 1,\ 2\}\}$ is called full 3-Shift.
Now define

$$G_l^n(\alpha) = S_l^n(graph\ \phi_{\alpha_n}) \quad \text{and} \quad G_r^n(\alpha) = S_r^n(graph\ \psi_{\alpha_n}),$$

we have

$$G_l(\alpha) = \bigcup_{n\ge1} G_l^n(\alpha) \quad \text{and} \quad G_r(\alpha) = \bigcup_{n\ge1} G_r^n(\alpha).$$

Therefore, we conclude that $G(\alpha) = G_l(\alpha) \oplus G_r(\alpha)$ is the graph of a C^{1+Lip} bimodal map b_α by using the same facts of Section 3.2.
The shift map $\sigma : \Sigma_3 \to \Sigma_3$ is defined as

$$\sigma(\alpha_1\alpha_2\alpha_3\ldots) = (\alpha_2\alpha_3\alpha_4\ldots).$$

Proposition 3.3.1. *The restricted maps $b_\alpha^3 : [y_1,\ z_1] \to [y_1,\ z_1]$ and $b_\alpha^3 : [y_1',\ z_1'] \to [y_1',\ z_1']$ are the unimodal maps for all $\alpha \in \Sigma_3$. In particular, b_α is a renormalizable map and $Rb_\alpha = b_{\sigma(\alpha)}$.*

Proof. We know that $b_\alpha : [y_1, z_1] \to I^1_{2,l}$ is a unimodal and onto, $b_\alpha : I^1_{2,l} \to I^1_{0,l}$ is onto and affine and also $b_\alpha : I^1_{0,l} \to [y_1, z_1]$ is onto and affine. Therefore b^3_α is a unimodal map on $[y_1, z_1]$. Analogously, b^3_α is a unimodal map on $[y'_1, z'_1]$. The above construction implies

$$Rb_\alpha = b_{\sigma(\alpha)}.$$

□

This gives us the following theorem.

Theorem 3.3.2. *The renormalization operator R acting on the space of C^{1+Lip} symmetric bimodal maps has unbounded topological entropy.*

Proof. From the above construction, we conclude that $\alpha \longmapsto b_\alpha \in C^{1+Lip}$ is injective. The domain of R contains two copies, namely Λ_1 and Λ_2, of the full 3-shift. As topological entropy h_{top} is an invariant of topological conjugacy. Hence $h_{top}(R|_{\Lambda_1 \cup \Lambda_2}) > \ln 3$. In fact, if we choose n different pairs of C^{1+Lip} maps, say, $\phi_0, \phi_1, \phi_2, \ldots \phi_{n-1}$ and $\psi_0, \psi_1, \psi_2, \ldots \psi_{n-1}$, which extends f_{s^*}, then it will be embedded two copies of the full $n-$shift in the domain of R. Hence, the topological entropy of R on C^{1+Lip} symmetric bimodal maps is unbounded. □

3.4 NON-RIGIDITY OF RENORMALIZATION

In this section, we use an ε perturbation on the construction of the scaling data as presented in Section 3.1, to obtain the following theorem

Theorem 3.4.1. *There exists a continuum of fixed points of the renormalization operator acting on C^{1+Lip} symmetric bimodal maps.*

Proof. Consider an ε variation on scaling data and we modify the construction which is described in section 3.1.

Let us define the neighborhoods N^l_ε and N^r_ε about the respective points $(b^3_c(0), b^4_c(0))$ and $(b^3_c(1), b^4_c(1))$ as

$$N^l_\varepsilon(b^3_c(0), b^4_c(0)) = \{(b^3_c(0), \varepsilon \cdot b^4_c(0)) \ : \ \varepsilon > 0 \ \text{ and } \varepsilon \text{ close to } 1\}$$
$$N^r_\varepsilon(b^3_c(1), b^4_c(1)) = \{(b^3_c(1), \varepsilon \cdot b^4_c(1)) \ : \ \varepsilon > 0 \ \text{ and } \varepsilon \text{ close to } 1\}$$

(i). The perturbed scaling data on I^l_0, then the scaling ratios are defined as

$$s_{2,l}(c,\varepsilon) = \frac{b^3_c(0)}{b_c(0)}$$
$$s_{0,l}(c,\varepsilon) = \frac{b_c(0) - \varepsilon b^4_c(0)}{b_c(0)}$$
$$s_{1,l}(c,\varepsilon) = \frac{b^2_c(0) - b_c(\varepsilon b^4_c(0))}{b_c(0)},$$

where $c \in (0, \frac{3-\sqrt{3}}{6})$. Also, we define

$$\mathscr{R}(c,\varepsilon) = \frac{b^2_c(0) - c}{s_{1,l}(c,\varepsilon)}.$$

From subsection 3.1.1, we know that the map $\mathscr{R}$ which is defined in Eqn. 3.5, has unique fixed point c^*. Consequently, for a given ε close to 1, $\mathscr{R}(c,\varepsilon)$ has only one unstable fixed point, namely c^*_ε. Therefore, we consider the perturbed scaling data $s^*_{l,\varepsilon} : \mathbb{N} \to T_3$ with

$$s^*_{l,\varepsilon} = \left(\frac{b_{c^*_\varepsilon}(0) - \varepsilon b^4_{c^*_\varepsilon}(0)}{b_{c^*_\varepsilon}(0)}, \frac{b^2_{c^*_\varepsilon}(0) - b_{c^*_\varepsilon}(\varepsilon b^4_{c^*_\varepsilon}(0))}{b_{c^*_\varepsilon}(0)}, \frac{b^3_{c^*_\varepsilon}(0)}{b_{c^*_\varepsilon}(0)} \right).$$

Then $\sigma(s^*_{l,\varepsilon}) = s^*_{l,\varepsilon}$ and using Lemma 3.1.1, we have

$$R^l f_{s^*_{l,\varepsilon}} = f_{s^*_{l,\varepsilon}}.$$

(ii). Considering the perturbed scaling data on I^r_0, one has the scaling data $s^*_{r,\varepsilon} : \mathbb{N} \to T_3$ with

$$s^*_{r,\varepsilon} = \left(\frac{\varepsilon b^4_{c^*_\varepsilon}(1) - b_{c^*_\varepsilon}(1)}{1 - b_{c^*_\varepsilon}(1)}, \frac{b_{c^*_\varepsilon}(\varepsilon b^4_{c^*_\varepsilon}(1)) - b^2_{c^*_\varepsilon}(1)}{1 - b_{c^*_\varepsilon}(1)}, \frac{1 - b^3_{c^*_\varepsilon}(1)}{1 - b_{c^*_\varepsilon}(1)} \right).$$

Then $\sigma(s^*_{r,\varepsilon}) = s^*_{r,\varepsilon}$ and using Lemma 3.1.4, we have

$$R^r f_{s^*_{r,\varepsilon}} = f_{s^*_{r,\varepsilon}}.$$

Moreover, $f_{s^*_{l,\varepsilon}}$ and $f_{s^*_{r,\varepsilon}}$ are the piece-wise affine maps which are infinitely renormalizable. For a given pair of proper scaling data $s^*_\varepsilon = (s^*_{l,\varepsilon},\ s^*_{r,\varepsilon})$, we have

$$Rf_{s^*_\varepsilon} = f_{s^*_\varepsilon}.$$

Now we use similar extension described in section 3.2, then we get $g_{s^*_\varepsilon}$ is the C^{1+Lip} extension of $f_{s^*_\varepsilon}$. This implies that $g_{s^*_\varepsilon}$ is a renormalizable map. As $Rg_{s^*_\varepsilon}$ is an extension of $Rf_{s^*_\varepsilon}$. Therefore $Rg_{s^*_\varepsilon}$ is renormalizable. Hence, for each ε close to 1, $g_{s^*_\varepsilon}$ is a fixed point of the renormalization. This proves the existence of a continuum of fixed points of the renormalization. □

Remark 3.4.1. *In particular, for two different perturbed scaling data $s_{\varepsilon^*_1}$ and $s_{\varepsilon^*_2}$, one can construct two infinitely renormalizable maps $g_{s_{\varepsilon^*_1}}$ and $g_{s_{\varepsilon^*_2}}$. Therefore, the respective Cantor attractors will have different scaling ratios. Consequently, it shows the non-rigidity for symmetric bimodal maps, whose smoothness is C^{1+Lip}.*

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