4
Pairwise period doubling renormalization of symmetric bimodal maps

In this chapter, we introduce a pairwise period doubling renormalization operator to study the dynamics of low smooth symmetric bimodal maps. Primarily, we construct a piece-wise affine infinitely renormalizable map, namely f_{s^*}, corresponding to a pair of proper scaling data $s^* = (s_l^*, s_r^*)$, where s_l^* and s_r^* are corresponding to the critical points c_l^* and c_r^* respectively. Further, we show that the nested sequences of periodic intervals corresponding to the scaling data s_l^* and s_r^* converge to the respective critical points c_l^* and c_r^*. Further, f_{s^*} is extended to a C^{1+Lip} symmetric bimodal map $\mathfrak{F}_{s^*}$. Also, we prove that the topological entropy of the pairwise period doubling renormalization operator acting on C^{1+Lip} symmetric bimodal maps is unbounded. Finally, we prove the continuum of fixed points of this renormalization operator by considering perturbed scaling data.

Before proceeding further, we give some basic definitions required.
Let $I = [a, b]$ be a closed interval.
A unimodal map $u : I \to I$ is called *period doubling renormalizable map* if there exists a proper subinterval J of I such that
(1) $J \cap u(J) = \emptyset$,
(2) J is u^2- invariant.
Then $u^2 : J \to J$ is called a renormalization of u.

A map $u : I \to I$ is *period doubling infinitely renormalizable* map if there exists an infinite sequence $\{J_n\}_{n=0}^{\infty}$ of nested intervals such that $u^2|_{J_n} : J_n \to J_n$ are renormalizations of u and the length of J_n tends to zero as $n \to \infty$.
Let $\mathscr{U}_\infty$ be the collection of period doubling infinitely renormalizable maps.

Definition 4.0.1. A bimodal map $b : I = [0, 1] \to I$, is a C^1 map with two critical points c_l and c_r, is said to be *pairwise period doubling renormalizable* if there exists a pair of disjoint intervals (I_l, I_r), with $I_l \ni c_l$ and $I_r \ni c_r$, such that
(i) $I_l \cap b(I_l) = \emptyset$, $I_r \cap b(I_r) = \emptyset$,
(ii) I_l and I_r are b^2-invariant,
(iii) The unimodal maps $\hat{b}_l : [0,\ b(0)] \to [0,\ b(0)]$ and $\hat{b}_r : [b(1), 1] \to [b(1), 1]$ are joined by $\oplus$ to generate a bimodal map $\hat{b}_l \oplus \hat{b}_r$, where $\oplus$ is defined in the definition 3.0.1 in chapter 3. The unimodal maps $\hat{b}_l$ and $\hat{b}_r$ are defined as

$$\hat{b}_l(x) = h_l^{-1} b^2 h_l(x)$$

and

$$\hat{b}_r(x) = h_r^{-1} b^2 h_r(x),$$

where $h_l : [0,\ b(0)] \to I_l$ and $h_r : [b(1), 1] \to I_r$ are the affine orientation reversing homeomorphisms.

The pairwise period doubling renormalization of a symmetric bimodal map is illustrated in Figure 4.1.

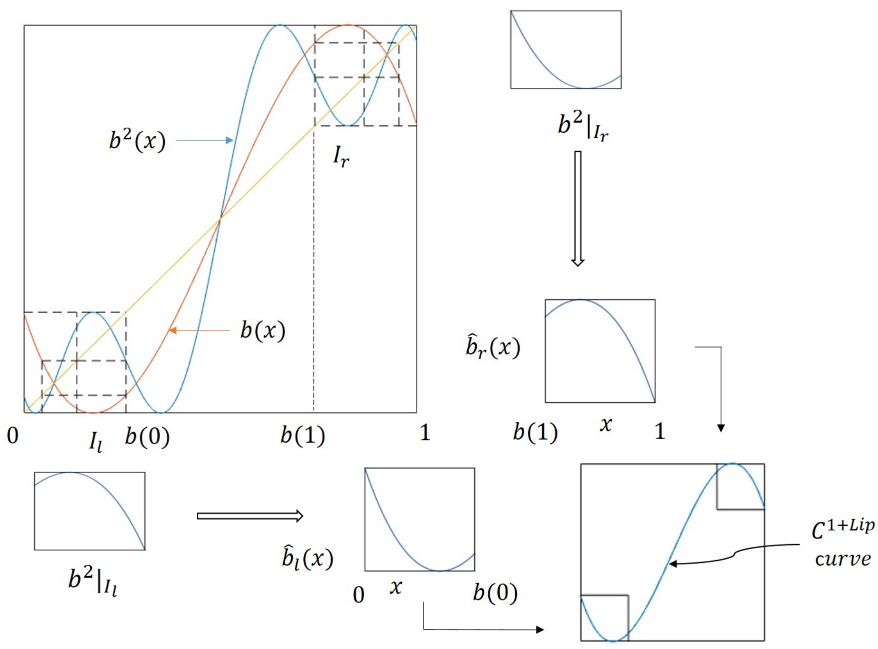


Figure 4.1 : Pairwise period doubling renormalization of a bimodal map

The construction of the pairwise period doubling renormalization operator has been explained in the next section.

4.1 CONSTRUCTION OF PIECE-WISE AFFINE RENORMALIZABLE MAPS

We recall a family of $-+-$ symmetric bimodal maps $\mathscr{B}_c : [0,1] \to [0,1]$ which is defined in chapter 3 as:

$$\mathscr{B}_c(x) = \begin{cases} 1 - \frac{1-6c+9c^2-4c^3+6cx-6c^2x-3x^2+2x^3}{(1-2c)^3}, & \text{if } c \in \left[0, \frac{1}{4}\right] \\ 1 - \frac{4c^3-3c^2+6cx-6c^2x-3x^2+2x^3}{(2c-1)^3}, & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases}$$

$$\equiv \begin{cases} b_c(x), & \text{if } c \in \left[0, \frac{1}{4}\right] \\ \tilde{b}_c(x), & \text{if } c \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{4.1}$$

We use the notion "$-+-$ bimodal map" for a bimodal map which is increasing on the interval between the critical points and decreasing elsewhere.

Define an open triangle

$$T_2 = \left\{(s_0, s_1) \in \mathbb{R}^2_+ \ : \ s_0 + s_1 < 1\right\}.$$

The element (s_0, s_1) of T_2 is called a scaling bi-factor. Two sets of affine maps (L_0, L_1) and (R_0, R_1) are induced by a pair of scaling bi-factors $(s_{0,l}, s_{1,l})$ and $(s_{0,r}, s_{1,r})$ respectively. For each $i = 0, 1$,

$$L_i : I_L = [0, b_c(0)] \to I_L$$

are defined as

$$L_0(t) = b_c(0) - s_{0,l} \cdot t,$$
$$L_1(t) = s_{1,l} \cdot t,$$

and

$$R_i : I_R = [\tilde{b}_c(1), 1] \to I_R$$

are defined as

$$R_0(t) = \tilde{b}_c(1) + s_{0,r} \cdot (1-t),$$
$$R_1(t) = 1 - s_{1,r} \cdot (1-t).$$

Note that ${I_L}^{\circ} \cap {I_R}^{\circ} = \phi$, for $c \in [0, \frac{3-\sqrt{3}}{6}]$.

The sequences of scaling bi-factors $s_l : \mathbb{N} \to T_2$ and $s_r : \mathbb{N} \to T_2$ are said to be a scaling data. Set scaling bi-factors $s_l(n) = (s_{0,l}(n), s_{1,l}(n)) \in T_2$ and $s_r(n) = (s_{0,r}(n), s_{1,r}(n)) \in T_2$, so that $s_l(n)$ and $s_r(n)$ induce the pair of affine maps $(L_0(n)(t), L_1(n)(t))$ and $(R_0(n)(t), R_1(n)(t))$ as described above. For $i = 0, 1$, let us define the intervals

$$I_{i,l}^n = L_0(1) \circ L_0(2) \circ L_0(3) \circ \ldots.. \circ L_0(n-1) \circ L_i(n)([0, b_c(0)]).$$

Also,

$$I_{i,r}^n = R_0(1) \circ R_0(2) \circ R_0(3) \circ \ldots.. \circ R_0(n-1) \circ R_i(n)([\tilde{b}_c(1), 1]).$$

Definition 4.1.1. For $j = l, r$, if $d(s_j(n), \partial T_2) \geq \varepsilon$, for some $\varepsilon > 0$, then the scaling data $s_j(n)$ is said to be proper scaling data.

A pair of proper scaling data $s_l : \mathbb{N} \to T_2$ and $s_r : \mathbb{N} \to T_2$, which is denoted by $s = (s_l, s_r)$, induce the sets $D_{s_l} = \bigcup\limits_{n\geq 1} I_{1,l}^n$ and $D_{s_r} = \bigcup\limits_{n\geq 1} I_{1,r}^n$, respectively. Consider a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0,1]$$

defined as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

where $f_{s_l}|_{I_{1,l}^n}$ and $f_{s_r}|_{I_{1,r}^n}$ be the affine extensions of $b_c|_{\partial I_{1,l}^n}$ and $b_c|_{\partial I_{1,r}^n}$, respectively. These affine extensions are shown in Figure 4.2. The end points of the intervals at each level are labeled by

$$u_{-1} = 0, \quad u_0 = b_c(0), \quad I_{0,l}^0 = I_L = [0, b_c(0)]$$
$$u'_0 = \tilde{b}_c(1), \quad u'_{-1} = 1, \quad I_{0,r}^0 = I_R = [\tilde{b}_c(1), 1]$$

and for $n \geq 1$,

$$u_n = \partial I_{0,l}^n \backslash \partial I_{0,l}^{n-1}, \qquad u'_n = \partial I_{0,r}^n \backslash \partial I_{0,r}^{n-1}$$
$$v_n = \partial I_{1,l}^n \backslash \partial I_{0,l}^{n-1}, \qquad v'_n = \partial I_{1,r}^n \backslash \partial I_{0,r}^{n-1}.$$

These points are illustrated in Figures 4.3a and 4.3b.

Definition 4.1.2. For a given pair of proper scaling data $s_l, s_r : \mathbb{N} \to T_2$, a map f_s is said to be *pairwise infinitely renormalizable* if for $n \geq 1$,

1(i) $[0, f_{s_l}(u_{n-1})]$ is the maximal interval containing 0 on which $f_{s_l}^{2^n-1}$ is defined affinely,

(ii) $[f_{s_r}(u'_{n-1}), 1]$ is the maximal interval containing 1 on which $f_{s_r}^{2^n-1}$ is defined affinely,

2(i) $f_{s_l}^{2^n-1}([0, f_{s_l}(u_{n-1})]) = I_{0,l}^n$,

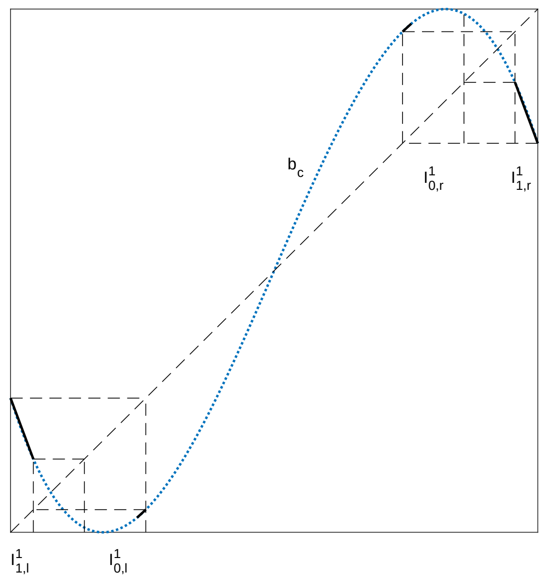
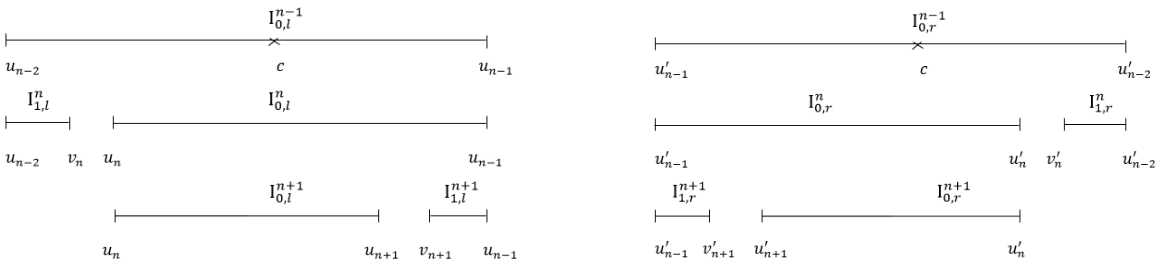


Figure 4.2 : Piece-wise affine extension f_s



(a) **(b)**

Figure 4.3 : Next generations intervals in I_l (Fig. 4.3a) and I_r (Fig. 4.3b).

(ii) $f_{s_r}^{2^n-1}([f_{s_r}(u'_{n-1}),1]) = I^n_{0,r}$.

Define $B_\infty = \{f_s \,:\, f_s \text{ is pairwise infinitely renormalizable}\}$.
Note that f_{s_l} and f_{s_r} are said to be infinitely renormalizable if f_{s_l} satisfies conditions $1(i)$ & $2(i)$ and f_{s_r} satisfies conditions $1(ii)$ & $2(ii)$, respectively.

4.1.1 Left sided renormalization operator R^l on left branch I_L

Let $f_{s_l} \in \mathscr{U}_\infty$ be given by the proper scaling data $s_l : \mathbb{N} \to T_2$ and define

$$\hat{I}^n_{0,l} = [0, b_c(u_{n-1})] = [0, f_{s_l}(u_{n-1})].$$

To construct the renormalization operator R^l, we have to define the homeomorphisms. Let

$$h_{s_l,n} : [0, b_c(0)] \to [0, b_c(0)]$$

be a homeomorphism which is defined by

$$h_{s_l,n} = L_0(1) \circ L_0(2) \circ L_0(3) \circ ..... \circ L_0(n).$$

Furthermore, let

$$\hat{h}_{s_l,n} : [0, b_c(0)] \to \hat{I}^n_{1,l}$$

be the affine orientation preserving homeomorphism. Then define

$$R^l_n f_s : h^{-1}_{s_l,n}(D_{s_l}) \to [0, b_c(0)]$$

by

$$R^l_n f_{s_l}(x) = \hat{h}^{-1}_{s_l,n} \circ f_{s_l} \circ h_{s_l,n}(x),$$

which is illustrated in Figure 4.4.

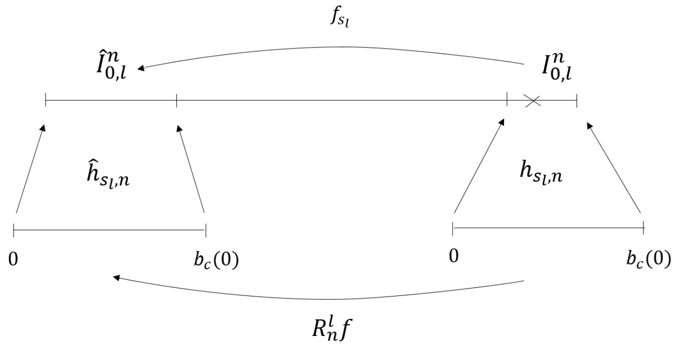


Figure 4.4 : Illustration of operator R^l_n

Let $\sigma : T_2^{\mathbb{N}} \to T_2^{\mathbb{N}}$ be the shift map defined as $\sigma(s^1_l s^2_l s^3_l s^4_l ....) = (s^2_l s^3_l s^4_l ....)$, where $s^i_l \in T_2$ for all $i \in \mathbb{N}$.

Lemma 4.1.1. *The piece-wise affine map f_{s_l} is infinitely renormalizable for a proper scaling data $s_l : \mathbb{N} \to T_2$. Then*

$$R^l_n f_{s_l} = f_{\sigma^n(s_l)}.$$

Let f_{s_l} be infinitely renormalizable map, then for $n \geq 0$, we have

$$f^{2^n}_{s_l} : D_{s_l} \cap I^n_{0,l} \to I^n_{0,l}$$

is well defined.
Define the left renormalization $R^l : \mathscr{U}_\infty \to \mathscr{U}_\infty$ by

$$R^l f_{s_l} = h^{-1}_{s_l,1} \circ f^2_{s_l} \circ h_{s_l,1}.$$

The map $f^{2^n-1}_{s_l} : \hat{I}^n_{0,l} \to I^n_{0,l}$ is the affine homeomorphism whenever $f_{s_l} \in \mathscr{U}_\infty$. Then this gives the following lemma,

Lemma 4.1.2. *One has $(R^l)^n f_{s_l} : D_{\sigma^n(s_l)} \to [0, b_c(0)]$ and $(R^l)^n f_{s_l} = R^l_n f_{s_l}$.*

The Lemma 4.1.1 and Lemma 4.1.2 give the following result.

Proposition 4.1.3. *One has $f_{s^*_l} \in \mathscr{U}_\infty$, where s^*_l is characterized by*

$$R^l f_{s^*_l} = f_{s^*_l}.$$

Proof. Let us consider a proper scaling data $s_l : \mathbb{N} \to T_2$ such that f_{s_l} be an infinitely renormalizable. Let c_n be the critical point of $f_{\sigma^n(s_l)}$. Then

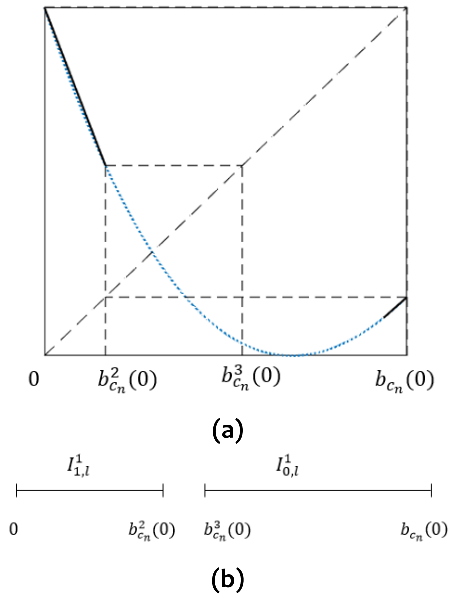


Figure 4.5 : Length of intervals.

The following scaling ratios, which are illustrated in Figure 4.5, can be written as

$$s_{0,l}(n) = \frac{b_{c_n}(0) - b^3_{c_n}(0)}{b_{c_n}(0)} \tag{4.2}$$

$$s_{1,l}(n) = \frac{b^2_{c_n}(0)}{b_{c_n}(0)} \tag{4.3}$$

$$c_{n+1} = \frac{b_{c_n}(0) - c_n}{s_{0,l}(n)} \equiv \mathscr{R}(c_n). \tag{4.4}$$

Since $(s_{0,l}(n), s_{1,l}(n)) \in T_2$, then

$$s_{0,l}(n),\ s_{1,l}(n) > 0 \tag{4.5}$$

$$s_{0,l}(n) + s_{1,l}(n) < 1 \tag{4.6}$$

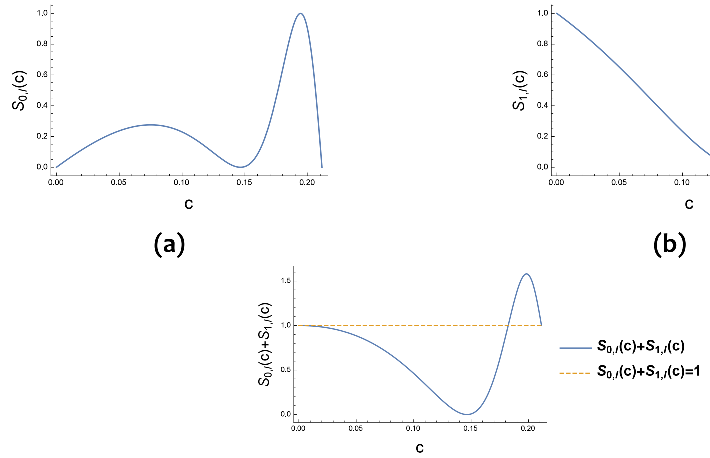
$$0 < c_n < \frac{3-\sqrt{3}}{6} \tag{4.7}$$

We solve Eqns (4.2) and (4.3) by using Mathematica, then we get the expressions for $s_{0,l}(n)$ and $s_{1,l}(n)$. Let $s_{i,l}(n) \equiv S_{i,l}(c_n)$ for $i = 0, 1$. The graphs of $S_{i,l}(c_n)$ are illustrated in Figure 4.6. From the conditions (4.5) to (4.7), we define the feasible domain F^l_d to be:

$$F^l_d = \left\{ c \in \left(0,\ \frac{3-\sqrt{3}}{6}\right)\ :\ S_{i,l}(c) > 0 \text{ for } i = 0, 1, S_{0,l}(n) + S_{1,l}(n) < 1 \right\}. \tag{4.8}$$

The feasible domain is same as the subinterval(s) in which the conditions of (4.8) are satisfied. By using Mathematica with the following command, we obtain the feasible domain

$$\text{N[Reduce[}\left\{ S_{0,l}(c) > 0, S_{1,l}(c) > 0, S_{0,l}(n) + S_{1,l}(n) < 1, 0 < c < \frac{3-\sqrt{3}}{6} \right\}, c]].$$



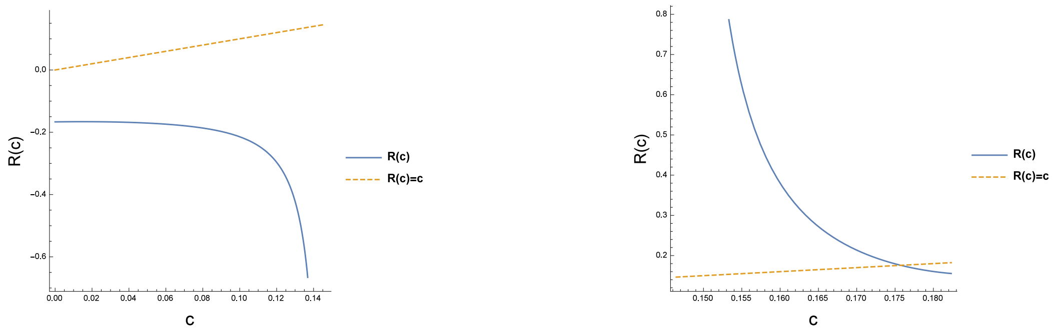
(a) (b) (c)

Figure 4.6 : (a), (b) and (c) show the graphs of $S_{0,l}(c)$, $S_{1,l}(c)$, and $(S_{0,l}+S_{1,l})(c)$.

This implies:

$$F_d^l = (0,\ 0.146447...) \cup (0.146447...,\ 0.182324...) \equiv F_{d_1}^l \cup F_{d_2}^l.$$

From the Eqn.(4.5), the graphs of $\mathscr{R}(c)$ are plotted in the sub-domains $F_{d_1}^l$ and $F_{d_2}^l$ of F_d^l which are shown in Figures 4.7a and 4.7b, respectively. Note that the map $\mathscr{R} : F_d^l \to \mathbb{R}$ is expanding in



(a) $\mathscr{R}$ has no fixed point in $F_{d_1}^l$. **(b)** $\mathscr{R}$ has only one fixed point in $F_{d_2}^l$.

Figure 4.7 : The graph of $\mathscr{R} : F_d^l \to \mathbb{R}$ and the diagonal $\mathscr{R}(c) = c$.

the neighborhood of fixed point c_l^* (see Figure 4.7b). By Mathematica computations, we have an unstable fixed point $c_l^* = 0.175749...$ in F_d^l such that

$$\mathscr{R}(c_l^*) = c_l^*$$

corresponds to an infinitely renormalizable maps $f_{s_l^*}$.
In other words, consider the scaling data $s_l{}^* : \mathbb{N} \to T_2$ with

$$\begin{aligned} s_l{}^*(n) &= (s_{0,l}^*(n), s_{1,l}^*(n)) \\ &= \left(\frac{b_{c_l^*}(0) - b_{c_l^*}^3(0)}{b_{c_l^*}(0)},\ \frac{b_{c_l^*}^2(0)}{b_{c_l^*}(0)} \right). \end{aligned}$$

Then $\sigma(s_l^*) = s_l^*$ and using Lemma 4.1.1 we have

$$R^l f_{s_l^*} = f_{s_l^*}.$$

□

4.1.2 Right sided renormalization operator R^r on right branch I_R

In subsection 4.1.1, the bimodal map $b_c(x)$ has two critical points $c \in I_L$ and $1-c \in I_R$ and we define the piece-wise renormalization on I_L. In similar fashion, to define the renormalization operator R^r on I_R with $c \in I_R$, from Eqn (4.1), we consider

$$\tilde{b}_c(x) = 1 - \frac{4c^3 - 3c^2 + 6cx - 6c^2x - 3x^2 + 2x^3}{(2c-1)^3}$$

where $x \in [0,1]$ and $c \in [\frac{3}{4}, 1]$.

Note that $I_L{}^o \cap I_R{}^o = \phi$, for $c \in [\frac{3+\sqrt{3}}{6}, 1]$.

Let $f_{s_r} \in \mathscr{U}_\infty$ be given by the proper scaling data $s_r : \mathbb{N} \to T_2$ and define

$$\hat{I}^n_{0,r} = [\tilde{b}_c(u'_{n-1}), 1] = [f_{s_r}(u'_{n-1}), 1].$$

Let us define homeomorphism

$$h_{s_r,n} : [\tilde{b}_c(1), 1] \to [\tilde{b}_c(1), 1]$$

as

$$h_{s_r,n} = R_0(1) \circ R_0(2) \circ R_0(3) \circ \ldots\ldots \circ R_0(n).$$

Further, let

$$\hat{h}_{s_r,n} : [\tilde{b}_c(1), 1] \to \hat{I}^n_{0,r}$$

be the affine orientation preserving homeomorphism. Then define

$$R^r_n f_s : h^{-1}_{s_r,n}(D_{s_r}) \to [\tilde{b}_c(1), 1]$$

by

$$R^r_n f_{s_r}(x) = \hat{h}^{-1}_{s_r,n} \circ f_{s_r} \circ h_{s_r,n}(x),$$

which is illustrated in Figure 4.8.

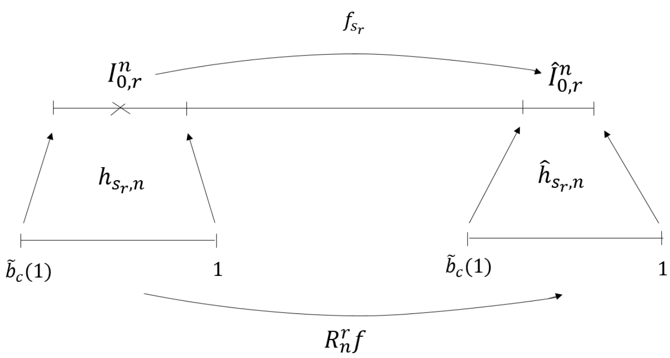


Figure 4.8 : Illustration of operator R^r_n

Let $\sigma : T_2^{\mathbb{N}} \to T_2^{\mathbb{N}}$ be the shift map defined as $\sigma(s_r^1 s_r^2 s_r^3 s_r^4 \ldots.) = (s_r^2 s_r^3 s_r^4 \ldots.)$, where $s_r^i \in T_2$ for all $i \in \mathbb{N}$.

Lemma 4.1.4. *The piece-wise affine map f_{s_r} is infinitely renormalizable for a proper scaling data $s_r : \mathbb{N} \to T_2$. Then*

$$R_n^r f_{s_r} = f_{\sigma^n(s_r)}.$$

Let f_{s_r} be infinitely renormalizable map, then for $n \geq 0$, we have

$$f_{s_r}^{2^n} : D_{s_r} \cap I_{0,r}^n \to I_{0,r}^n$$

is well defined.
Define the renormalization $R^r : \mathscr{U}_\infty \to \mathscr{U}_\infty$ by

$$R^r f_{s_r} = h_{s_r,1}^{-1} \circ f_{s_r}^2 \circ h_{s_r,1}.$$

The map $f_{s_r}^{2^n-1} : \tilde{I}_{0,r}^n \to I_{0,r}^n$ is the affine homeomorphism whenever $f_{s_r} \in \mathscr{U}_\infty$. Then

Lemma 4.1.5. *One has $(R^r)^n f_{s_r} : D_{\sigma^n(s_r)} \to [\tilde{b}_c(1), 1]$ and $(R^r)^n f_{s_r} = R_n^r f_{s_r}$.*

The Lemma 4.1.4 and Lemma 4.1.5 give the following result.

Proposition 4.1.6. *One has a map $f_{s_r^*} \in \mathscr{U}_\infty$, where s_r^* is characterized by*

$$R^r f_{s_r^*} = f_{s_r^*}.$$

Proof. The proof of the Proposition 4.1.6 is similar to the proof of the Proposition 4.1.3. In fact, one has an unstable fixed points $c_r^* = 0.824251...$ in the feasible domain corresponds to an infinitely renormalizable maps $f_{s_r^*}$.
In other words, consider the scaling data $s_r^* : \mathbb{N} \to T_2$ with

$$\begin{aligned} s_r^*(n) &= (s_{0,r}^*(n), s_{1,r}^*(n)) \\ &= \left(\frac{b_{c_r^*}^3(1) - b_{c_r^*}(1)}{1 - b_{c_l^*}(1)}, \frac{1 - b_{c_r^*}^2(1)}{1 - b_{c_r^*}(1)} \right). \end{aligned}$$

Then $\sigma(s_r^*) = s_r^*$ and using Lemma 4.1.4 we have

$$R^r f_{s_r^*} = f_{s_r^*}.$$

□

Remark 4.1.1. *By Matlab computations, we observe that the nested sequence of periodic intervals $I_{0,l}^n$ converges to c_l^* corresponding to $f_{s_l^*}$ and $I_{0,r}^n$ converges to c_r^* corresponding to $f_{s_r^*}$, i.e.,*

$$\{c_l^*\} = \bigcap_{n \geq 1} I_{0,l}^n, \qquad \{c_r^*\} = \bigcap_{n \geq 1} I_{0,r}^n.$$

Now we are in a position to introduce renormalization operator, which is a pair of period doubling renormalizations (R^l, R^r) defined in subsections 4.1.1 and 4.1.2.
For a given pair of proper scaling data $s = (s_l, s_r)$, we defined a map

$$f_s : D_{s_l} \cup D_{s_r} \to [0, 1]$$

as

$$f_s(x) = \begin{cases} f_{s_l}(x), & \text{if } x \in D_{s_l} \\ f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

Then, the pairwise period doubling renormalization of f_s is defined as

$$Rf_s(x) = \begin{cases} R^l f_{s_l}(x), & \text{if } x \in D_{s_l} \\ R^r f_{s_r}(x), & \text{if } x \in D_{s_r} \end{cases}$$

From proposition 4.1.3 and 4.1.6, for a given pair of proper scaling data s_l^* and s_r^*, one can easily conclude that $f_{s_l^*}$ and $f_{s_r^*}$ are period doubling infinitely renormalizable maps. Then, this implies

$$\begin{aligned} Rf_{s^*}(x) &= \begin{cases} R^l f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ R^r f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= \begin{cases} f_{s_l^*}(x), & \text{if } x \in D_{s_l^*} \\ f_{s_r^*}(x), & \text{if } x \in D_{s_r^*} \end{cases} \\ &= f_{s^*}(x) \end{aligned}$$

The above construction will imply the following theorem,

Theorem 4.1.7. *One has $B_\infty = \{f_{s^*}\}$, where $s^* = (s_l^*, s_r^*)$ is characterized by*

$$Rf_{s^*} = f_{s^*}.$$

Remark 4.1.2. *The scaling data holds the following inequality conditions,*

(i) $(s_{0,l}^*)^2 > s_{1,l}^*$

(ii) $(s_{0,r}^*)^2 > s_{1,r}^*$

Note that the above conditions are in contrast to the case of period doubling renormalization acting on unimodal maps with low smoothness where the equality occurs [Chandramouli et al., 2009].

4.2 EXTENSION OF f_{s^*} TO A C^{1+Lip} SYMMETRIC BIMODAL MAP

In this section, our goal is to extend the piece-wise affine infinitely renormalizable map f_{s^*} to a C^{1+Lip} symmetric bimodal map. Define a pair of proper scaling maps

$$\xi_l : [0,\ b_{c_l^*}(0)]^2 \to [0,\ b_{c_l^*}(0)]^2$$

$$\xi_r : [\tilde{b}_{c_r^*}(1),\ 1]^2 \to [\tilde{b}_{c_r^*}(1),\ 1]^2$$

as

$$\xi_l \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{c_l^*}(0) - s_{0,l}^* \cdot x \\ s_{1,l}^* \cdot y \end{pmatrix};$$
$$\xi_r \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{b}_{c_r^*}(1) + s_{0,r}^* \cdot (1-x) \\ 1 - s_{1,r}^* \cdot (1-y) \end{pmatrix}.$$

Let $\mathfrak{F}_{s^*}$ be an extension of f_{s^*} and Γ be the graph of $\mathfrak{F}_{s^*}$, where $f_{s^*} : D_{s_l^*} \cup D_{s_r^*} \to [0,\ 1]$. Let $\mathfrak{F}_{s^*}|_{[u_{-1},\, u_1]}$ and $\mathfrak{F}_{s^*}|_{[u_1',\, u_{-1}']}$ are the C^{1+Lip} extension of $f_{s^*}|_{[u_{-1},\, u_1]}$ and $f_{s^*}|_{[u_1',\, u_{-1}']}$ respectively. Suppose Γ_l^1 and Γ_r^1 be the graphs of $\mathfrak{F}_{s^*}|_{[u_{-1},\, u_1]}$ and $\mathfrak{F}_{s^*}|_{[u_1',\, u_{-1}']}$, respectively, which are shown in Figure 4.9. Also for symmetricity, let Γ_r^1 be the reflection of Γ_l^1 across the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ respectively. Define

$$\Gamma_l = \cup_{n\geq 1} \xi_l^n(\Gamma_l^1) \text{ and } \Gamma_r = \cup_{n\geq 1} \xi_r^n(\Gamma_r^1).$$

Then, the unimodal map $\mathfrak{F}_{s_l^*}$ is the extension of $f_{s_l^*}$ with the graph Γ_l and the unimodal map $\mathfrak{F}_{s_r^*}$ is the extension of $f_{s_r^*}$ with the graph Γ_r. Therefore, $\Gamma = graph(\mathfrak{F}_{s^*})$, where $\mathfrak{F}_{s^*} = \mathfrak{F}_{s_l^*} \oplus \mathfrak{F}_{s_r^*}$. We claim that $\mathfrak{F}_{s^*}$ is a C^{1+Lip} symmetric bimodal map.

Let $\mathscr{B}_l^0 = [0,\, b_{c_l^*}(0)] \times [0,\, b_{c_l^*}(0)]$ and $\mathscr{B}_r^0 = [\tilde{b}_{c_r^*}(1),\, 1] \times [\tilde{b}_{c_r^*}(1),\, 1]$.
For $n \in \mathbb{N}$, define

$$\mathscr{B}_l^n = \xi_l^n(\mathscr{B}_l^0) \quad \text{and} \quad \mathscr{B}_r^n = \xi_r^n(\mathscr{B}_r^0)$$

as

$$\mathscr{B}_l^n = \begin{cases} [u_n,\, u_{n-1}] \times [0,\, \tilde{u}_{n-1}], & \text{if } n \text{ is odd} \\ [u_{n-1},\, u_n] \times [0,\, \tilde{u}_{n-1}], & \text{if } n \text{ is even} \end{cases}$$

and

$$\mathscr{B}_r^n = \begin{cases} [u'_{n-1},\, u'_n] \times [\tilde{u'}_{n-1},\, 1], & \text{if } n \text{ is odd} \\ [u'_n,\, u'_{n-1}] \times [\tilde{u'}_{n-1},\, 1], & \text{if } n \text{ is even.} \end{cases}$$

Let $p_l^n = (u_{n-1}, \tilde{u}_{n-1}) = \xi_l(0,\, b_{c_l^*}(0))$ and $p_r^n = \left(u'_{n-1}, \tilde{u'}_{n-1}\right) = \xi_r(\tilde{b}_{c_r^*}(1),\, 1)$,
where $\tilde{u}_n = b_{c_l^*}(u_n)$, and $\tilde{u'}_n = \tilde{b}_{c_r^*}(u'_n)$.

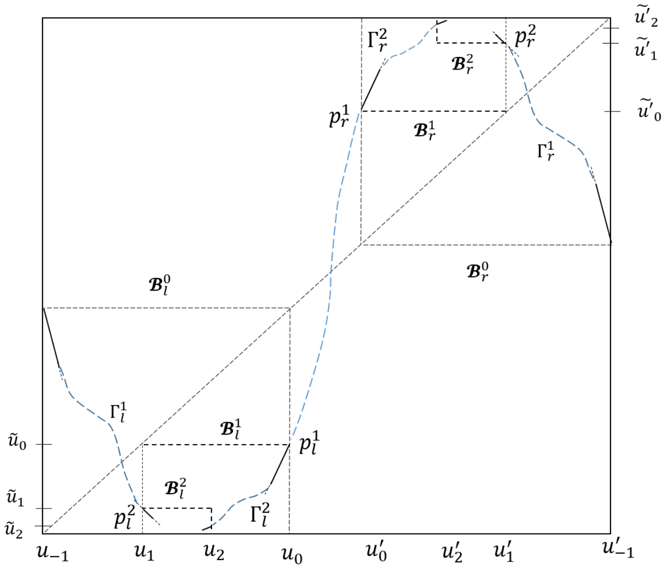


Figure 4.9 : Extension of f_{s^*}

Then the above construction will lead to the following proposition,

Proposition 4.2.1. *Γ is the graph of C^1 extension $\mathscr{F}_{s^*}$ of f_{s^*}.*

Proof. Let $\Gamma_l^m = \xi_l^{m-1}(\Gamma_l^1)$ for each $m \in \mathbb{N}$. Therefore, Γ_l^m is the graph of a C^1 function defined

$$\text{on } [u_{m-2},\, u_m] \text{ if } m \in 2\mathbb{N}-1,$$
$$\text{on } [u_m,\, u_{m-2}] \text{ if } m \in 2\mathbb{N}.$$

Also, we have $\Gamma_r^m = \xi_r^{m-1}(\Gamma_r^1)$ for each $m \in \mathbb{N}$. Note that Γ_r^m is the graph of a C^1 function defined

$$\text{on } [u'_m,\, u'_{m-2}] \text{ if } m \in 2\mathbb{N}-1,$$
$$\text{on } [u'_{m-2},\, u'_m] \text{ if } m \in 2\mathbb{N}.$$

We prove the proposition by checking the continuous differentiability of $\mathfrak{F}_{s^*}$ at the points p_l^m and p_r^m. Choose a small neighborhood $(u_1 - \varepsilon, u_1 + \varepsilon)$ of u_1 containing p_l^2. Clearly, on the subinterval $(u_1, u_1 + \varepsilon)$ the slope is given by an affine pieces of $f_{s_l^*}$ and on the subinterval $(u_1 - \varepsilon, u_1)$ the slope is given by the chosen C^1 extension on $(u_1 - \varepsilon, u_1)$. This implies, Γ_l^1 is C^1 at p_l^2 .
Let $\gamma \subset \Gamma_l$ be the graph over the interval $(u_1 - \varepsilon, u_1 + \varepsilon)$, then, for $m > 1$, the graph Γ_l is equal to $\xi_l^{m-2}(\gamma)$ locally around p_l^m. This implies, Γ_l^m is C^1 at p_l^{m+1} for $m \in \mathbb{N}$. Hence Γ_l is a graph of a C^1 function $\mathfrak{F}_{s_l^*}$ on $[0, b_{c_l^*}(0)] \setminus \{c_l^*\}$.

We note that the horizontal contraction of ξ_l is weaker than the vertical contraction of ξ_l. Therefore, for large n, the slope of Γ_l^n tends to zero. Hence, Γ_l is the graph of a C^1 function $\mathfrak{F}_{s_l^*}$ on $[0, b_{c_l^*}]$. In similar way, Γ_r is the graph of a C^1 function $\mathfrak{F}_{s_r^*}$ on $[\tilde{b}_{c_r^*}, 1]$. Therefore, $\Gamma = \Gamma_l \oplus \Gamma_r$ is the graph of a C^1 symmetric bimodal map $\mathfrak{F}_{s^*} = \mathfrak{F}_{s_l^*} \oplus \mathfrak{F}_{s_r^*}$ which is a C^1 extension of f_{s^*}. □

Proposition 4.2.2. *One has $\mathfrak{F}_{s^*}$ is a C^{1+Lip} symmetric bimodal map.*

Proof. We need to prove, for $i \in \{l,\ r\}$, Γ_i^n is the graph of a C^{1+Lip} function

$$\mathfrak{F}_{s_i^*}^n : Dom(\Gamma_i^n) \to [0,1]$$

with an uniform Lipschitz bound.

That is, for $n \geq 1$,

$$Lip((\mathfrak{F}_{s_i^*}^{n+1})') \leq Lip((\mathfrak{F}_{s_i^*}^{n})')$$

Assume that $\mathfrak{F}_{s_l^*}^n$ is C^{1+Lip} with Lipschitz constant λ_n for its derivatives. We show that $\lambda_{n+1} \leq \lambda_n$.

For given (x,y) on the graph of $\mathfrak{F}_{s_l^*}^n$, i.e., $\mathfrak{F}_{s_l^*}^n(x) = y$.

Then, there is $(\bar{x}, \bar{y}) = \xi_l(x,y)$ on the graph of $\mathfrak{F}_{s_l^*}^{n+1}$, this implies

$$\mathfrak{F}_{s_l^*}^{n+1}(\bar{x}) = \bar{y} = s_{1,l}^* \cdot y = s_{1,l}^* \cdot \mathfrak{F}_{s_l^*}^n(x)$$

Since $\bar{x} = \xi_l(x)$, this implies $x = \frac{b_{c_l^*}(0) - \bar{x}}{s_{0,l}^*}$, we have

$$\mathfrak{F}_{s_l^*}^{n+1}(\bar{x}) = s_{1,l}^* \cdot \mathfrak{F}_{s_l^*}^n \left(\frac{b_{c_l^*}(0) - \bar{x}}{s_{0,l}^*} \right)$$

Differentiate both sides with respect to $\bar{x}$, we have

$$\left(\mathfrak{F}_{s_l^*}^{n+1}\right)'(\bar{x}) = -\frac{s_{1,l}^*}{s_{0,l}^*} \cdot \left(\mathfrak{F}_{s_l^*}^n\right)' \left(\frac{b_{c_l^*}(0) - \bar{x}}{s_{0,l}^*} \right)$$

Therefore,

$$\begin{aligned} \left| \left(\mathfrak{F}_{s_l^*}^{n+1}\right)'(\bar{x}_1) - \left(\mathfrak{F}_{s_l^*}^{n+1}\right)'(\bar{x}_2) \right| &= \left| \frac{s_{1,l}^*}{s_{0,l}^*} \right| \cdot \left| \left(\mathfrak{F}_{s_l^*}^n\right)' \left(\frac{b_{c_l^*}(0) - \bar{x}_1}{s_{0,l}^*} \right) - \left(\mathfrak{F}_{s_l^*}^n\right)' \left(\frac{b_{c_l^*}(0) - \bar{x}_2}{s_{0,l}^*} \right) \right| \\ &\leq \frac{s_{1,l}^*}{(s_{0,l}^*)^2} \cdot \lambda \left(\mathfrak{F}_{s_l^*}^n\right)' |\bar{x}_1 - \bar{x}_2| \end{aligned}$$

From remark 4.1.2, we have $(s_{0,l}^*)^2 > s_{1,l}^*$. Then,

$$\lambda(\mathfrak{F}_{s_l^*}^{n+1})' < \lambda(\mathfrak{F}_{s_l^*}^{n})' < \lambda(\mathfrak{F}_{s_l^*}^{1})'.$$

Analogously, one can prove that

$$\lambda(\mathfrak{F}_{s_r^*}^{n+1})' < \lambda(\mathfrak{F}_{s_r^*}^{n})' < \lambda(\mathfrak{F}_{s_r^*}^{1})'.$$

Therefore, choose $\lambda = max\{\lambda(\mathfrak{F}_{s_l^*}^1)',\ \lambda(\mathfrak{F}_{s_r^*}^1)'\}$ is the uniform Lipschitz bound. □

In this section, we have claimed that $\mathfrak{F}_{s^*}$ is a C^{1+Lip} extension of a piece-wise affine infinitely renormalizable map f_{s^*} corresponding to a pair of proper scaling data $s^* = (s_l^*, s_r^*)$. This implies $\mathfrak{F}_{s^*}$ is also renormalizable map. Since $R\mathfrak{F}_{s^*}$ is an extension of Rf_{s^*}, therefore $R\mathfrak{F}_{s^*}$ is renormalizable. In fact, $\mathfrak{F}_{s^*}$ is infinitely renormalizable map which is not a C^2 map. Then we have the following theorem,

Theorem 4.2.3. *For a given proper scaling data $s^* = (s_l^*, s_l^*)$, one has a pairwise infinitely renormalizable C^{1+Lip} symmetric bimodal map $\mathfrak{F}_{s^*}$ such that*

$$R\mathfrak{F}_{s^*} = \mathfrak{F}_{s^*}.$$

4.3 TOPOLOGICAL ENTROPY OF RENORMALIZATION OPERATOR

Let us opt two pairs of C^{1+Lip} maps, say, $\eta_i : [0,u_1] \to [0, b_{c_l^*}(0)]$ and $\varphi_i : [u_1', 1] \to [\tilde{b}_{c_r^*}(1), 1]$, for $i = 0,1$, which extend f_{s^*}. Because of symmetricity,
$\varphi_i(x) = 1 - \eta_i(1-x)$.
For a sequence $\beta = \{\beta_n\}_{n\geq 1} \in \Sigma_2$, where $\Sigma_2 = \{\{x_n\}_{n\geq 1} : x_n \in \{0,\ 1\}\}$ is called full 2-Shift.
Now define

$$\Gamma_l^n(\beta) = \xi_l^n(graph\ \eta_{\beta_n}) \quad \text{and} \quad \Gamma_r^n(\beta) = \xi_r^n(graph\ \varphi_{\beta_n}),$$

then, we have

$$\Gamma_l(\beta) = \bigcup_{n\geq 1} \Gamma_l^n(\beta) \quad \text{and} \quad \Gamma_r(\beta) = \bigcup_{n\geq 1} \Gamma_r^n(\beta).$$

Therefore, one can prove that $\Gamma(\beta) = \Gamma_l(\beta) \oplus \Gamma_r(\beta)$ is the graph of a C^{1+Lip} symmetric bimodal map b_β by using the same arguments of Section 4.2.
The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined as

$$\sigma(\beta_1\beta_2\beta_3\ldots) = (\beta_2\beta_3\beta_4\ldots).$$

Proposition 4.3.1. *For all $\beta \in \Sigma_2$, the restricted maps $b_\beta^2 : [u_1,\ u_0] \to [u_1,\ u_0]$ and $b_\beta^2 : [u_0',\ u_1'] \to [u_0',\ u_1']$ are unimodal maps. In particular, b_β is a renormalizable map and $Rb_\beta = b_{\sigma(\beta)}$.*

Proof. Since $b_\beta : [u_1,\ u_0] \to I_{1,l}^1$ is a unimodal and onto, and also $b_\beta : I_{1,l}^1 \to [u_1,\ u_0]$ is onto and affine. This implies, b_β^2 is a unimodal map on $[u_1,\ u_0]$. Analogously, one can prove that b_β^2 is a unimodal map on $[u_0',\ u_1']$. The above construction implies

$$Rb_\beta = b_{\sigma(\beta)}.$$

□

Then the above construction yields a result on topological entropy.

Theorem 4.3.2. *The pairwise period doubling renormalization operator R acting on the space of C^{1+Lip} symmetric bimodal maps has unbounded topological entropy.*

Proof. From Section 4.1.1, one can conclude that the domain of R^l contains a copy, namely Λ_l, of the full 2-shift. Analogously, in Section 4.1.2, the domain of R^r contains a copy, namely Λ_r, of the full 2-shift. As topological entropy h_{top} is an invariant of topological conjugacy. Hence $h_{top}(R|_{\Lambda_l \cup \Lambda_r}) > \ln 2$. Opt n different symmetric pairs, namely $(\eta_i,\ \varphi_i)$, of C^{1+Lip} maps η_i and φ_i for $i = 0,1,2,\ldots.,n-1$, which extends f_{s^*}. Then it will be embedded two copies of the full $n-$shift in the domain of R. Since, n is an arbitrary natural number, hence, the topological entropy of R acting on the space of C^{1+Lip} symmetric bimodal maps is unbounded. □

4.4 CONTINUUM OF FIXED POINTS OF RENORMALIZATION

This section describes the construction of the fixed point of renormalization by perturbing the scaling data as presented in Section 4.1, to get the following result:

Theorem 4.4.1. *The renormalization operator acting on C^{1+Lip} symmetric bimodal maps has a continuum of its fixed points.*

Proof. Let us perturb the scaling data and modify the construction as described in Section 4.1. Define a neighborhoods N_ε^l and N_ε^r about the respective points $(b_c^2(0),\ b_c^3(0))$ and $(b_c^2(1),\ b_c^3(1))$ as

$$N_\varepsilon^l(b_c^2(0),\ b_c^3(0)) = \{(b_c^2(0),\ \varepsilon \cdot b_c^3(0)) \ :\ \varepsilon > 0 \ \text{and}\ \varepsilon \text{ close to } 1\}$$
$$N_\varepsilon^r(b_c^2(1),\ b_c^3(1)) = \{(b_c^2(1),\ \varepsilon \cdot b_c^3(1)) \ :\ \varepsilon > 0 \ \text{and}\ \varepsilon \text{ close to } 1\}$$

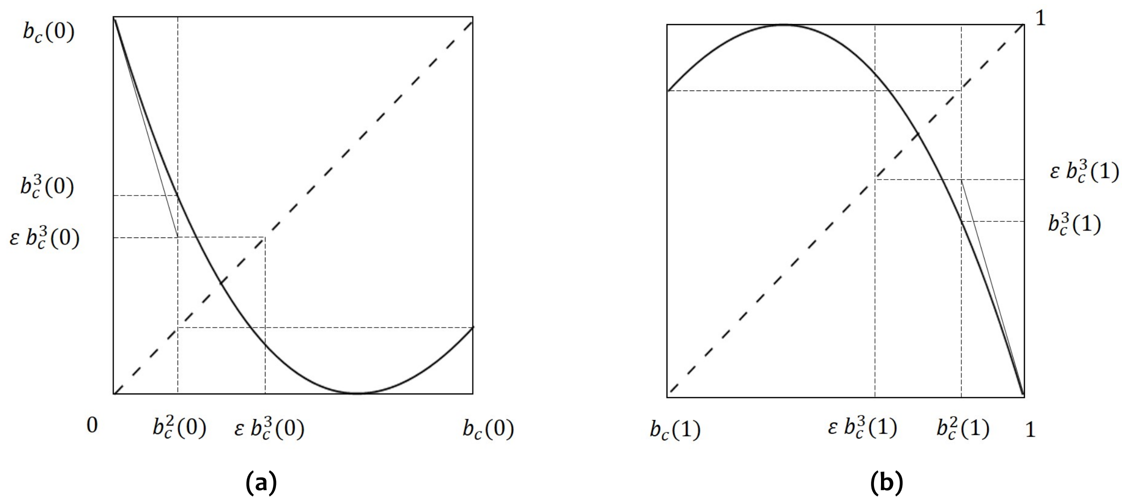


Figure 4.10 : The perturbed scaling data (a) on $I_L = [0, b_c(0)]$; (b) on $I_R = [b_c(1), 1]$.

(i). From Figure 4.10a, the perturbed scaling data on $I_L = [0, b_c(0)]$ defines the scaling ratios as

$$s_{1,l}(c,\varepsilon) = \frac{b_c^2(0)}{b_c(0)}$$
$$s_{0,l}(c,\varepsilon) = \frac{b_c(0) - \varepsilon b_c^3(0)}{b_c(0)},$$

where $c \in (0, \frac{3-\sqrt{3}}{6})$. Also, define

$$\mathscr{R}(c,\varepsilon) = \frac{b_c(0) - c}{s_{0,l}(c,\varepsilon)}.$$

From subsection 4.1.1, the map $\mathscr{R}$ which is defined in Eqn. 4.4, has unique fixed point c^*. Thus, $\mathscr{R}(c,\varepsilon)$ has exactly one unstable fixed point, namely c_ε^*, for a given ε close to 1. Therefore, corresponding to c_ε^*, we have the perturbed scaling data $s_{l,\varepsilon}^* : \mathbb{N} \to \Delta^3$ with

$$s_{l,\varepsilon}^* = \left(\frac{b_{c_\varepsilon^*}(0) - \varepsilon b_{c_\varepsilon^*}^3(0)}{b_{c_\varepsilon^*}(0)}, \frac{b_{c_\varepsilon^*}^2(0)}{b_{c_\varepsilon^*}(0)} \right).$$

Then $\sigma(s_{l,\varepsilon}^*) = s_{l,\varepsilon}^*$ and using Lemma 4.1.1, we get

$$R^l f_{s_{l,\varepsilon}^*} = f_{s_{l,\varepsilon}^*}.$$

(ii). Similarly, by considering the perturbed scaling data on I_R, we get the scaling data $s_{r,\varepsilon}^* : \mathbb{N} \to \Delta^3$ with

$$s_{r,\varepsilon}^* = \left(\frac{\varepsilon b_{c_\varepsilon^*}^3(1) - b_{c_\varepsilon^*}(1)}{1 - b_{c_\varepsilon^*}(1)}, \frac{1 - b_{c_\varepsilon^*}^2(1)}{1 - b_{c_\varepsilon^*}(1)} \right).$$

Then $\sigma(s_{r,\varepsilon}^*) = s_{r,\varepsilon}^*$ and using Lemma 4.1.4, we have

$$R^r f_{s_{r,\varepsilon}^*} = f_{s_{r,\varepsilon}^*}.$$

It follows that $f_{s_{l,\varepsilon}^*}$ and $f_{s_{r,\varepsilon}^*}$ are the piece-wise affine infinitely renormalizable maps. For a given pair of perturbed scaling data $s_\varepsilon^* = (s_{l,\varepsilon}^*, s_{r,\varepsilon}^*)$, we get

$$R f_{s_\varepsilon^*} = f_{s_\varepsilon^*}.$$

Further, the piece-wise renormalizable map $f_{s_\varepsilon^*}$ is extended to a C^{1+Lip} symmetric bimodal map, namely $g_{s_\varepsilon^*}$. Therefore, $g_{s_\varepsilon^*}$ is also a renormalizable map. As $Rg_{s_\varepsilon^*}$ is an extension of $Rf_{s_\varepsilon^*}$. This implies $Rg_{s_\varepsilon^*}$ is also renormalizable with $Rg_{s_\varepsilon^*} = g_{s_\varepsilon^*}$. Hence, for every ε close to 1, $g_{s_\varepsilon^*}$ is a fixed point of the renormalization. This proves the existence of a continuum of fixed points of the renormalization. $\square$

Moreover, we opt two different perturbed scaling data $s_{\varepsilon_1^*}$ and $s_{\varepsilon_2^*}$, one can construct two infinitely renormalizable maps $g_{s_{\varepsilon_1^*}}$ and $g_{s_{\varepsilon_2^*}}$. Therefore, their respective Cantor attractors have different scaling ratios. Consequently, this proves the non-rigidity for C^{1+lip} symmetric bimodal maps.

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