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Summary and future scope

5.1 SUMMARY

Renormalization is a tool to study the geometric properties of attractors on a smaller scale. The renormalization operator acts like a microscope. Given a one dimensional dynamical system, the image under the renormalization operator is a similar system describing the geometry and dynamics on micro scale. For infinitely renormalizable systems, one can study the dynamical properties at arbitrary small scale by applying the renormalization operator again and again. The present study describes the renormalizations of unimodal maps and symmetric bimodal maps whose smoothness are below C^2. From the study of this thesis, the important conclusions are briefly described as follows:

At the beginning, this thesis discusses the period tripling and period quintupling renormalizations of unimodal maps below C^2 smoothness. Firstly, the piece-wise affine renormalization's fixed points, namely f_{s^*} and g_{s^*}, have been constructed corresponding to the proper scaling data. Also, it has been observed that the geometry of the invariant Cantor set of the map g_{s^*} is more complex than the geometry of the invariant Cantor set of f_{s^*}. Furthermore, the piece-wise affine period tripling and quintupling renormalizations fixed points are extended to the space of C^{1+Lip} unimodal maps (which is not C^2). Further, the period tripling and period quintupling renormalization operators defined on this space have positive entropy. In fact, the topological entropy of both renormalization operators R are unbounded. Moreover, the existence of continuum of fixed points of period tripling and period quintupling renormalizations are described by considering $\varepsilon-$variation of scaling data. Consequently, there exists an infinitely renormalizable C^{1+Lip} unimodal map k with quadratic tip such that the critical points of $R^n k$ are dense in a Cantor set. This shows non-rigidity of period tripling and quintupling renormalizations. In addition, the similar results are obtained in case of period quadrupling renormalization by considering period 4 combinatorics. This study reveals the fact that as n increases for period $n-$renormalization, the number of possible combinatorics are increased. Therefore, each such combinatorics yields the construction of renormalization fixed point. In addition, for period p combinatorics ($p \geq 6$), the renormalization operator corresponding to a proper scaling data can be defined. Simultaneously, the scaling ratios $s_i(n)$ and c_{n+1}, for $i = 1, 2, \dots p$ and $n \in \mathbb{N}$, can be derived. Since c_{n+1} and the scaling ratios will be the higher degree rational polynomial functions, therefore it is a very challenging task to compute the feasible domain and the fixed point of c_{n+1}. Therefore, due to computational limitations, we only able to prove the results for period p combinatorics, where $p \leq 5$.

In the context of renormalization of symmetric bimodal maps with period tripling combinatorics, the renormalization operator R is a pair of period tripling renormalization operators R^l and R^r which are defined on piece-wise affine period tripling infinitely renormalizable maps corresponding to a pair of proper scaling data s_l and s_r, respectively. For a given pair of proper scaling data $s^* = (s_l^*,\ s_r^*)$, the construction of the piece-wise affine infinitely renormalizable map f_{s^*}, which is the only fixed point of the renormalization, has been described. One can observe that the geometry of invariant Cantor set is more complex than the geometry of the Cantor set of piece-wise affine period doubling renormalizable map [Chandramouli et al., 2009] and different from the geometry of the Cantor set of piece-wise affine pairwise period doubling renormalizable map. Further, there exists a C^{1+Lip} symmetric bimodal map which extends f_{s^*}. Moreover, the renormalization operator acting on the space of C^{1+Lip} symmetric bimodal maps has infinite topological entropy. Finally, the existence a continuum of fixed points of renormalization has been proven by considering a small perturbation on the scaling data. Consequently, it showed the non-rigidity of the Cantor attractors of infinitely renormalizable symmetric bimodal maps with low smoothness.

Finally, in the case of period doubling combinatorics, the renormalization operator R is a pairwise period doubling renormalization operators R^l and R^r which are defined on piece-wise affine period doubling infinitely renormalizable maps corresponding to a pair of proper scaling data s_l and s_r, respectively. The pairwise period doubling renormalization operator R has a fixed point f_{s^*} in the space of piece-wise affine infinitely renormalizable maps. The geometry of invariant Cantor set is different from the geometry of the Cantor set of piece-wise affine period doubling renormalizable map [Chandramouli et al., 2009]. Further, the fixed point f_{s^*} is extended to a C^{1+Lip} symmetric bimodal map. Also, the topological entropy of pairwise period doubling renormalization operator R acting on C^{1+Lip} symmetric bimodal maps, is unbounded.

5.2 FUTURE SCOPE

The study investigated in the thesis can be continued in following manner.

1. The developed theory for unimodal maps can be extended to study the renormalization fixed point(s), with bounded combinatorics of length $n \geq 6$, whose smoothness is below C^2.

2. The existing theory of symmetric bimodal maps can be helpful to construct a low-smooth renormalization fixed point by considering a pair of bounded combinatorics of length $n \geq 4$.

3. The renormalization operator introduced in chapter 3, can motivate to generalize the renormalization operator for multimodal maps with a finite number of critical points.

4. The developed theory can be helpful to describe the renormalization of asymmetric unimodal or asymmetric bimodal maps with low smoothness. Asymmetric in the sense that the critical exponents are not equal around the neighborhood of the critical point.

5. In the context of circle maps, the developed theory may be useful to construct the low smooth renormalization fixed point.

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