Appendix A
List of publications

Gupta D., Chandramouli V.V.M.S., "Topological entropy of one-dimensional deformed maps", *AIP Conference Proceedings*, 2435 (1), 020019, 2022. (doi: https://doi.org/10.1063/5.0083735)

Gupta D., Chandramouli V.V.M.S., "An improved q-deformed logistic map and its implications", *Pramana - Journal of Physics*, 95 (4), 2021. (doi: 10.1007/s12043-021-02209-7)

Gupta D., Chandramouli V.V.M.S., "Stochastically stable chaos for q-deformed unimodal maps" *International Journal of Dynamics & Control* 2022. (doi: https://doi.org/10.1007/s40435-022-00968-8)

Gupta D., Chandramouli V.V.M.S., "Dynamics of deformed Hénon-like map" *Chaos, Solitons & Fractals*, 155, 111760, 2021. (doi: https://doi.org/10.1016/j.chaos.2021.111760)

Appendix B

Relation between different types of deformations

We discuss that the deformations used in the thesis are not topologically conjugate with each other, which can be observe by following proposition:

Lemma B.0.1. *Deformed logistic map with type-1 deformation* $[x]_q = \frac{1-q^x}{1-q}$ *is not topologically conjugate to deformed logistic map with type-3 deformation* $[x]_q = \frac{x}{1+(1-q)(1-x)}$.

Proof. Let $f_1(x) = \frac{1-q^x}{1-q}$ and $f_2(x) = \frac{x}{1+(1-q)(1-x)}$.
Find the value of q such that

$$f_1(x)|_{q=q_1} = f_2(x)|_{q=q_2},$$

which implies

$$q_2 = \frac{(x-2)(q_1^x-1)+x(q_1-1)}{(x-1)(q_1^x-1)}. \tag{B.1}$$

Take the logistic map $f_a(x) = ax(1-x)$, where $a \in [0,4]$ and $x \in [0,1]$.
Now define the composition

$$\tilde{\mathscr{F}}(x) = f_1 \circ f_a(x) = \frac{1-q^{(ax(1-x))}}{1-q},$$

and

$$\tilde{\mathscr{G}}(x) = f_2 \circ f_a(x) = \frac{ax(1-x)}{1+(1-q)(1-ax(1-x))}.$$

Now

$$\tilde{\mathscr{F}}(x)|_{q=q_1} = \frac{1-q_1^{(ax(1-x))}}{1-q_1}.$$

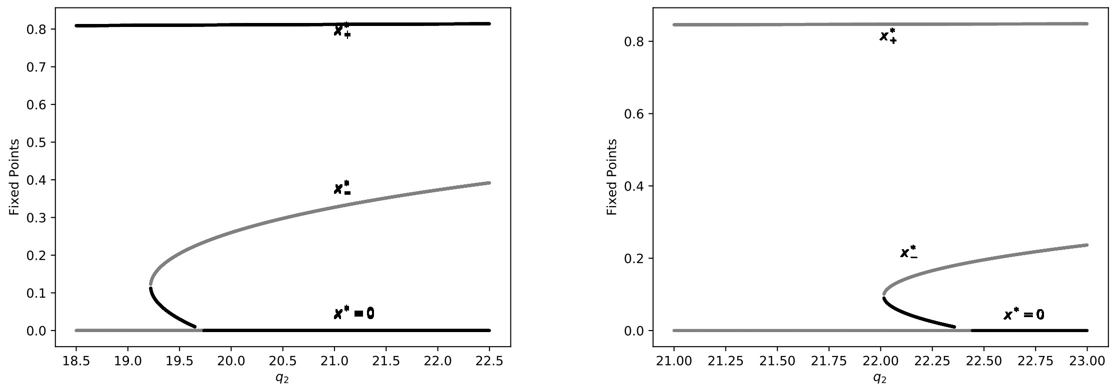
By substituting the value $q = q_2$ Eq. (B.1) in $\mathscr{G}(x)$ and simplification, we obtain

$$\tilde{\mathscr{G}}(x)|_{q=q_2} = \frac{-ax(1-x)^2(q_1^x-1)}{(x-1)(q_1^x-1)+(q_1^x-1-xq_1+x)(1-ax+ax^2)}.$$

At $q_1 = 0.02$ and $a = 2.957$, $\tilde{\mathscr{F}}(x)$ has two unstable fixed points 0 and 0.8263, whereas $\tilde{\mathscr{G}}(x)$ has one stable fixed point 0.0858 and one unstable fixed point 0. Therefore, the map $\tilde{\mathscr{F}}(x)$ is not topologically conjugate to $\tilde{\mathscr{G}}(x)$. □

Remark B.0.2. *Similar to the Proposition B.0.1, one can show that deformed logistic map with type-1 deformation* $[x]_q = \frac{1-q^x}{1-q}$ *is not topologically conjugate to deformed logistic map with type-2 deformation* $[x]_q = \frac{q^x-q^{-x}}{q-q^{-1}}$.

The zoom part of the Figs. 2.6(i) and 2.6(ii) (page no 19) around the bifurcation threshold are shown in the following Fig. B.1.



(i) $a = 3.3$ (ii) $a = 3.6$

Figure B.1. : Fixed points of q-logistic map of type-1 i.e. $\mathsf{H}_{a,q_2}(x)$.

The coexistence of the fixed point $x^* = 0$ with the chaotic attractor for fixed 'a' within the region R_4 (Fig. 2.7, page no 18) is shown in the Fig. B.2.

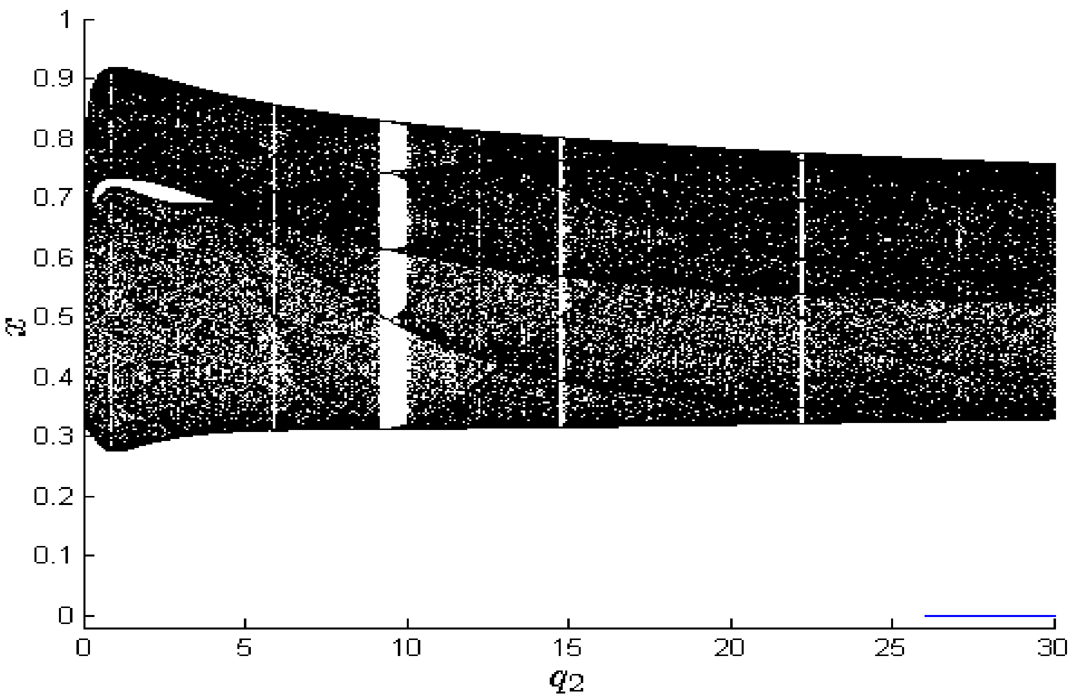


Figure B.2. : Coexistence of attractor of q-logistic map type-2 at $a = 3.6735$ and $q_2 \in (0, 30)$.

The superstable periodic orbits of period-2^n of q-Gaussian map of type-1 ($G_{c,q_1}(x)$) are given by the following tables:

Period	Value of a	Value of δ	Value of a in reverse bif.	Value of δ
2^1	- 0.85469370593149	-	-0.00055775297404	-
2^2	-0.79520432749891	4.00000000000	-0.24253406661085	-
2^3	-0.77954637235113	4.00000000000	-0.26794589708880	4.00000000000
2^4	-0.77601238220907	4.43067312538	-0.27180589094423	6.58338625129
2^5	-0.77524647520234	4.61412431407	-0.27259592978116	4.88582798087
2^6	-0.77508202097225	4.65726546702	-0.27276405928896	4.69898976849
2^7	-0.77504678050774	4.66663060188	-0.27280002143645	4.67517986387
2^8	-0.77503923218953	4.66865115448	-0.27280772138566	4.67043956838
2^9	-0.77503761553033	4.66908437281	-0.27280937038507	4.66946756017
2^{10}	-0.77503726928986	4.66918024695	-0.27280972354582	4.66926002150
2^{11}	-0.77503719513541	4.66918019958	-0.27280979918203	4.66920222716

Table B.1. : Feigenbaum ratios of the superstable periodic orbits of q-Gaussian map of type-1 at $b = 7.5$ and $q_1 = 1.01$.

Appendix C

Figures and tables related to Chapter 4

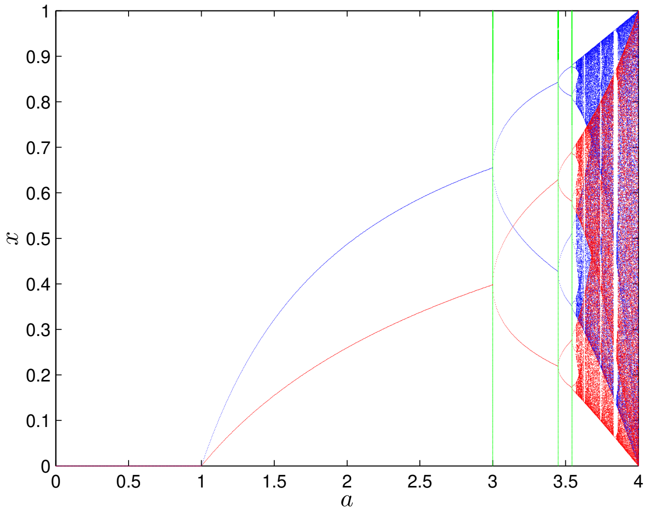


Figure C.1. : Bifurcation diagram of q-logistic map of type-A (4.6) in which red color is for $q = 0.1$ and blue color is for $q = 0.9$ and green vertical lines shows period doubling bifurcation points.

Per	$q = 0.01$	δ for $q = 0.01$	Per	$q = 0.15$	δ for $q = 0.15$
2^0	0.593417243762	-	2^0	1.166790171614	-
2^1	2.155456474286	-	2^1	2.978015520938	-
2^2	2.879435954753	-	2^2	3.399813047742	-
2^3	3.048791037611	4.27492029320	2^3	3.487720711575	4.79818833091
2^4	3.085469745532	4.61725868919	2^4	3.506542761766	4.67046166282
2^5	3.093347621967	4.65591307811	2^5	3.510572364181	4.67094473702
2^6	3.095035678955	4.66683085537	2^6	3.511435348436	4.66938115410
2^7	3.095397252764	4.66863733834	2^7	3.511620170878	4.66926117617
2^8	3.095474692682	4.66908824694	2^8	3.511659754100	4.66921172884
2^9	3.095491278030	4.66917641043	2^9	3.511668231609	4.66920411917
2^{10}	3.095494830108	4.66919618653	2^{10}	3.511670047232	4.66920204996
2^{11}	3.095495590855	4.66920070785	2^{11}	3.511670436082	4.66920150540
2^{12}	3.095495753783	4.66920071311	2^{12}	3.511670519362	4.66920151615

Table C.1. : Periodic points and corresponding Feigenbaum ratios of q-logistic map of type-B (4.12) for different values of q.

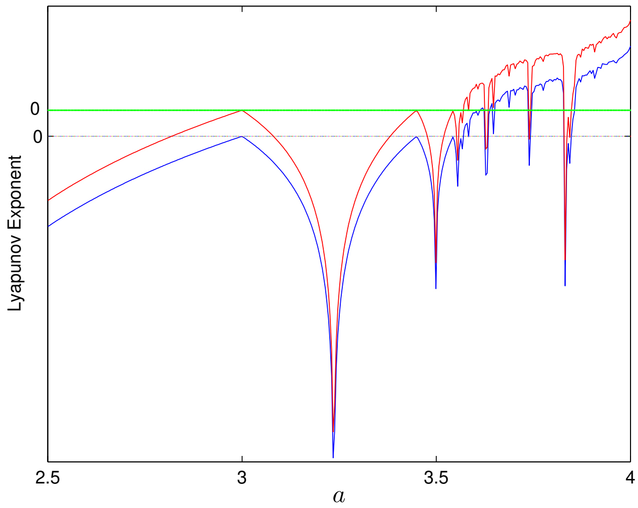


Figure C.2. : Lyapunov exponent of q-logistic map of type-A (4.6) is shown by red color for $q = 0.15$; and Lyapunov exponent of logistic map is shown by blue color for $a \in (2,4)$. For both cases, we consider initial condition $x_0 = 0.1$.

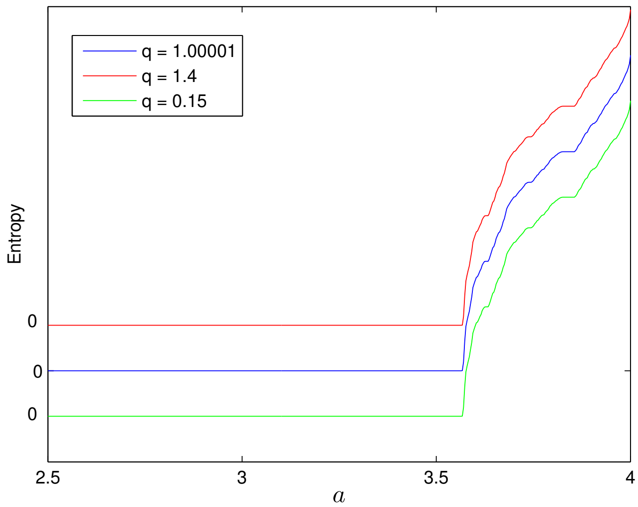


Figure C.3. : Topological entropy of q-logistic map of type-A for different values of q and $a \in (2.5,4)$

Per	$q = 1.4$	δ for $q = 1.4$	Per	$q = 1.05$	δ for $q = 1.05$
2^0	2.167448448345	-	2^0	2.024392662801	-
2^1	3.263598572094	-	2^1	3.240205330936	-
2^2	3.502251124298	-	2^2	3.499235788929	-
2^3	3.553643162657	4.64376506213	2^3	3.554635537699	4.67566123937
2^4	3.564671363996	4.66005623010	2^4	3.566517226753	4.66261560244
2^5	3.567034192331	4.66737307090	2^5	3.569062429784	4.66826768405
2^6	3.567540282707	4.66878733057	2^6	3.569607565715	4.66893280172
2^7	3.567648673774	4.66911514732	2^7	3.569724318385	4.66915172829
2^8	3.567671887916	4.66918266467	2^8	3.569749323294	4.66918989699
2^9	3.567676859678	4.66919770043	2^9	3.569754678582	4.66919936023
2^{10}	3.567677924478	4.66920075327	2^{10}	3.569755825521	4.66920056115
2^{11}	3.567678152525	4.66920075737	2^{11}	3.569756071161	4.66920357486
2^{12}	3.567678201366	4.66920692382	2^{12}	3.569756123769	4.66919786553

Table C.2. : Periodic points and corresponding Feigenbaum ratio of q-logistic map of type-B (4.12) for different values of q.

Appendix D

Superstable periodic attractors of degenerate q-Hénon map

The values of the parameter a for various ε at which the degenerate q-Hénon map accumulates to the period doubling.

Period	$a_0^{2^n}$ **for** $\varepsilon = -0.2$	δ **for** $\varepsilon = -0.2$	$a_0^{2^n}$ **for** $\varepsilon = 0$	δ **for** $\varepsilon = 0$
2	1.0000000000000000	-	1.0000000000000000	-
2^2	1.4368382203993642	-	1.3107026413368323	-
2^3	1.5457778755381502	4.00991007217	1.3815474844320617	4.38567759857
2^4	1.5700700835207542	4.48455139264	1.3969453597042405	4.60094927663
2^5	1.5753157700037295	4.63089207894	1.4002530812147824	4.65513049485
2^6	1.5764413067567378	4.66060878861	1.4009619629448415	4.66611194827
2^7	1.5766824553940413	4.66739835478	1.4011138049397998	4.66854858074
2^8	1.5767341063759470	4.66881031891	1.4011463258269454	4.66906066486
2^9	1.5767451686316347	4.66911842972	1.4011532908499202	4.66917155382
2^{10}	1.5767475378371645	4.66918363508	1.4011547825466173	4.66919514441
2^{11}	1.5767480452488623	4.66919769523	1.4011551020224629	4.66920024723
2^{12}	1.5767481539208992	4.66920205168	1.4011551704444150	4.66920097617
Period	$a_0^{2^n}$ **for** $\varepsilon = 0.3$	δ **for** $\varepsilon = 0.3$	$a_0^{2^n}$ **for** $\varepsilon = 0.5$	δ **for** $\varepsilon = 0.5$
2	1.0000000000000000	-	1.0000000000000000	-
2^2	1.2603986201527453	-	1.2502397789658288	-
2^3	1.3203860743505751	4.34088466722	1.3099753316731226	4.18912636821
2^4	1.3333789702912560	4.61694255628	1.3229657108855148	4.59844564432
2^5	1.3361695827919216	4.65592981382	1.3257591422155437	4.65033060693
2^6	1.3367675779401160	4.66661395012	1.3263578725178429	4.66559203585
2^7	1.3368956662431861	4.66861636747	1.3264861248884932	4.66837610302
2^8	1.3369230995492167	4.66908009290	1.3265135936213326	4.66903119995
2^9	1.3369289749566236	4.66917511095	1.3265194766297081	4.66916432646
2^{10}	1.3369302332904842	4.66919598247	1.3265207365920826	4.66919369573
2^{11}	1.3369305027871357	4.66920035535	1.3265210064375363	4.66919993233
2^{12}	1.3369305605050643	4.66920171430	1.3265210642301721	4.66920136242

Table D.1. : The value of the parameter a at which the deformed Hénon-like map has superstable periodic attractor of period-2^n and corresponding Feigenbaum ratio at $b = 0$ and for different ε.

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