

Declaration

I hereby declare that the work presented in this thesis titled *Dynamics of q-deformed nonlinear maps* submitted to the Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of Dr. V.V.M.S. Chandramouli. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to, any other Institute or University in India or abroad for the award of any degree or diploma.



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Certificate

This is to certify that the thesis titled *Dynamics of q -deformed nonlinear maps*, submitted by *Divya Gupta (P15MA001)* to the Indian Institute of Technology Jodhpur for the award of the degree of *Doctor of Philosophy*, is a bonafide record of the research work done by her under my supervision. To the best of my knowledge, the contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.



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Ph.D. Thesis Supervisor

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List of Symbols

\mathbb{N}	Set of natural numbers
$[x]_q$	Deformation on real number x with deformed parameter q
$\omega(x^*)$	Size of basin of fixed point x^*
F^n	Composition of the function F with itself n times
$h(f)$	Topological entropy of the function f
$K(f)$	Kneading sequence of the function f
$H_{a,b}(x)$	Hénon-like map with parameters a and b
$\mathcal{H}_{a,b,\varepsilon}(x)$	q -Hénon map with parameters a and b , and deformed parameter ε
$J(\mathcal{H})$	Jacobian of \mathcal{H}
$(J(\phi))^{-1}$	Inverse of the matrix $J(\phi)$
$\det J(\mathcal{H})$	Determinant of the matrix $J(\mathcal{H})$
\mathcal{D}_ε	Domain $\varepsilon \in (-0.2, 0) \cup (0, 0.5)$
$\mathcal{D}_\varepsilon^*$	Domain $\varepsilon \in (-0.2, \varepsilon_*) \setminus \{0\}$
$\gamma_{2^n, \varepsilon}$	most attracting curves in the parameter plane (a, b) and deformed parameter ε
$W^u(\alpha)$	Unstable manifold of the fixed point α
$W^s(\alpha)$	Stable manifold of the fixed point α
P_{2^i}	Saddle periodic points of period 2^i for $i \geq 1$
$W_{loc}^u(P_{2^i})$	Local unstable manifold the saddle periodic point P_{2^i}
$W_{loc}^s(P_{2^i})$	Local stable manifold the saddle periodic point P_{2^i}
b_*	Parameter value of b on curve $\gamma_{2^\infty, \varepsilon}$, where hetroclinic bifurcation occurs