1
Introduction

Dynamical system is a system whose state is uniquely determined by a set of variables and the behaviour is described by predefined rule. It can be characterized in discrete time steps or continuous time line. Dynamical system evolves with discrete time steps is called discrete dynamical system. A well known example of one-dimensional discrete dynamical system is logistic map, which is given by $x_{n+1} = ax_n(1-x_n)$, where the parameter $a \in [0,4]$. This map undergoes period doubling route to chaos as the parameter a varies, and it has Cantor-like attractor at the transition from periodic to chaotic region.

The interesting theory of q-deformed physical system related to quantum group structure has attracted the considerable interest of mathematicians and physicist in the direction of particular branch of mathematics (q-mathematics). A q-deformed physical system in quantum group structure is an exploration of the possible deformation in the well-known physical phenomena models. The expectation is that deviation from the exact system would be detect as a result of the deformed system. This deviation may help to observe the changes in physical behaviour of the system. Deformation of a function introduces an additional parameter q into the function's definition in such a way that the original function can be recovered under the limit $q \to 1$. Therefore several deformations can be defined for the same function.

The q-deformation of nonlinear maps can be helpful in modelling of several phenomena that are not exactly modelled with the canonical maps, but could benefit from the q-deformed variant in quantum computing applications. The newly introduced parameter q of deformed nonlinear map can be varied according to our needs, allowing us to fit a wide range of functional forms that are identical in nature. For example, the deformed logistic map and the experimentally constructed one-dimensional map for the Belousov Zhabotinskii reaction in a stirred chemical reaction are strikingly similar. Another advantage of a deformed nonlinear system is that it has a wide range of deformed parameter q so that the system leads to chaos which is useful in cryptography.

In this thesis work, we introduce different types of deformations and apply it to one-dimensional maps and two-dimensional Hénon-like maps to obtain the q-deformed maps. We discuss the various dynamical properties of these q-deformed maps and present comparisons with respect to their canonical maps. We consider the following deformations.

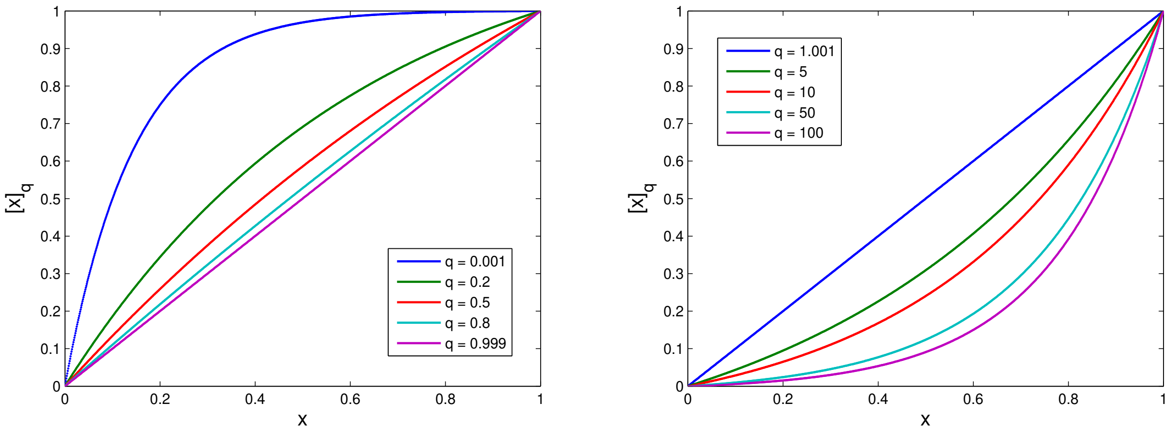
Heine 1846 [Heine, 1846] deformed a real number x as

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.1}$$

such that $[x]_q \to x$ when $q \to 1$. This we refer it as type-1 deformation.

The deformation $[x]_q$ given by Eq. (1.1) for $x \in [0,1]$ is illustrated in Fig. 1.1. We observe that when $q \in (0,1)$, then the deformation $[x]_q$ covers the region above the diagonal line in an unit square. When $q \in (1,\infty)$, then $[x]_q$ covers the region below the diagonal line in an unit square.

There is an another deformation of real number x, it is mentioned in [Jaganathan and Sinha,



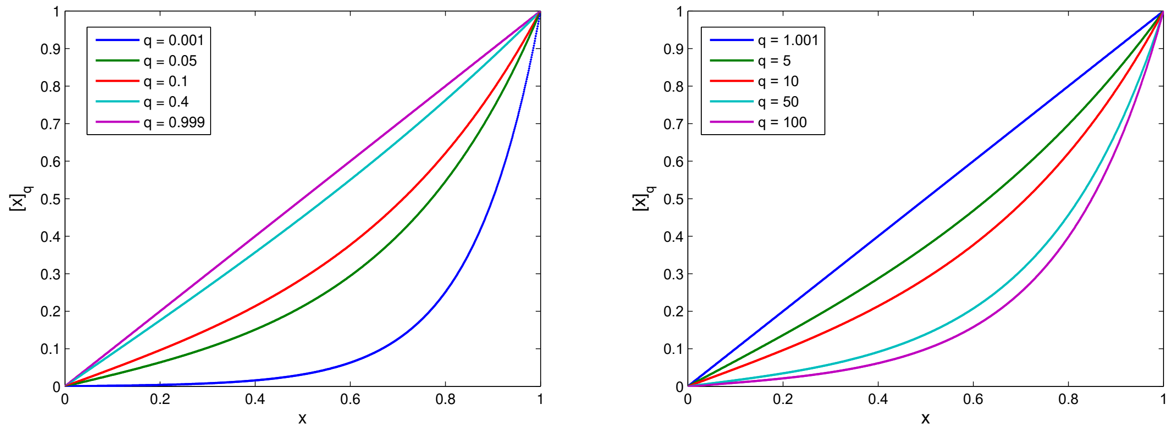
(i) (ii)

Figure 1.1. : Graph of type-1 deformation $[x]_q$; (i) $q \in (0.001, 0.999)$; (ii) $q \in (1.001, 100)$.

2005], which is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{1.2}$$

with the required property that as $q \to 1$ then $[x]_q \to x$. We refer this as type-2 deformation. The graph of this deformation $[x]_q$ is shown in Fig. 1.2(i) and Fig. 1.2(ii) for $q \in (0,1)$ and $q \in (1,\infty)$ respectively. Note that, the type-2 deformation $[x]_q$ covers the region below the diagonal line in the unit square whenever $q \in (0,1)$ or $q \in (1,\infty)$. Therefore, we choose $q \in (0,1)$ in our discussion.

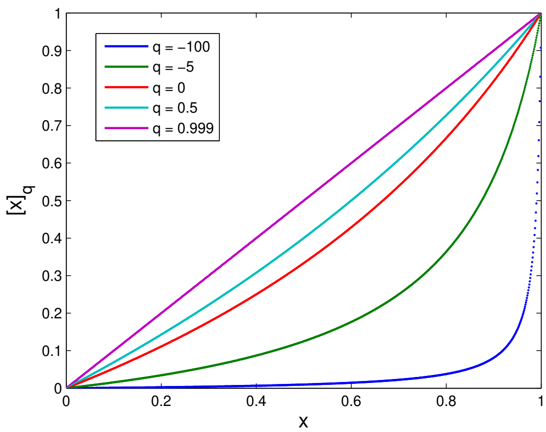


(i) (ii)

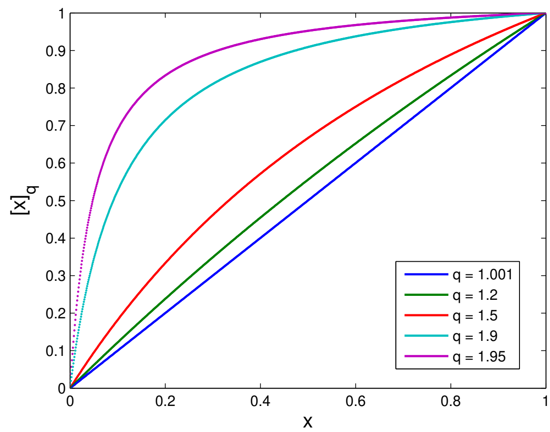
Figure 1.2. : Graph of type-2 deformation $[x]_q$; (i) $q \in (0.001, 0.999)$; (ii) $q \in (1.001, 100)$.

In the non-extensive statistical mechanics of Tsallis, a new q-exponential function has been introduced as

$$e_q^x = (1 + (1-q)x)^{\frac{1}{1-q}}, \tag{1.3}$$



(i)



(ii)

Figure 1.3. : Graph of type-3 deformation $[x]_q$; (i) $q \in (-100, 0.999)$; (ii) $q \in (1.001, 1.95)$.

which approaches to e^x as $q \to 1$.

Corresponding to Eq. (1.3), Jaganathan [Jaganathan and Sinha, 2005] introduced the deformation of nonlinear map considering logistic map as an example. The deformation of real number x with the parameter $q \in (-\infty, 2)$ is given by

$$[x]_q = \frac{x}{1+(1-x)(1-q)} \tag{1.4}$$

The graph of the deformation $[x]_q$ (Eq. (1.4)) is shown in Fig. 1.3, by considering $x \in [0,1]$. Note that the deformation $[x]_q$ covers the region below the diagonal line when $q \in (-\infty, 1)$, and it covers the region above the diagonal line when $q \in (1,2)$.
Let $1-q = \varepsilon$, then the Eq. (1.4) can be rewritten as

$$[x]_\varepsilon = \frac{x}{1+\varepsilon(1-x)}. \tag{1.5}$$

We refer it as type-3 deformation.

The q-exponential function introduced in the context of the generalization of Boltzmann-Gibbs entropy [Tsallis, 1988]. By the inspiration of q-deformed physical system related to quantum group structure and Tsallis statistical mechanics, the notion of q-deformation on the nonlinear maps was established in [Jaganathan and Sinha, 2005] and explained the dynamics of deformed logistic map generated by type-3 deformation. The deformed logistic map exhibits coexistence of attractor, which distinguishes it from the canonical logistic map. The coexistence of attractor is known as the Allee effect in population dynamics, which is related to the extinction of small population (see [Schreiber, 2003]). The Allee effect has been widely explored for unimodal maps. When the Allee effect is sufficiently strong then the population experience rapid extinction below the Allee threshold. A strong Allee effect can be divided into two categories: bistability and essential extinction. In bistability, there is an interval of initial population densities for which the population persists, while for initial densities outside this interval the extinction occurs. Population experiences extinction for almost every initial population density in the case of essential extinction.

Parrondo's paradox that involves games of chance was introduced in [Harmer and Abbott, 1999]. In this, the remarkable observation is that the randomly switching between two fair (unbiased) games with losing expectation produces a winning expectation, which leads to a game that is no longer fair. The Parrondo's paradox phenomenon in discrete systems was first discussed in [Almeida *et al.*, 2005], by showing the periodic mixing of two chaotic dynamics leading to the formation of an ordered dynamics. In [Cánovas *et al.*, 2006], it was shown that the effect of changing the compositions of functions i.e. $f_1 \circ f_2$ or $f_2 \circ f_1$ can have robustness of the Parrondo's effect under small perturbation.

The deformation of the logistic map using type-1 deformation (Eq. (1.1)) was discussed by calculating the Lyapunov exponent to analyze the stability [Banerjee and Parthasarathy, 2011]. The implications of Parrondo's paradox examined by considering growth side as a winning strategy and decay side as a losing strategy. If one can start from the 'losing' side then the deformed logistic map allows transit to the 'winning' side with fewer iterations than a canonical logistic map. In [Shrimali and Banerjee, 2012], the q-deformation of one-dimensional map with delay feedback studied and concluded that the map suppresses to be chaotic for certain region of deformation parameter and feedback parameter. Recently, the deformed logistic map using type-3 deformation analyzed, and existence of chaotic region was shown by computing the topological entropy and Lyapunov exponent [Cánovas and Muñoz-Guillermo, 2019]. It was also showed that there exist Parrondo's paradox, which occurs when two simple maps are combined to produce a complicated dynamical behaviour.

The deformation of type-3 applied on a one-dimensional Gaussian map was investigated in [Patidar and Sud, 2009]. They analyzed the variations in dynamical behaviour between the Gaussian map and and its q-deformed version, focusing on the region of parameter space where attracting fixed point that lies on the boundary of invariant interval coexist with other attractors. The q-deformed Gaussian map obtained by applying type-3 deformation on Gaussian map was also analyzed in [Cánovas and Muñoz-Guillermo, 2020]. They computed the topological entropy and Lyapunov exponent to analyze the chaos and highlighted the region of attractors that coexist with chaotic attractors.

In the first part of the thesis, we discuss the dynamical properties of the q-deformed versions of one-dimensional maps: logistic map, Gaussian map and Ricker map. Here we apply the deformations on these models and obtain q-deformed maps which are different than the study done in the literature. First we examine the periodic attractors, chaotic attractors and the boundary where the transition takes place from simple to chaotic region and then explains the Parrondo's paradox. We compute the topological entropy to highlight the region of Li-Yorke chaos. Further, we show that there exists a positive measure set of parameter values for which these deformed maps exhibit stochastically stable chaos, in the sense of [Baladi and Viana, 1996].

In the Chapter-2, we apply the type-1 deformation given by Eq. (1.1) to deform the logistic map $f_a(x) = ax(1-x)$ and Gaussian map $g_c(x) = e^{-bx^2} + c$, then we obtain the deformed maps

$$f_a([x]_{q_1}) = f_a(x) \circ [x]_{q_1} = \frac{a(1-q_1^x)(q_1^x - q_1)}{(1-q_1)^2}$$

and

$$g_c([x]_{q_1}) = g_c(x) \circ [x]_{q_1} = e^{-b\left(\frac{1-q_1^x}{1-q_1}\right)^2} + c$$

respectively. Further we apply type-2 deformation given by Eq. (1.2) on logistic map and obtain the deformed map as

$$f_a([x]_{q_2}) = f_a(x) \circ [x]_{q_2} = \frac{q_2^{ax(1-x)} - q_2^{ax(x-1)}}{q_2 - q_2^{-1}}.$$

We analyze the basic dynamics of the deformed maps including fixed points, periodic attractors, and the transition from periodicity to chaos. Further, we compute the topological entropy of deformed maps with two different methods given in [Block *et al.*, 1989] and [Dilao and Amigó, 2012] to illustrate the presence of Li-Yorke chaos. Topological conjugacy allow us to study the dynamical behaviour of an unknown map, which is associated with the map of known dynamics. The topological entropy of the tent map with slope s is given by $\log(s)$. We use the tent map as a standard map and apply the method given in [Block *et al.*, 1989], to calculate the topological entropy of deformed logistic map with type-1 and type-2 deformations. From this, we highlight the region where the chaos found to be unobservable. We also calculate the parameter value at which the phase transition (period doubling route to chaos) happens for q-deformed logistic maps. To investigate the chaotic behaviour of the deformed Gaussian map, we compute the topological entropy using the *Lap Number Method* and show that there is a region of physically observable chaos that is separated from a region of non-physically observable chaos (say R_2). In the region R_2, the basin size of the fixed point $x^* = 0$ is 1, which implies that all the trajectories in this region are converges to $x^* = 0$. But there is a set of disconnected points (say Γ) which are not converges to $x^* = 0$. Finally, we conclude that in the region R_2, the states belonging to the set Γ of totally disconnected points do not converge to 0 and the states outside Γ converges to 0.

The results described in the Chapter 2 are based on the following article:

Gupta D., Chandramouli V.V.M.S., "Topological entropy of one-dimensional deformed maps", *AIP Conference Proceedings*, (2022), 2435 (1), (doi: https://doi.org/10.1063/5. 0083735).

There are so many aspects by which one can determine the chaotic behaviour of nonlinear maps, in which the sensitive dependence on initial condition (SDIC) is one of the weakest. The SDIC is weak in the sense that it can be implied by the existence of a period 3-cycle [Li and Yorke, 1975]. In this case, we have sensitive dependence on an invariant uncountable set, which could be of measure zero. Therefore chaos can coexist with a stable periodic attractor whose basin of attraction has full measure. Another way to say that f is chaotic if it admits the absolutely continuous invariant probability (acip) measure. Almost all orbits distribute themselves according to this measure over the entire interval in S-unimodal maps with a nonflat critical point. In fact, it is similar to have a positive Lyapunov exponent almost everywhere.

Jakobson [Jakobson, 1981] proved that for a given class of unimodal map f_λ, there is a set of parameter values λ so that $f_\lambda(x)$ has an invariant measure absolutely continuous with respect to Lebesgue measure. This invariant measure exists for the parameters which are chosen in such a way that the critical point of the map is contained in the preimage of an unstable periodic orbit. Then we obtain the set $\Lambda_1 = \{\lambda : f_\lambda$ has an acip with respect to Lebesgue measure$\}$ has a positive measure. The map corresponding to each parameter of the set Λ_1 has sensitive dependence on initial conditions. This leads to the stochastically stable chaotic behaviour of the map.

Hans Thunberg [Thunberg, 2001] discussed the concept of stochastically stable chaos in one-dimensional discrete time models, the Ricker and Hassell families. They showed that these maps admits acip measure, which describes the asymptotic distribution for almost all orbits under the maps. They observed that strong chaotic properties appear with positive frequency in parameter space for these maps. From the motivation of this work, we examine the strong chaotic properties of one-dimensional deformed maps: (*i*) q-deformed logistic map with type-1 deformation and type-3 deformation, (*ii*) q-deformed Gaussian map with type-1 deformation, and (*iii*) q-deformed Ricker map with type-3 deformation.

In the Chapter-3, we consider the deformed logistic map with type-3 deformation (Eq. (1.4)), which is given by

$$F_{a,\varepsilon}(x) = \frac{ax(1+\varepsilon)(1-x)}{(1+\varepsilon-\varepsilon x)^2}.$$

and the deformed Gaussian map with type-1 deformation (Eq. (1.1)), which is given by

$$G_{c,q}(x) = e^{-b\left(\frac{1-q^x}{1-q}\right)^2} + c.$$

First, we analyze the basic dynamics of these deformed maps, such as the existence of a fixed point, periodic attractor, and their stability. The stable fixed point lies on the boundary of invariant interval coexists with other attractors in these maps. We emphasize on typical asymptotic behaviour, which can be observed for large sets of maps and for almost all initial conditions. We compute the parameters (a_n^*, ε_n) such that the deformed map $F_{a_n^*,\varepsilon_n}$ is a Misiurewicz map. Then we show that in the neighborhood of a_n^*, the map $F_{a,\varepsilon}$ has strong chaotic properties with positive frequency in parameter space. This result leads to the Theorem 1.0.1. In case of the deformed Gaussian map we show that $G_{c,q}(x)$, for a given sequence of parameters $\{q_n\} \to q^*$, there exist two sequences of Misiurewicz parameters $\{c_{1n}\}$ and $\{c_{2n}\}$ which are converging to c^* in such a way that the corresponding sequences of Misiurewicz maps G_{c_{1n},q_n} and G_{c_{2n},q_n} converges to the Misiurewicz map G_{c^*,q^*}. Further we prove that the map $G_{c,q}(x)$ has strong chaotic properties with positive frequency in the neighborhood of Misiurewicz parameters, which leads to Theorem 1.0.2. The deformed maps have chaotic behaviour for a large space of deformed parameter q than the canonical maps, which is intended to be used in cryptography. A similar analysis has been carried out to describe the stochastically stable chaos in the type-3 deformed Ricker map.

Theorem 1.0.1. *In the parameter plane, $F_{a,\varepsilon}$ are strongly chaotic with a positive frequency: For each $\varepsilon = \varepsilon_n \in (-0.95, 2) \setminus \{0\}$, there exists a Misiurewicz parameter a_n^* and positive measure subsets $\Lambda_n(\varepsilon) \subset (0,4]$ containing a_n^* such that if $a \in \Lambda_n(\varepsilon)$ then we have*

1. *There is no periodic attractor for F_{a,ε_n} and transitive interval attractor is the only metric attractor.*

2. *F_{a,ε_n} exists an acip μ_{a,ε_n} which is strongly stochastic stable and has density in L^p, $1 \le p < 2$. The asymptotic distribution of almost all orbits under F_{a,ε_n} is defined by μ_{a,ε_n}.*

3. *The Lyapunov Exponent of F_{a,ε_n} is positive almost everywhere, especially at the critical points.*

Theorem 1.0.2. *In the parameter plane, $G_{c,q}(x)$ is strongly chaotic with a positive frequency: For each $q = q_n \in (0, q^*)$, there exists two positive measure subsets Λ_{jn} containing c_{jn} for $j = \{1,2\}$ such that for each $c \in \Lambda_{jn}$, we have*

1. *There is no periodic attractor for $G_{c,q_n}(x)$ and transitive interval attractor is the only metric attractor.*

2. *$G_{c,q_n}(x)$ exists an acip μ_{c,q_n} which is strongly stochastic stable and has density in L^p, $1 \le p < 2$. The asymptotic distribution of almost all orbits under $G_{c,q_n}(x)$ is defined by μ_{c,q_n}.*

3. *The Lyapunov Exponent of $G_{c,q_n}(x)$ is positive almost everywhere, especially at the critical points.*

The results described in the Chapter 3 are based on the following article:

Gupta D., Chandramouli V.V.M.S., "Stochastically stable chaos for q-deformed unimodal maps", *Int. J. of Dynamics & Control* 2022, (doi: https://link.springer.com/article/10.1007/s40435-022-00968-8).

In Chapter 4, we show that the q-deformation scheme applied on both sides of the difference equation of logistic map is topologically conjugate to the canonical logistic map and therefore there is no dynamical changes by this q-deformation. We propose an improved variation of q-deformation scheme and apply type-1 deformation on logistic map to describe the dynamical changes. We illustrate the Parrondo's paradox by assuming chaotic dynamics as a gain. Finally we show that in the neighborhood of particular parameter value, the q-logistic map has stochastically stable chaos.

The results described in the Chapter 4 are based on the following article:

Gupta D., Chandramouli V.V.M.S., "An improved q-deformed logistic map and its implications", *Pramana - Journal of Physics* 2021, (doi: 10.1007/s12043-021-02209-7).

In the second part of the thesis, we study the dynamics of the two-dimensional deformed maps. In particular we consider deformed Hénon-like map, which is obtained by applying the type-3 deformation on the Hénon-like maps. The reason to choose the Hénon-like maps is that these maps are good models for creating chaos in real time applications. The two-dimensional Hénon-like maps $H(x,y) = (f_a(x) - by, x)$ are real continuous maps, where $f_a(x) = a - x^2$ is an unimodal map and b is the small perturbation. The Hénon-like maps are the higher-dimensional extension of unimodal maps. Many dynamical properties of unimodal maps are shared by strongly dissipative Hénon-like maps. But these maps are not rigid as compare to the one-dimensional maps, when there is an enough smoothness.

The period doubling renormalization operator was introduced to study the asymptotic small scale geometry of the attractor of one-dimensional systems, which are at the transition from simple to chaotic dynamics [Feigenbaum, 1978] and [Coullet and Tresser, 1978]. The attractors of the maps at the transition have a special property. They are Cantor sets and on arbitrary small scale the attractor can be identified with a rescaled version of the attractor of another one-dimensional map. The microscopic geometry of an attractor at the transition to chaos is universal. The fine scale geometrical structure can not be deformed. Therefore, the attractors are rigid. The renormalization theory for dissipative two-dimensional maps was initiated in [Collet *et al.*, 1981], and proved that for nearby dissipative two-dimensional maps, the one-dimensional renormalization fixed point f_* serves as a hyperbolic fixed point. Later, it was shown that similar to the one-dimensional scenario, the infinitely renormalizable two-dimensional maps which are close to f_* have an attracting Cantor set on which the map works as an adding machine [Gambaudo *et al.*, 1989]. The geometric properties of real Hénon-like maps H that are infinitely renormalizable have been studied in [De Carvalho *et al.*, 2005]. The appropriately defined renormalizations $R^n H$ converge exponentially to the one-dimensional renormalization fixed point f_*. The average Jacobian and universal function control the rate of convergence to a one-dimensional system, which occurs at a super-exponential rate. It was also proved that the universality features coexist with unbounded geometry. This is due to the lack of rigidity, which distinguish from the well-known one-dimensional renormalization theory. In [Lyubich and Martens, 2011], they showed that the infinitely renormalizable Hénon-like maps form a curve in the parameter space, which is parametrized by the average Jacobian. For an infinitely renormalizable Hénon-like maps, the average Jacobian on the attracting Cantor set is topologically invariant. The deformation on the two-dimensional Hénon map analyzed numerically by applying the type-3 deformation on the both state variables [Patidar *et al.*, 2011]. They focused on the region of periodic motion and strange chaotic motion in the parameter space. It was also showed that the q-deformed Hénon map suppresses chaos as compared to the canonical Hénon map for specific values of deformed parameters.

The above research work motivates to investigate the dynamical properties of q-Hénon maps for various deformation parameter q. First, we construct the "most attracting curve" on the parameter space, where the transition take place from simple to chaotic dynamics. The dynamical properties of these maps are very interesting to study. We use heteroclinic web and renormalization concept to describe the similarities and differences between the q-Hénon map and the canonical

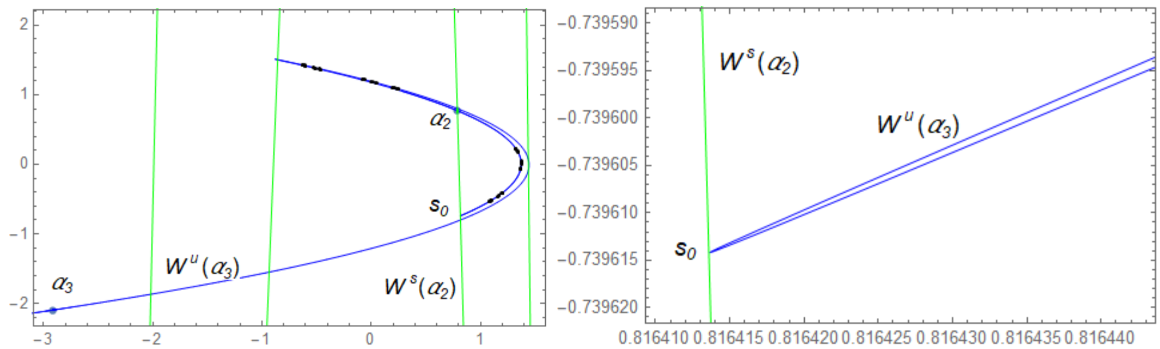
Hénon-like map.

In Chapter-5, we apply the type-3 deformation (Eq. (1.4)) on Hénon-like maps given by $H_{a,b}(x,y) = (f_a(x) - by, x)$ and obtain the deformed Hénon map, namely q-Hénon map, which is,

$$\mathscr{H}_{a,b,\varepsilon}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - \frac{x^2}{(1+\varepsilon-\varepsilon x)^2} - by \\ \frac{x}{1+\varepsilon-\varepsilon x} \end{pmatrix}.$$

We analyze some fundamental properties and stability of the fixed points of q-Hénon system. We observe that as ε increases from -0.2 to 0.5, initially the q-Hénon map $\mathscr{H}_{a,b,\varepsilon}$ has three fixed points, namely, α_1 (stable), α_2 (flip saddle) and α_3 (regular saddle), in which the stable fixed point α_1 coexists with the other two saddle fixed points. Later, at some $\varepsilon = \varepsilon_*$, the bifurcation occurs, where two fixed points α_1 and α_3 vanishes, and the system has only one fixed point α_2 (flip saddle) for $\varepsilon > \varepsilon_*$. We propose an algorithm for constructing a curve with parameter (a,b) such that the q-Hénon map has 2^n-superstable periodic orbits on this curve. For each ε, these curves are denoted by $\gamma_{2^\infty,\varepsilon}$. We show that for $\varepsilon > 0$, the phase transition in the q-Hénon maps occurs earlier than the canonical Hénon-like maps, which explains the Parrondo's paradoxical behaviour. We trace the unstable manifold of the fixed points to describe the location of periodic attractors. All the periodic points of period 2^{11} exists near the flip saddle α_2. We define the hetroclinic web and describe the hetroclinic bifurcation on each curve $\gamma_{2^\infty,\varepsilon}$ for various ε values. The stable manifold $W^s(\alpha_2)$ and unstable manifold $W^u(\alpha_3)$ are plotted in Fig. 1.4 to explain the heteroclinic bifurcation in the q-Hénon map at $\varepsilon = 0.1$ and corresponding hetroclinic bifurcation $b_* = 0.03000426$. We further show that all q-Hénon maps on the curve $\gamma_{2^\infty,\varepsilon}$ have a special property which is stated as the following result.

Proposition 1.0.3. *All q-Hénon maps $\mathscr{H}_{a,b,\varepsilon}$ are infinitely renormalizable and having Cantor set as an attractor, before the hetroclinic bifurcation on the curve $\gamma_{2^\infty,\varepsilon}$.*



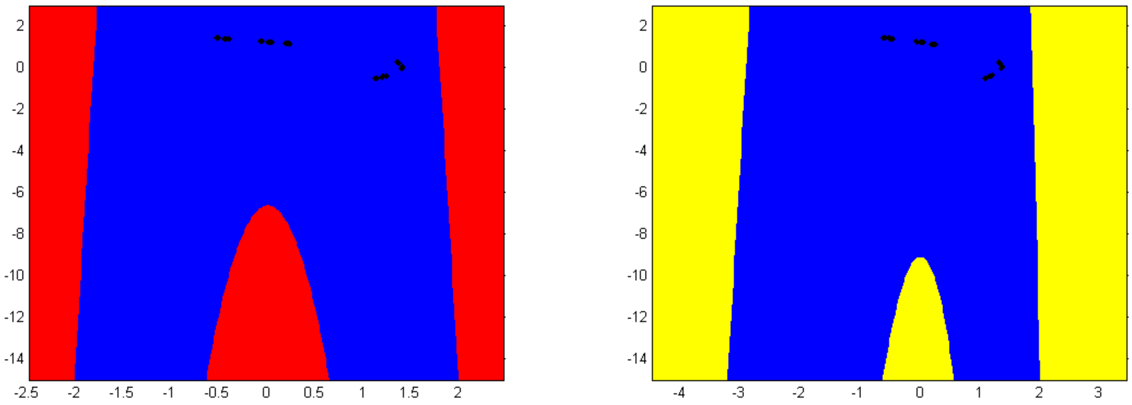
(i) (ii)

Figure 1.4. : (i) Bifurcation moment of q-Hénon map at $b_* = 0.03000426$ on $\gamma_{2^n,\varepsilon}$ for $\varepsilon = 0.1$ and 2^8 period; (ii) Magnification around the point s_0, where $W^u(\alpha_3)$ touches $W^s(\alpha_2)$.

We investigate the changes in the basin of attraction of the maps $\mathscr{H}_{a,b,\varepsilon}$ on the curve $\gamma_{2^n,\varepsilon}$ for various ε values. In the Fig. 1.5(i) and Fig. 1.5(ii), the blue region represents the basin of periodic attractor of period 2^n, the yellow region represents the basin of coexisting fixed point α_1 and the red colour region indicate the escaping region, where the black points in the blue region represents the periodic attractor. When $\varepsilon = 0$ is the case of canonical Hénon-like map, and the basin contains attracting region and escaping region, which is shown in Fig. 1.5(i). For $\varepsilon = 0.1$, the basin of the q-Hénon map contain two regions including the basin corresponding to the periodic attractor and the basin corresponding to the coexisting fixed point α_1, which is shown in Fig. 1.5(ii). Finally we conclude that the basin of attraction of the q-Hénon maps do not have escape region for each $\varepsilon \in (0, \varepsilon^*)$. This is an interesting dynamical behaviour and it illustrates the similarity of q-Hénon map to the Lorentz system, in which all trajectories are bounded.

The results described in the Chapter 5 of the thesis work are based on the following article:

Gupta D., Chandramouli V.V.M.S., "Dynamics of deformed Hénon-like map" *Chaos, Solitons & Fractals* 2021, (doi: https://doi.org/10.1016/j.chaos.2021.111760).



(i) $\varepsilon = 0$

(ii) $\varepsilon = 0.1$

Figure 1.5. : Basin of attraction of q-Hénon map for period 2^8 and for different ε, where b=0.04999 and (i) $a = 1.478456522436630$; (ii) $a = 1.436441173331582$.